Universal Thermal Transport at Quantum Critical Points

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"Quantum Field Theory, String Theory and Condensed Matter Physics"

Orthodox Academy of Crete, Kolymbari

4th September 2014

Outline

- AdS/CMT and far from equilibrium dynamics
- Quenches and thermalization
- Recent work on heat flow between CFTs
- Exact results for average current and fluctuations in $1+1$D
- Numerical simulations in lattice models
- Beyond integrability
- Potential for AdS/CFT to offer new insights
- Higher dimensions and non-equilibrium fluctuations
- Current status and future developments

M. J. Bhaseen, Benjamin Doyon, Andrew Lucas, Koenraad Schalm

"Far from equilibrium energy flow in quantum critical systems"

arXiv:1311.3655

Progress in AdS/CMT

Transport Coefficients

Viscosity, Conductivity, Hydrodynamics, Bose–Hubbard, Graphene

Strange Metals

Non-Fermi liquid theory, instabilities, cuprates

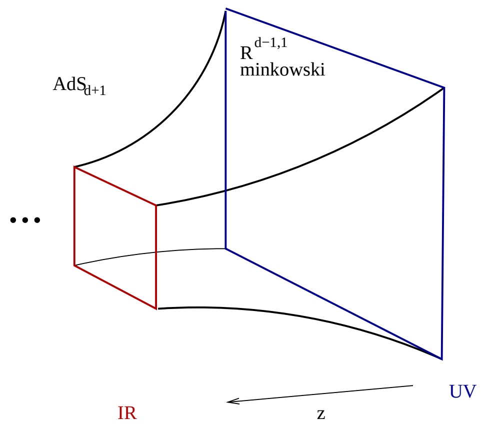
Holographic Duals

Superfluids, Fermi Liquid, $O(N)$, Luttinger Liquid

Equilibrium or close to equilibrium

AdS/CFT Correspondence

For an overview see for example John McGreevy, *Holographic duality with a view toward many body physics*, arXiv:0909.0518



Generating function for correlation functions

$$Z[\phi_0]_{\rm CFT} \equiv \langle e^{-\int d\mathbf{x}dt\, \phi_0(\mathbf{x},t)\mathcal{O}(\mathbf{x},t)} \rangle_{\rm CFT}$$

Gubser–Klebanov–Polyakov–Witten

$$Z[\phi_0]_{\rm CFT} \simeq e^{-S_{\rm AdS}[\phi]}|_{\phi\sim\phi_0 \; at\; z=0}$$

$$\phi(z) \sim z^{d-\Delta}\phi_0(1+\dots) + z^{\Delta}\phi_1(1+\dots)$$

Fields in AdS $\leftrightarrow$ operators in dual CFT $\phi \leftrightarrow \mathcal{O}$

Utility of Gauge-Gravity Duality

Quantum dynamics **Classical Einstein equations**

Finite temperature **Black holes**

Real time approach to finite temperature quantum dynamics in interacting systems, with the possibility of anchoring to $1+1$ and generalizing to higher dimensions

Non-Equilibrium **Beyond linear response**

Temporal dynamics in strongly correlated systems

Combine analytics with numerics

Dynamical phase diagrams

Organizing principles out of equilibrium

Progress

Simple protocals and integrability

Methods of integrability and CFT have been invaluable in classifying equilibrium phases and phase transitions in 1+1

Do do these methods extend to non-equilibrium problems?

Quantum quench

Parameter in H abruptly changed

$$H(g) \to H(g')$$

System prepared in state $|\Psi_g\rangle$ but time evolves under $H(g')$

Quantum quench to a CFT

Calabrese & Cardy, PRL **96**, 136801 (2006)

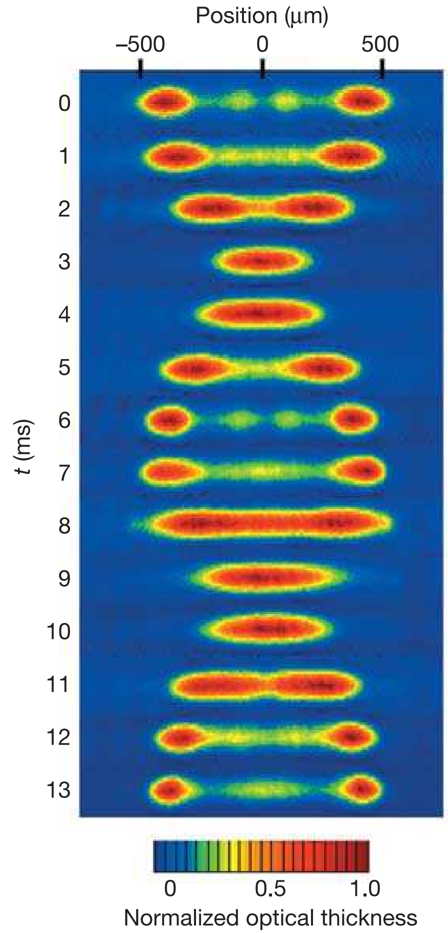
Spin chains, BCS, AdS/CFT ...

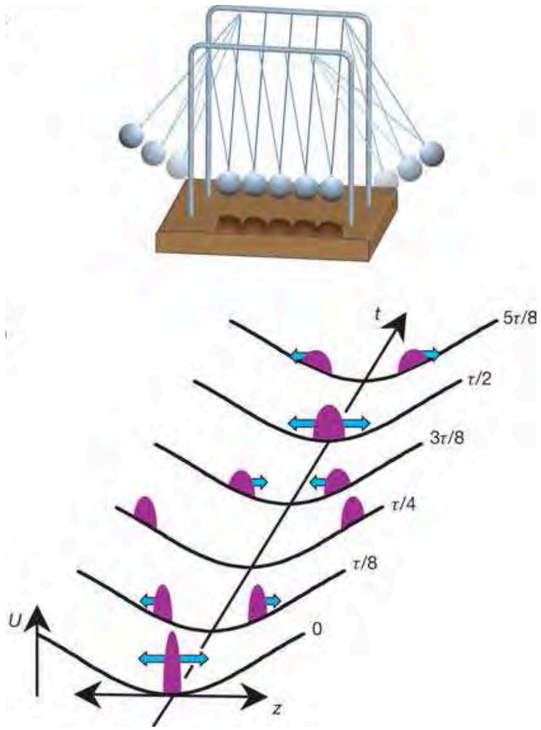
Thermalization

Experiment

Weiss *et al* “A quantum Newton’s cradle”, Nature **440**, 900 (2006)

Non-Equilibrium 1D Bose Gas





Integrability and Conservation Laws

Non-Equilibrium Holography

Chesler & Yaffe, *"Horizon formation and far-from-equilibrium isotropization in supersymmetric Yang–Mills plasma"*, PRL (2009)

$$\boxed{ds^2 = -dt^2 + e^{B_0(t)}d\mathbf{x}_\perp^2 + e^{-2B_0(t)}d\mathbf{x}_\parallel^2}$$

The dependent shear of the geometry $\quad B_0(t) = \frac{1}{2}c\,[1 - \tanh(t/\tau)]$

Jan de Boer & Esko Keski-Vakkuri *et al*, *"Thermalization of Strongly Coupled Field Theories"*, PRL (2011)

$$\boxed{ds^2 = \frac{1}{z^2}\left[-(1-m(v))z^d dv^2 - 2dzdv + d\mathbf{x}^2\right]}$$

Vaidya metric quenches $\quad m(v) = \frac{1}{2}M[1 + \tanh(v/v_0)]$

Aparício & López, *"Evolution of Two-Point Functions from Holography"*, JHEP (2011)

Albash & Johnson, *"Evolution of Holographic Entanglement Entropy after Thermal and Electromagnetic Quenches"*, NJP (2011)

Basu & Das, *"Quantum Quench across a Holographic Critical Point"*, JHEP (2012)

Non-Equilibrium AdS/CMT

Current Noise

Sonner and Green, *"Hawking Radiation and Nonequilibrium Quantum Critical Current Noise"*, PRL **109**, 091601 (2013)

Hawking Radiation

Quenches in Holographic Superfluids

Bhaseen, Gauntlett, Simons, Sonner & Wiseman, *"Holographic Superfluids and the Dynamics of Symmetry Breaking"* PRL (2013)

Quasi-Normal-Modes

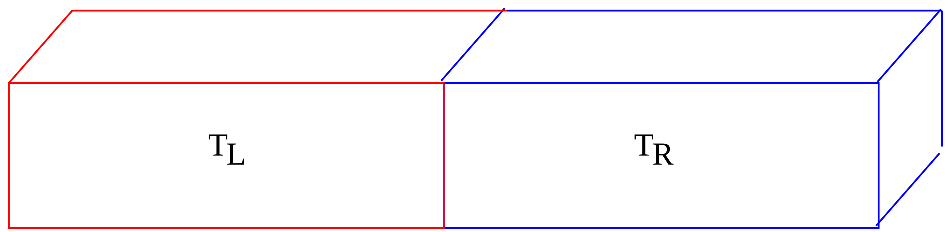
Amado, Kaminski, Landsteiner (09); Murata, Kinoshita, Tanahashi (10)

Superfluid Turbulence

Chesler, Liu and Adams, *"Holographic Vortex Liquids and Superfluid Turbulence*, Science **341**, 368 (2013); also arXiv:1307.7267

Fractal Horizons

Thermalization

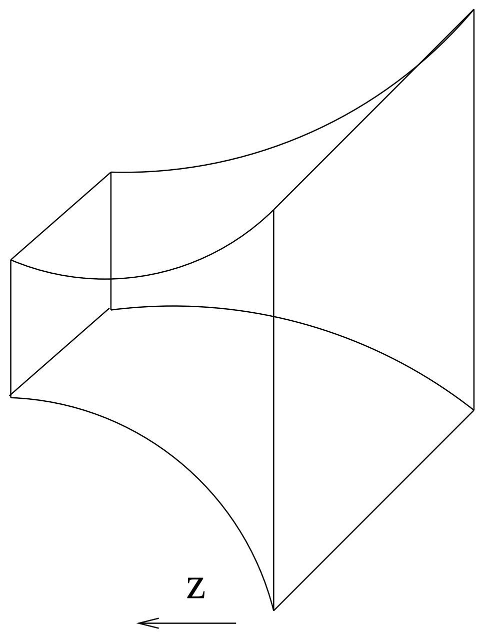


Why not connect two strongly correlated systems together and see what happens?

AdS/CFT

Heat flow may be studied within pure Einstein gravity

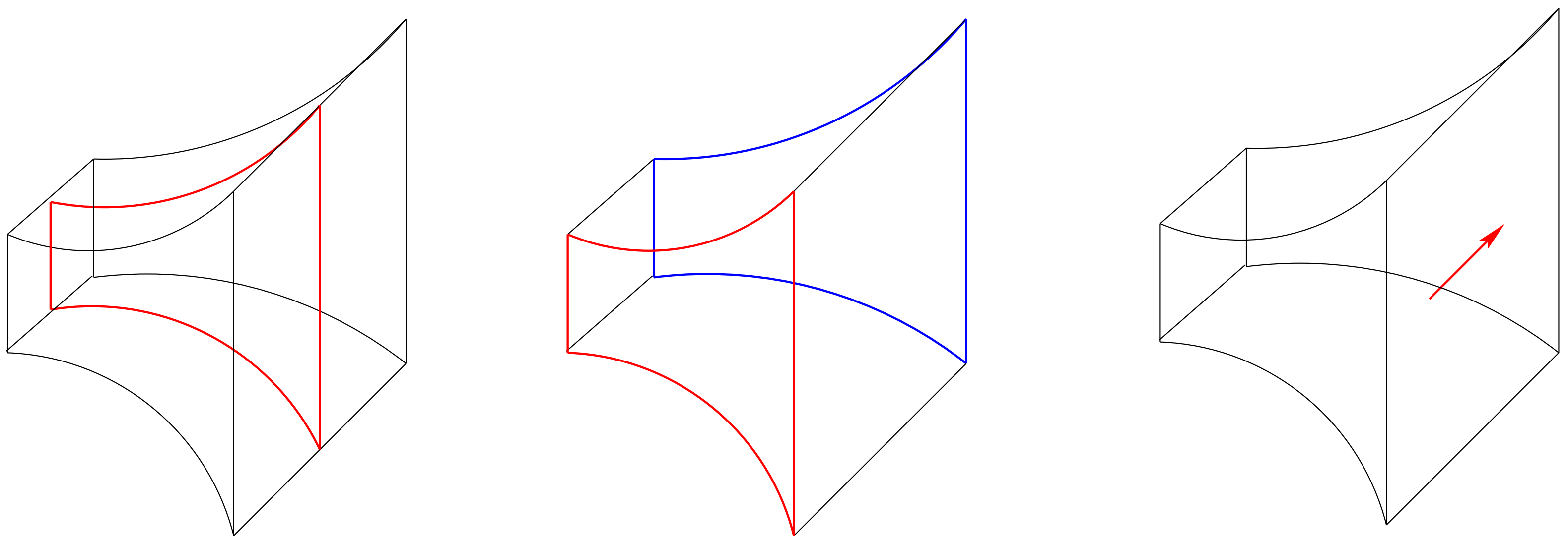
$$S = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g}(R - 2\Lambda)$$



$$g_{\mu\nu} \leftrightarrow T_{\mu\nu}$$

Possible Setups

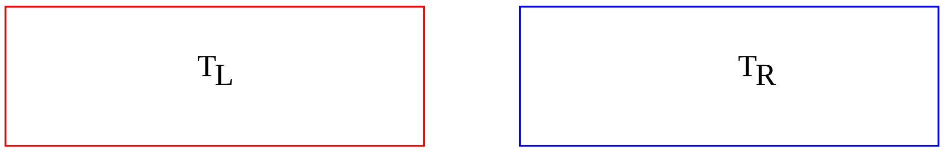
Local Quench **Driven Steady State** **Spontaneous**



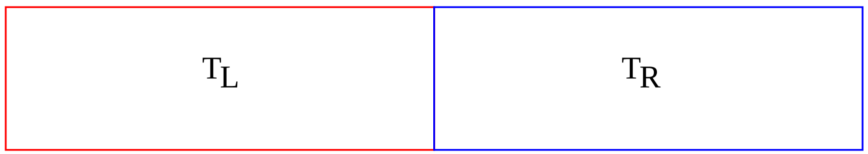
Non-Equilibrium CFT

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45**, 362001 (2012)

Two critical 1D systems (central charge c)
at temperatures T_L & T_R



Join the two systems together



Alternatively, take one critical system and impose a step profile

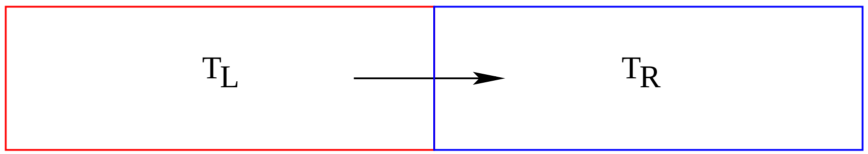
Local Quench

Steady State Heat Flow

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45** 362001 (2012)

If systems are very large ($L \gg vt$) they act like heat baths

For times $t \ll L/v$ a steady heat current flows



Non-equilibrium steady state

$$J = \frac{c\pi^2 k_B^2}{6h}(T_{\mathrm{L}}^2 - T_{\mathrm{R}}^2)$$

Universal result out of equilibrium

Direct way to measure central charge; velocity doesn't enter

Sotiriadis and Cardy. J. Stat. Mech. (2008) P11003.

Heuristic Interpretation of CFT Result

$$J = \sum_m \int \frac{dk}{2\pi}\, \hbar\omega_m(k) v_m(k) [n_m(T_{\rm L}) - n_m(T_{\rm R})] \mathbb{T}_m(k)$$

$$v_m(k) = \partial\omega_m/\partial k \quad n_m(T) = \frac{1}{e^{\beta\hbar\omega_m}-1}$$

$$J = f(T_{\rm L}) - f(T_{\rm R})$$

Consider just a single mode with $\omega = vk$ and $\mathbb{T} = 1$

$$f(T) = \int_0^\infty \frac{dk}{2\pi} \frac{\hbar v^2 k}{e^{\beta\hbar v k}-1} = \frac{k_B^2 T^2}{h} \int_0^\infty dx\, \frac{x}{e^x - 1} = \frac{k_B^2 T^2}{h}\frac{\pi^2}{6} \qquad x \equiv \frac{\hbar v k}{k_B T}$$

Velocity cancels out

$$J = \frac{\pi^2 k_B^2}{6h}(T_{\rm L}^2 - T_{\rm R}^2)$$

For a 1+1 critical theory with central charge c

$$J = \frac{c\pi^2 k_B^2}{6h}(T_{\rm L}^2 - T_{\rm R}^2)$$

Linear Response

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45**, 362001 (2012)

$$J = \frac{c\pi^2 k_B^2}{6h}(T_{\rm L}^2 - T_{\rm R}^2)$$

$$T_{\rm L} = T + \Delta T/2 \qquad T_{\rm R} = T - \Delta T/2 \qquad \Delta T \equiv T_{\rm L} - T_{\rm R}$$

$$J = \frac{c\pi^2 k_B^2}{3h} T \Delta T \equiv g \Delta T \qquad g = c g_0 \qquad g_0 = \frac{\pi^2 k_B^2 T}{3h}$$

Quantum of Thermal Conductance

$$g_0 = \frac{\pi^2 k_B^2 T}{3h} \approx (9.456 \times 10^{-13}\,\mathrm{WK}^{-2})\, T$$

Free Fermions

Fazio, Hekking and Khmelnitskii, PRL **80**, 5611 (1998)

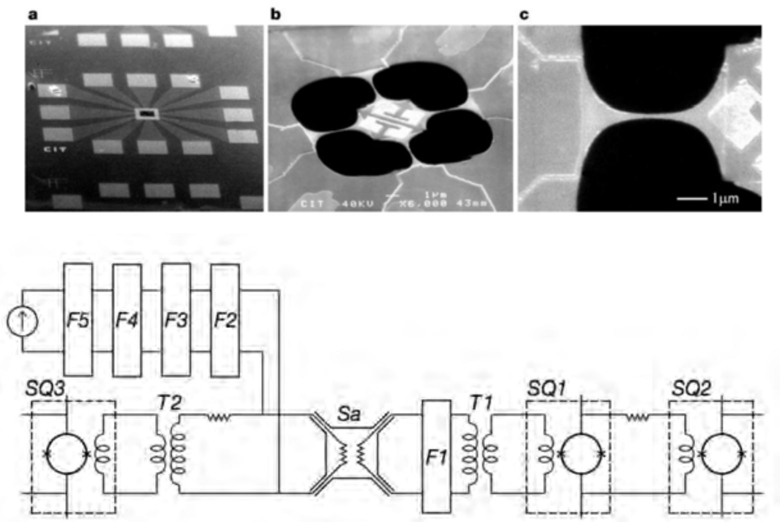
Wiedemann–Franz $\quad \frac{\kappa}{\sigma T} = \frac{\pi^2}{3e^2} \qquad \sigma_0 = \frac{e^2}{h} \qquad \kappa_0 = \frac{\pi^2 k_B^2 T}{3h}$

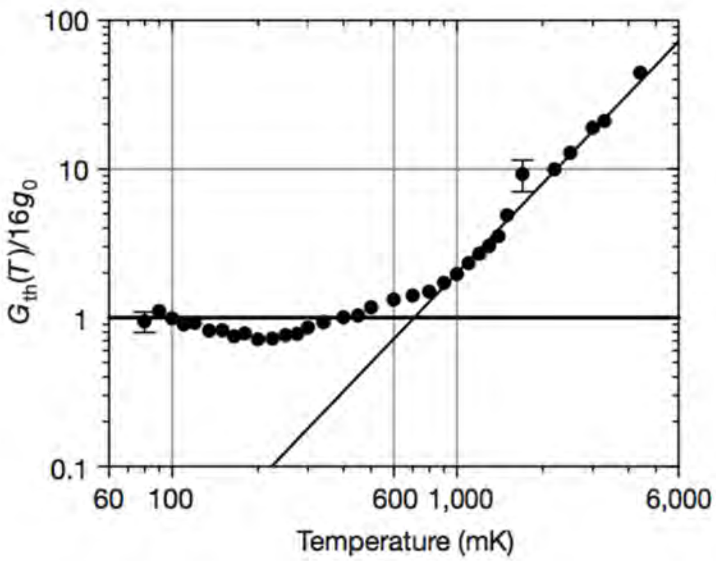
Conformal Anomaly

Cappelli, Huerta and Zemba, Nucl. Phys. B **636**, 568 (2002)

Experiment

Schwab, Henriksen, Worlock and Roukes, *Measurement of the quantum of thermal conductance*, Nature **404**, 974 (2000)





Quantum of Thermal Conductance

Energy Current Fluctuations

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45**, 362001 (2012)

Generating function for all moments

$$\mathrm{F(z)} \equiv \lim_{t\to\infty} t^{-1} \ln\langle e^{z\Delta_t Q}\rangle$$

Exact Result

$$F(z) = \frac{c\pi^2}{6h}\left(\frac{z}{\beta_l(\beta_l - z)} - \frac{z}{\beta_r(\beta_r + z)}\right)$$

$$F(z) = \frac{c\pi^2}{6h}\left[z\left(\frac{1}{\beta_l^2} - \frac{1}{\beta_r^2}\right) + z^2\left(\frac{1}{\beta_l^3} + \frac{1}{\beta_r^3}\right) + \dots\right]$$

$$\langle J\rangle = \frac{c\pi^2}{6h} k_B^2 (T_\mathrm{L}^2 - T_\mathrm{R}^2) \qquad \langle \delta J^2\rangle \propto \frac{c\pi^2}{6h} k_B^3 (T_\mathrm{L}^3 + T_\mathrm{R}^3)$$

Poisson Process $\int_0^\infty e^{-\beta\epsilon}(e^{z\epsilon} - 1)d\epsilon = \frac{z}{\beta(\beta - z)}$

Non-Equilibrium Fluctuation Relation

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45**, 362001 (2012)

$$F(z) \equiv \lim_{t\to\infty} t^{-1} \ln\langle e^{z\Delta_t Q}\rangle = \frac{c\pi^2}{6h}\left(\frac{z}{\beta_l(\beta_l - z)} - \frac{z}{\beta_r(\beta_r + z)}\right)$$

$$F(-z) = F(z + \beta_l - \beta_r)$$

Irreversible work fluctuations in isolated driven systems

Crooks relation $$\frac{P(W)}{\tilde{P}(-W)} = e^{\beta(W-\Delta F)}$$

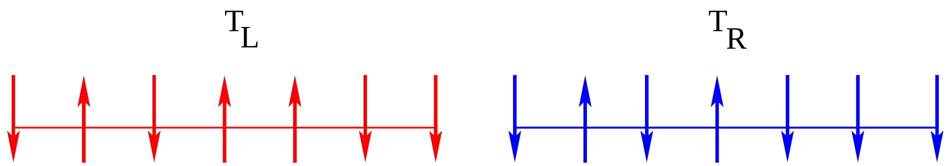
Jarzynski relation $$\langle e^{-\beta W}\rangle = e^{-\beta\Delta F}$$

Entropy production in non-equilibrium steady states

$$\frac{P(S)}{P(-S)} = e^{S}$$

Esposito *et al*, *"Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems"*, RMP **81**, 1665 (2009)

Lattice Models



Quantum Ising Model

$$H = J\sum_{\langle ij\rangle} S_i^z S_j^z + \Gamma \sum_i S_i^x$$

$\Gamma = J/2$ Critical $c = 1/2$

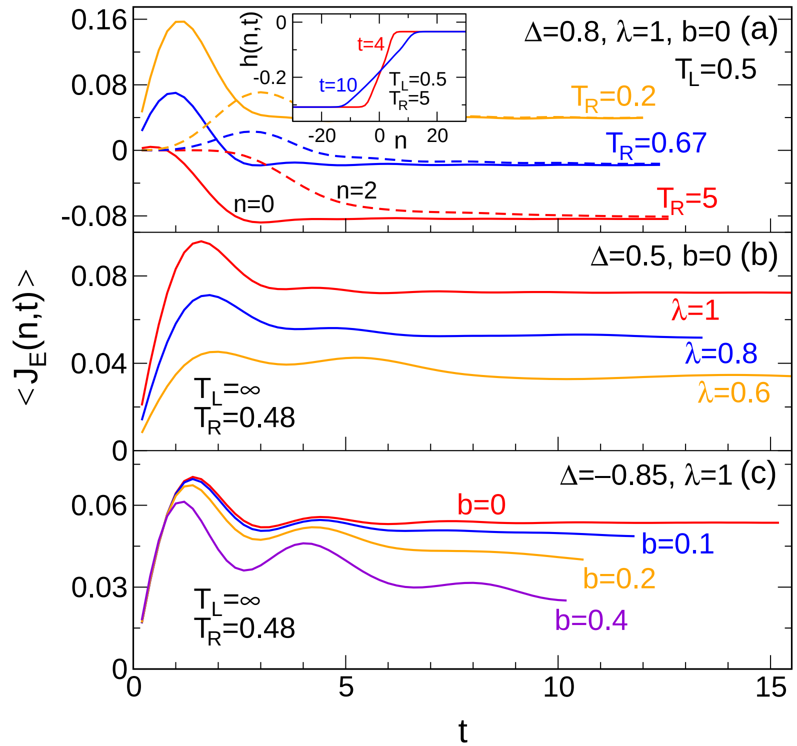
Anisotropic Heisenberg Model (XXZ)

$$H = J\sum_{\langle ij\rangle} \left(S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z\right)$$

$-1 < \Delta < 1$ Critical $c = 1$

Time-Dependent DMRG

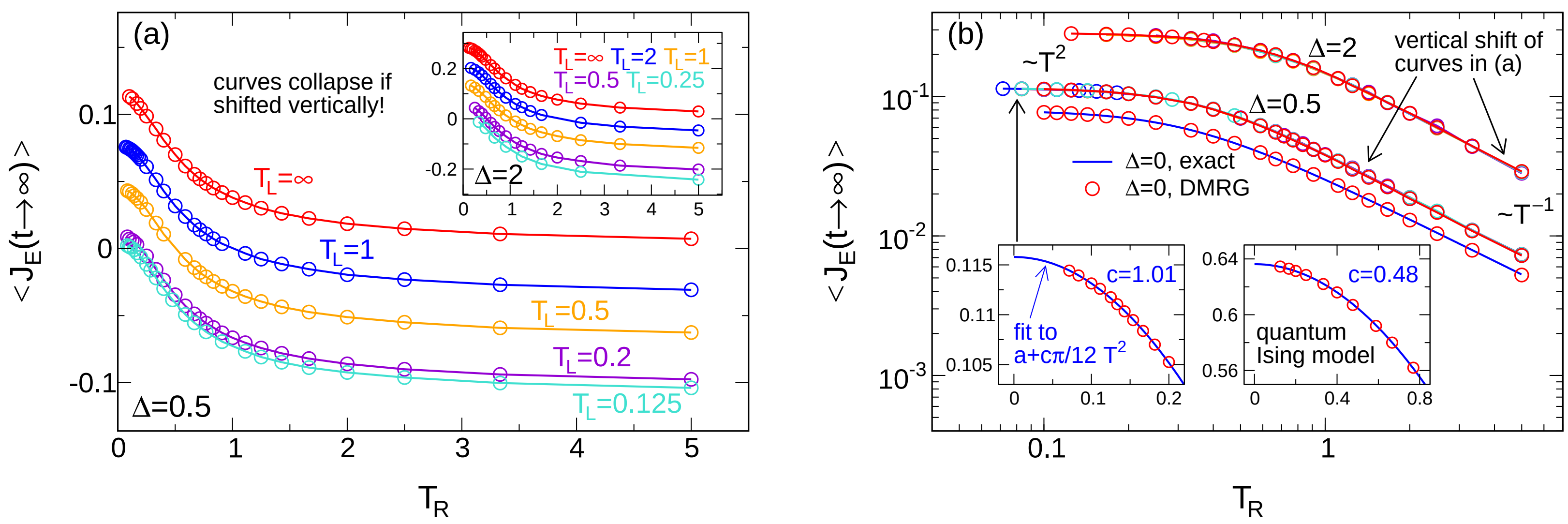
Karrasch, Ilan and Moore, *Non-equilibrium thermal transport and its relation to linear response*, `arXiv:1211.2236`



$$\text{Dimerization } J_n = \begin{cases} 1 & n \,\text{odd} \\ \lambda & n \,\text{even} \end{cases} \quad \Delta_n = \Delta \quad \text{Staggered } b_n = \frac{(-1)^n b}{2}$$

Time-Dependent DMRG

Karrasch, Ilan and Moore, *Non-equilibrium thermal transport and its relation to linear response*, `arXiv:1211.2236`



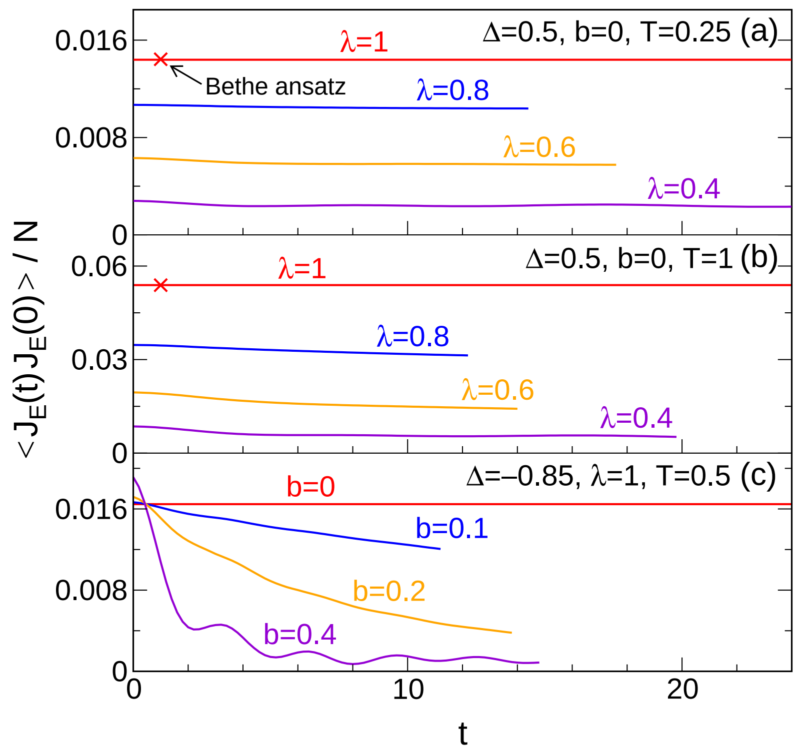
$$\lim_{t\to\infty}\langle J_E(n,t)\rangle = f(T_L) - f(T_R)$$

$$f(T) \sim \begin{cases} T^2 & T \ll 1 \\ T^{-1} & T \gg 1 \end{cases}$$

Beyond CFT to massive integrable models (Doyon)

Energy Current Correlation Function

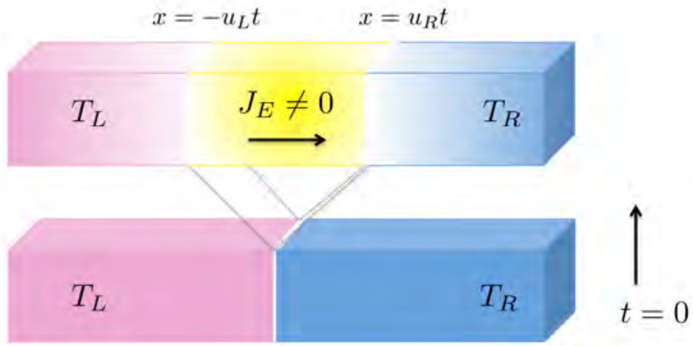
Karrasch, Ilan and Moore, *Non-equilibrium thermal transport and its relation to linear response*, `arXiv:1211.2236`

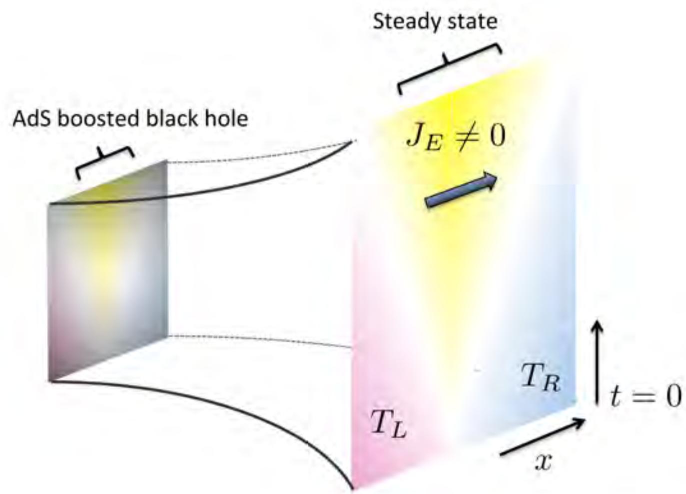


Beyond Integrability

Importance of CFT for pushing numerics and analytics

AdS/CFT





Steady State Region

General Considerations

$$\partial_\mu T^{\mu\nu} = 0 \quad \partial_0 T^{00} = -\partial_x T^{x0} \quad \partial_0 T^{0x} = -\partial_x T^{xx}$$

Stationary heat flow $\Longrightarrow$ **Constant pressure**

$$\partial_0 T^{0x} = 0 \Longrightarrow \partial_x T^{xx} = 0$$

In a CFT

$$P = u/d \Longrightarrow \partial_x u = 0$$

No energy/temperature gradient

Stationary homogeneous solutions

Solutions of Einstein Equations

$$S = \frac{1}{16\pi G_N}\int d^{d+2}x\sqrt{-g}(R-2\Lambda)$$

$$\Lambda = -d(d+1)/2L^2$$

Unique homogeneous solution = boosted black hole

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left[\frac{\mathrm{d}z^2}{f(z)} - f(z)(\mathrm{d}t\cosh\theta - \mathrm{d}x\sinh\theta)^2 + (\mathrm{d}x\cosh\theta - \mathrm{d}t\sinh\theta)^2 + \mathrm{d}y_\perp^2\right]$$

$$f(z) = 1 - \left(\frac{z}{z_0}\right)^{d+1} \qquad z_0 = \frac{d+1}{4\pi T}$$

Fefferman–Graham Coordinates

$$\langle T_{\mu\nu}\rangle_{\mathrm{s}} = \frac{L^d}{16\pi G_{\mathrm{N}}}\lim_{Z\to 0}\left(\frac{d}{dZ}\right)^{d+1}\frac{Z^2}{L^2}\,g_{\mu\nu}(z(Z))$$

$$z(Z) = Z/R - (Z/R)^{d+2}/[2(d+1)z_0^{d+1}] \qquad R = (d!)^{1/(d-1)}$$

Boost Solution

Lorentz boosted stress tensor of a finite temperature CFT

$$\boxed{\langle T^{\mu\nu}\rangle_{\rm s} = a_d\, T^{d+1}\left(\eta^{\mu\nu} + (d+1)u^\mu u^\nu\right)}$$

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$$

$$u^\mu = (\cosh\theta, \sinh\theta, 0, \ldots, 0)$$

$$\boxed{\langle T^{tx}\rangle_{\rm s} = \tfrac{1}{2} a_d\, T^{d+1}(d+1)\sinh 2\theta}$$

$$a_d = (4\pi/(d+1))^{d+1} L^d / 16\pi G_{\rm N}$$

One spatial dimension

$$a_1 = \frac{L\pi}{4G_{\rm N}} \qquad c = \frac{3L}{2G_{\rm N}}$$

$$T_{\rm L} = Te^{\theta} \qquad T_{\rm R} = Te^{-\theta} \qquad \langle T_{tx}\rangle = \frac{c\pi^2 k_B^2}{6h}(T_{\rm L}^2 - T_{\rm R}^2)$$

Can also obtain complete steady state density matrix

Steady State Density Matrix

$$\langle \mathcal{O} \ldots \rangle = \frac{\mathrm{Tr}(\rho_s \mathcal{O} \ldots)}{\mathrm{Tr}(\rho_s)}$$

$$\rho_s = e^{-\beta E \cosh \theta + \beta P_x \sinh \theta}$$

$$\beta = \sqrt{\beta_{\mathrm{L}} \beta_{\mathrm{R}}} \qquad e^{2\theta} = \frac{\beta_{\mathrm{R}}}{\beta_{\mathrm{L}}}$$

Lorentz boosted thermal density matrix

Describes all the cumulants of the energy transfer process

The non-equilibrium steady state (NESS)
is a Lorentz boosted thermal state

Shock Solutions

Rankine–Hugoniot

Energy-Momentum conservation across shock

$$\langle T^{tx}\rangle_{\mathrm{s}} = a_d \left(\frac{T_{\mathrm{L}}^{d+1} - T_{\mathrm{R}}^{d+1}}{u_{\mathrm{L}} + u_{\mathrm{R}}} \right)$$

Invoking boosted steady state gives $u_{\mathrm{L,R}}$ in terms of $T_{\mathrm{L,R}}$:

$$u_{\mathrm{L}} = \frac{1}{d}\sqrt{\frac{\chi+d}{\chi+d^{-1}}} \qquad u_{\mathrm{R}} = \sqrt{\frac{\chi+d^{-1}}{\chi+d}} \qquad \chi \equiv (T_{\mathrm{L}}/T_{\mathrm{R}})^{(d+1)/2}$$

Steady state region is a boosted thermal state with $T = \sqrt{T_{\mathrm{L}} T_{\mathrm{R}}}$

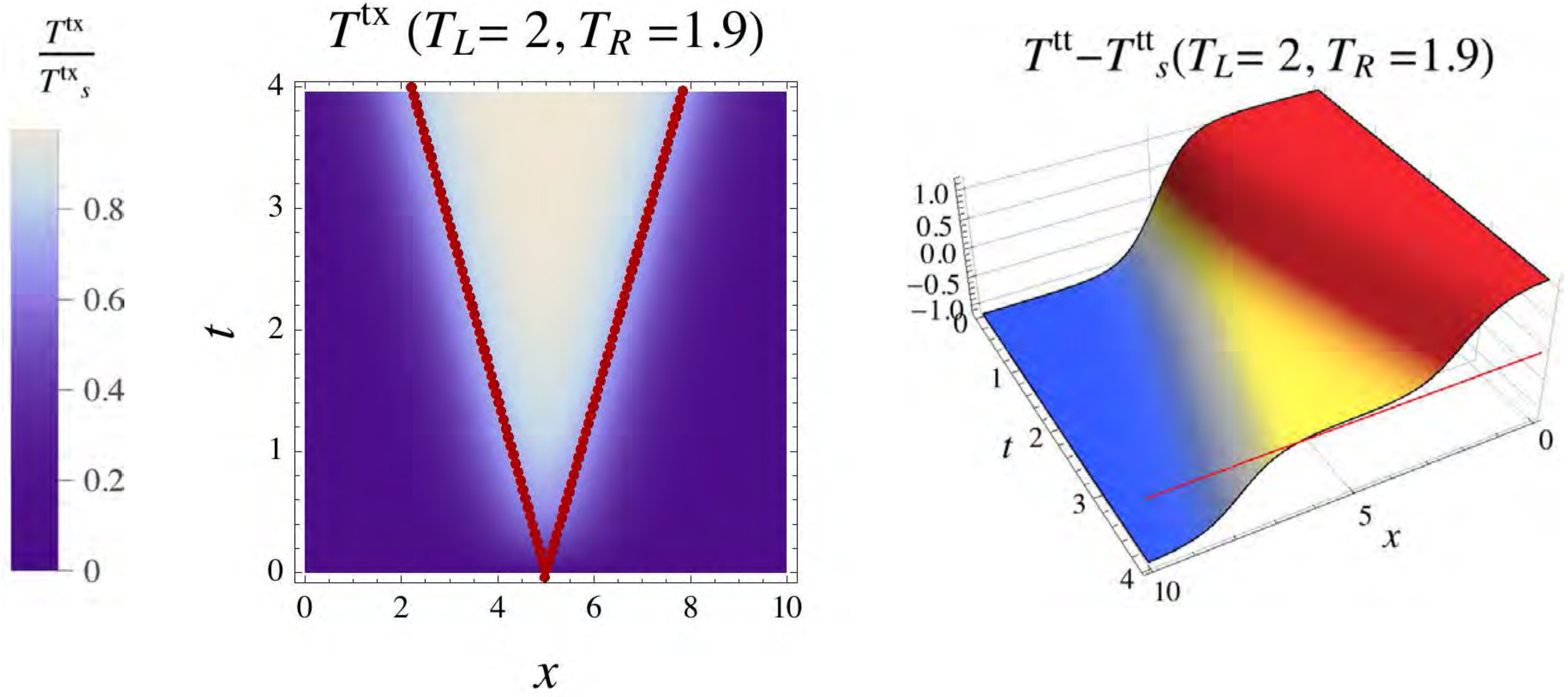
Boost velocity $(\chi - 1)/\sqrt{(\chi + d)(\chi + d^{-1})}$ Agrees with $d = 1$

Shock waves are non-linear generalizations of sound waves

EM conservation: $u_{\mathrm{L}} u_{\mathrm{R}} = c_{\mathrm{s}}^2$, where $c_{\mathrm{s}} = v/\sqrt{d}$ is speed of sound

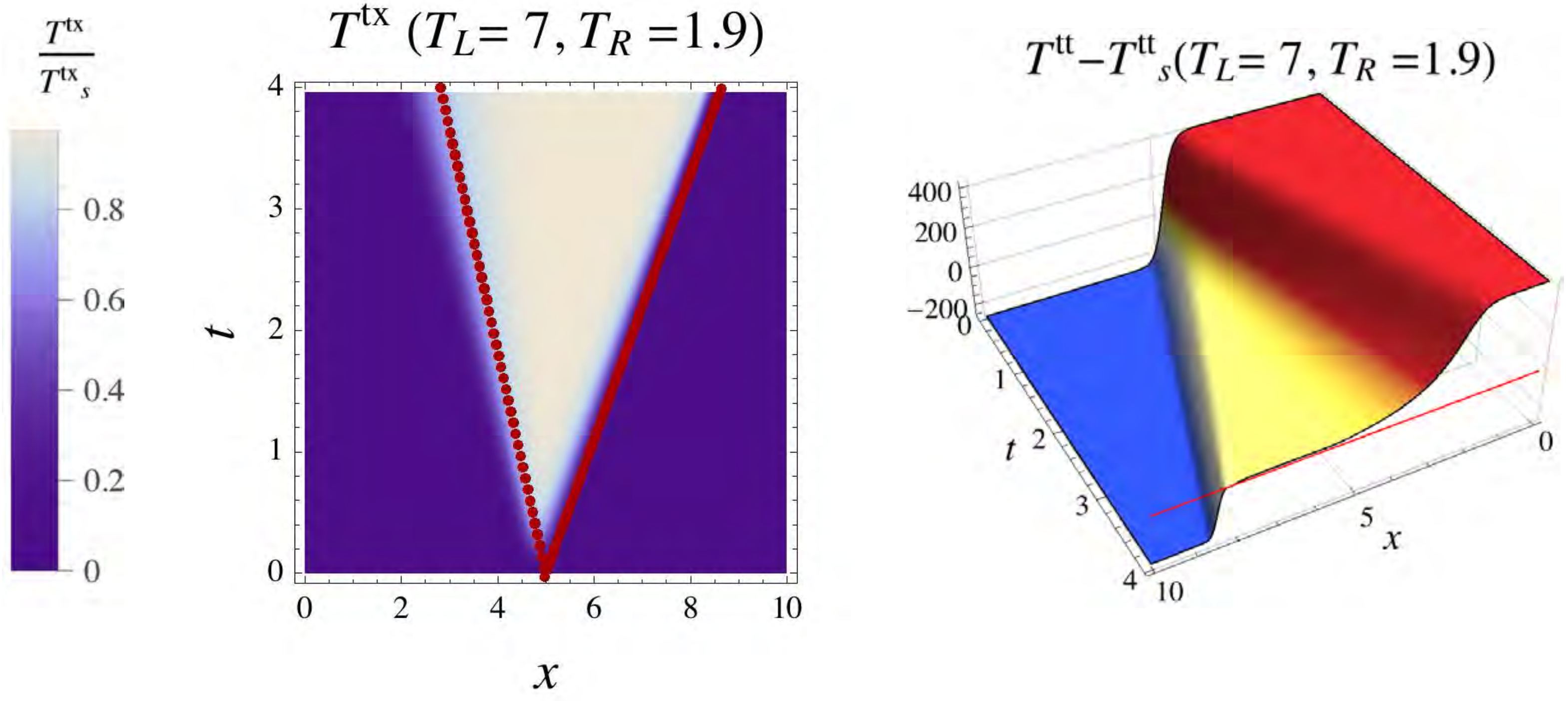
$c_s < u_{\mathrm{R}} < v \quad c_s < u_{\mathrm{L}} < c_s^2/v$ reinstated microscopic velocity v

Numerics I



Excellent agreement with predictions

Numerics II



Excellent agreement far from equilibrium

Asymmetry in propagation speeds

Conclusions

Average energy flow in arbitrary dimension

Lorentz boosted thermal state

Energy current fluctuations

Exact generating function of fluctuations

Generalizations

Other types of charge noise   Non-Lorentz invariant situations

Different central charges   Fluctuation theorems   Numerical GR

Acknowledgements

B. Benenowski, D. Bernard, P. Chesler, A. Green
D. Haldane C. Herzog, D. Marolf, B. Najian, C.-A. Pillet
S. Sachdev, A. Starinets

Stefan–Boltzmann

Cardy, *The Ubiquitous 'c': from the Stefan-Boltzmann Law to Quantum Information*, `arXiv:1008.2331`

Black Body Radiation in $3+1$ dimensions

$$dU = TdS - PdV$$

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

$$\boxed{u = T\left(\frac{\partial P}{\partial T}\right)_V - P}$$

For black body radiation $P = u/3$

$$\frac{4u}{3} = \frac{T}{3}\left(\frac{\partial u}{\partial T}\right)_V \qquad \frac{du}{4u} = \frac{dT}{T} \qquad \frac{1}{4}\ln u = \ln T + \text{const.}$$

$$\boxed{u \propto T^4}$$

Stefan–Boltzmann and CFT

Cardy, *The Ubiquitous 'c': from the Stefan-Boltzmann Law to Quantum Information*, `arXiv:1008.2331`

Energy-Momentum Tensor in $d+1$ Dimensions

$$T_{\mu\nu} = \begin{pmatrix} u & & & \\ & P & & \\ & & P & \\ & & & \ldots \end{pmatrix} \qquad \text{Traceless} \qquad P = u/d$$

Thermodynamics

$$u = T\left(\frac{\partial P}{\partial T}\right)_V - P \qquad \boxed{u \propto T^{d+1}}$$

For $1+1$ Dimensional CFT

$$\boxed{u = \frac{\pi c k_B^2 T^2}{6\hbar v} \equiv \mathcal{A} T^2} \qquad \boxed{J = \frac{\mathcal{A} v}{2}(T_{\mathrm{L}}^2 - T_{\mathrm{R}}^2)}$$

Stefan–Boltzmann and AdS/CFT

Gubser, Klebanov and Peet, *Entropy and temperature of black 3-branes*, Phys. Rev. D **54**, 3915 (1996).

Entropy of $SU(N)$ SYM = Bekenstein–Hawking $S_{\rm BH}$ of geometry

$$\boxed{S_{\rm BH} = \frac{\pi^2}{2} N^2 V_3 T^3}$$

Entropy at Weak Coupling = $8N^2$ free massless bosons & fermions

$$\boxed{S_0 = \frac{2\pi^2}{3} N^2 V_3 T^3}$$

Relationship between strong and weak coupling

$$S_{\rm BH} = \tfrac{3}{4} S_0$$

Gubser, Klebanov, Tseytlin, *Coupling constant dependence in the thermodynamics of $\mathcal{N}=4$ supersymmetric Yang-Mills Theory*, Nucl. Phys. B **534** 202 (1998)

Full Counting Statistics

A large body of results in the mesoscopic literature

Free Fermions

Levitov & Lesovik, *"Charge Distribution in Quantum Shot Noise"*, JETP Lett. **58**, 230 (1993)

Levitov, Lee & Lesovik, *"Electron Counting Statistics and Coherent States of Electric Current"*, J. Math. Phys. **37**, 4845 (1996)

Luttinger Liquids and Quantum Hall Edge States

Kane & Fisher, *"Non-Equilibrium Noise and Fractional Charge in the Quantum Hall Effect"*, PRL **72**, 724 (1994)

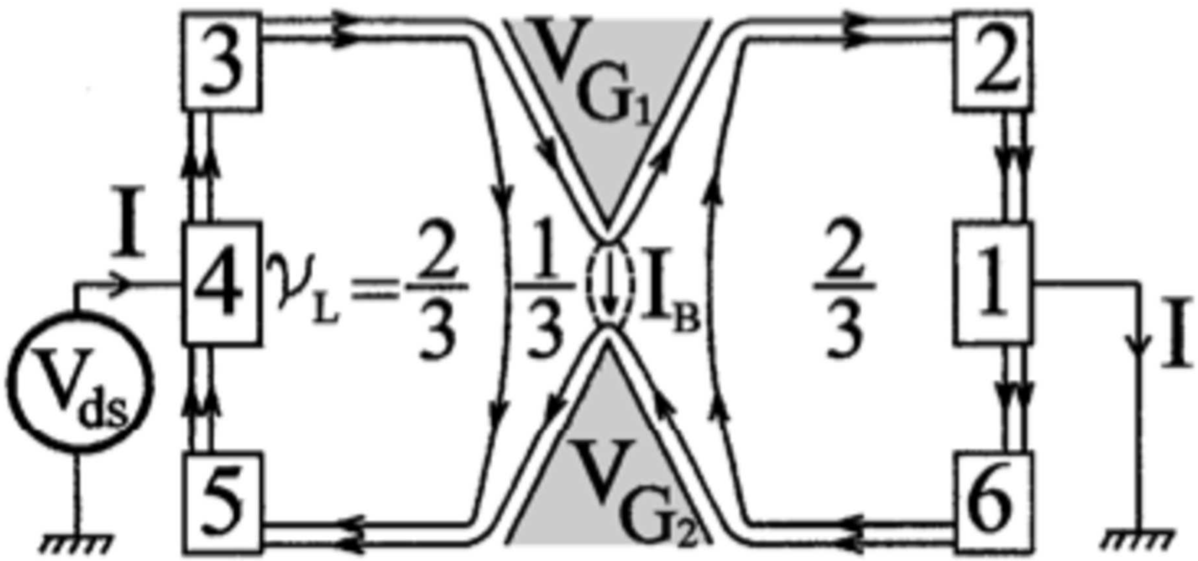
Fendley, Ludwig & Saleur, *"Exact Nonequilibrium dc Shot Noise in Luttinger Liquids and Fractional Quantum Hall Devices"*, PRL (1995)

Quantum Impurity Problems

Komnik & Saleur, *"Quantum Fluctuation Theorem in an Interacting Setup: Point Contacts in Fractional Quantum Hall Edge State Devices"*, PRL **107**, 100601 (2011)

Experimental Setup

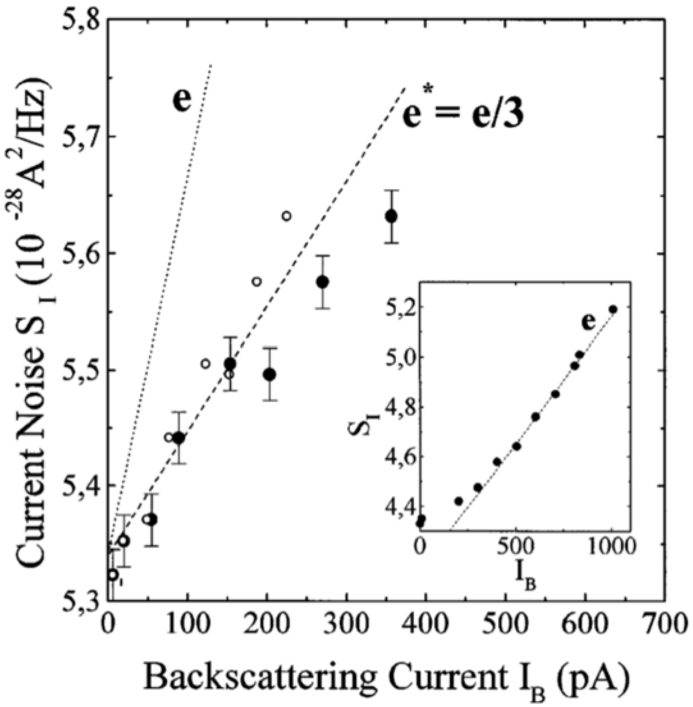
Saminadayar *et al*, PRL **79**, 2526 (1997)

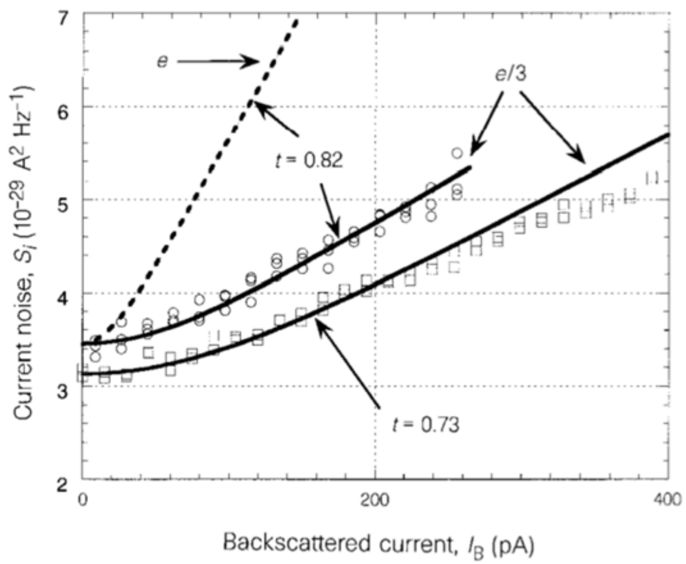


Shot Noise in the Quantum Hall Effect

Saminadayar *et al*, *"Observation of the e/3 Fractionally Charged Laughlin Quasiparticle"*, PRL **79**, 2526 (1997)

R. de-Picciotto *et al*, *"Direct observation of a fractional charge"*, Nature **389**, 162 (1997)





$S_I = 2QI_B$

$Q = e/3$