Lifshitz Field Theories

Yaron Oz (Tel-Aviv University)

- With Carlos Hoyos and Bom Soo Kim : arXiv:1304.7481, arXiv:1309.6794, arXiv:1312.6380.
- With Shira Chapman and Carlos Hoyos : arXiv:1402.2981.
- With Chris Eling : arXiv:1408.0268.
- With Igal Arav and Shira Chapman : to appear.

Kolymbari 2014

Outline

Lifshitz Scaling Symmetry

Lifshitz Hydrodynamics

Strange Metals

Scale Anomalies

Open Problems

Lifshitz Scaling Symmetry

- Lifshitz scaling :

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \quad i = 1, ...d \ . \tag{1}$$

  z is a dynamic exponent that measures the anisotropy.
- When $z = 1$ the spacetime symmetry can be enhanced to include the Lorentz group, and when $z = 2$ the Galilean group. For all other values of z, boost invariance will be explicitly broken.
- Lifshitz scaling shows up in condensed matter systems (quantum critical points), in gravity models that break local Lorentz invariance (Horava- Lifshitz) and in holographic models (Einstein gravity with matter)(N. Obers talk).

Lifshitz Algebra

- The generators of Lifshitz symmetry are time translation $P_0 = \partial_t$, spatial translations $P_i = \partial_i$, the scaling transformation $D = -zt\partial_t - x^i\partial_i$ and rotations.
- The subalgebra involving D, P_i and P_0 has commutation relations

$$[D, P_i] = P_i \ , \quad [D, P_0] = zP_0 \tag{2}$$

Lifshitz Scaling Symmetry

There are many questions concerning quantum field theories (QFTs) and gravity with Lifshitz scaling, such as :

- What values of z are allowed ?
- Is $z = 1$ realized only in a CFT ?
- Do the QFTs have a consistent UV behaviour ?
- Does the gravity that breaks local Lorentz invariance have a consistent UV behaviour ?
- Basic field theory questions : Spin-Statistics Theorem ? Euclidean rotation ? Representations ?
- Entanglement entropy ?
- What is the structure of the hydrodynamic limit ?
- What is the structure of anomalies ?
- Supersymmetric extension ?

Field Theory Realizations

- Free scalar field theory in $d+1$ dimensions (z even) :

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{\kappa}{2z}((\nabla^2)^{\frac{z}{2}}\phi)^2$$
$$\phi \to \lambda^{\frac{z-d}{2}}\phi, \qquad \kappa \quad \textit{dimensionsless} \qquad (3)$$

- Free scalar field theory in $2+1$ dimensions :

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{\kappa}{4}(\nabla^2\phi)^2, \qquad z = 2 \qquad (4)$$

- The dimension of the scalar is zero as in $(1+1)$-dimensional relativistic scalar theory. Here the dispersion relation is $p^4 \sim \omega^2$.

Field Theory Realizations

- Free fermion field theory in $2+1$ dimensions :

$$\mathcal{L} = -i\psi^\dagger \partial_t \psi + \psi^\dagger \nabla^2 \psi, \ \psi \to \lambda^{-1}\psi, \quad z = 2 \tag{5}$$

- ψ is a one component field.
- We can combine the scalar and fermion to a WZ-type supersymmetric model.
- Interaction in $2+1$ dimensions :

$$\mathcal{L} = g\psi^\dagger \psi \nabla^2 \phi, \qquad z = 2 \tag{6}$$

Gravity Realizations

- The gravitational background (Kachru and collaborators,Taylor...):

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2\delta_{ij}dx^i dx^j \tag{7}$$

has the isometry

$$r \to \lambda^{-1} r, \quad t \to \lambda^z t, \quad x^i \to \lambda x^i \tag{8}$$

- It is a solution of Einstein gravity with matter (scalars and massive vector fields), or Horava-Lifhshitz gravity (Horava and collaborators).
- Lifshitz as a deformation of AdS (Skenderis and collaborators).

Einstein Gravity with Matter

- The action reads

$$\mathcal{L} = R - \sum_j \left(\frac{Z_j(\phi)}{4} F_{jAB} F_j^{AB} + \frac{1}{2} m_j^2 V_{jA} V_j^A \right) - \sum_i \left(\frac{1}{2} (\partial \phi_i)^2 - V(\phi_i) \right) \tag{9}$$

- V_{jA} are massive or massless (for $m_j = 0$) vector fields, F_{jAB} their field strengths, and ϕ_i are the scalar fields. $Z_i(\phi)$ parametrize the couplings between scalar fields and vector fields.
- In Lifshitz solutions rotational invariance is not broken, so only the V_{jr} and V_{jt} components of the vector fields can be non-zero.

Einstein-aether and Horava-Lifshitz

- A generic, generally covariant model of local Lorentz violating gravity is Einstein-aether theory (Jacobson and collaborators). In this theory the symmetry is broken by the aether covector v_A, which is a dynamical field that is constrained to be unit timelike $v_A v^A = -1$.
- As a consequence, the theory has in general spin-2, spin-1, and spin-0 gravitational wave polarizations traveling at different speeds.
- A particular choice for the aether field is to be hypersurface orthogonal, thus determining a preferred time foliation of space-time. In this case the Einstein-aether theory is reduced the Horava-Lifshitz theory.

Einstein-aether Gravity

- The action for Einstein-aether theory is given in four-dimensional space-time by

$$S_{ae} = \frac{1}{16\pi G_{ae}} \int d^4x \sqrt{-g} L_{ae} \tag{10}$$

where $L_{ae} = R + L_{vec}$,

$$-L_{vec} = K^{AB}{}_{CD} \nabla_A v^C \nabla_B v^D - \lambda(v^2 + 1) \tag{11}$$

with “kinetic" tensor defined as

$$K^{AB}{}_{CD} = c_1 g^{AB} g_{CD} + c_2 \delta^A_C \delta^B_D + c_3 \delta^A_D \delta^B_C - c_4 v^A v^B g_{CD} \tag{12}$$

- This is the most general effective action for a timelike unit vector field at 2nd order in derivatives.

Horava-Lifshitz Gravity

- When the aether field is hypersurface orthogonal

$$v_{[A}\nabla_B v_{C]} = 0 \tag{13}$$

- Hypersurface orthogonality implies the co-vector is the gradient of a scalar

$$v_A = \frac{-\partial_A \phi}{\sqrt{g^{CD}\partial_C\phi\partial_D\phi}} \tag{14}$$

- Geometrically the aether field determines a foliation of space-time.

Horava-Lifshitz Gravity

- Choosing coordinates such that $\phi = \tau$, where τ is the preferred foliation of time, the Einstein-aether action reduces to the generic 3+1 form of the Horava-Lifshitz action (Griffin).
- In Horava-Lifshitz the spin-1 mode is non-propagating.
- The black holes have a space-like hypersurface called the universal horizon and it is the causal boundary of space-time (and not the null hypersurface Killing horizon).
- The black hole thermodynamics is associated with the universal horizon.

Lifshitz Stress-Energy Tensor

- Since boost invariance is explicitly broken in Lifshitz field theories, the conserved stress-energy tensor is not necessarily symmetric. The Lorentz current is $J^{\mu\alpha\beta} = x^{\alpha} T^{\mu\beta} - x^{\beta} T^{\mu\alpha}$, and

$$\partial_{\mu} J^{\mu\alpha\beta} = T^{\alpha\beta} - T^{\beta\alpha} \tag{15}$$

- In order to see its asymmetric part, we have to construct it not as a response of the action S to a change in a background metric $h_{\mu\nu}$, but rather as a response to a change in the vielbein e^{μ}_{a} (by a we denote tangent space indices)

$$T^{a}_{\mu} = \frac{1}{e} \frac{\delta S}{\delta e^{\mu}_{a}} \tag{16}$$

Stress-Energy Tensor

- The vielbein encodes both the metric data $h_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, and the foliation data $v_\mu = e^a_\mu v_a$, where $v_a = (1, 0.., 0)$.
- Using (16) one has

$$T_{\mu\nu} = \Theta_{\mu\nu} + J_\mu v_\nu \tag{17}$$

where

$$\Theta_{\mu\nu} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h^{\mu\nu}}, \qquad J_\mu = \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta v^\mu} \tag{18}$$

- We see from (17) that the asymmetric part of the stress-energy tensor arises from $J_\mu v_\nu$ and is directly connected to the foliation data.

Lifshitz Ward Identity

- The relativistic $z = 1$ (CFT) Ward identity is

$$T_\mu^\mu = 0 \tag{19}$$

- The Lifshitz Ward identity is

$$zT_0^0 + T_i^i = 0 \tag{20}$$

- Quantum anomalies modify the CFT Ward identity in D space-time dimensions

$$\langle T_\mu^\mu \rangle_g = -(-)^{\frac{D}{2}} aE_D + \sum_i c_i I_i \tag{21}$$

  where E_D is the Euler density (A-type anomaly) and I_i are Weyl invariant terms (B-type Anomaly).

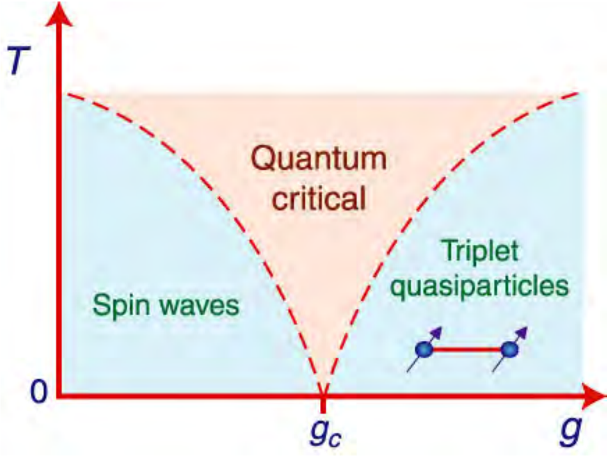
- Lifshitz anomalies seem to be all of B-type.

Experimental Setups: Strange Metals

- Heavy fermion compounds and other materials including high T_c superconductors have a metallic phase (dubbed as ‘strange metal’) whose properties cannot be explained within the ordinary Landau-Fermi liquid theory.
- In this phase some quantities exhibit universal behaviour such as the resistivity, which is linear in the temperature $\rho \sim T$.
- Such universal properties are believed to be the consequence of quantum criticality (Coleman:2005,Sachdev:2011).
- At the quantum critical point there is a Lifshitz scaling (Hornreich:1975,Grinstein:1981) symmetry.

Quantum Critical Points

- Phase transitions at zero temperature are driven by quantum fluctuations.



Quantum Critical Points

- Systems with ordinary critical points have a hydrodynamic description with transport coefficients whose temperature dependence is determined by the scaling at the critical point (Hohenberg:1977).
- Quantum critical systems also have a hydrodynamic description, e.g. conformal field theories at finite temperature, fermions at unitarity and graphene.
- At quantum critical regime the hydrodynamic description will be appropriate if the characteristic length of thermal fluctuations $\ell_T \sim 1/T$ is much smaller than the size of the system $L \gg \ell_T$ and both are smaller than the correlation length of quantum fluctuations $\xi \gg L \gg \ell_T$.

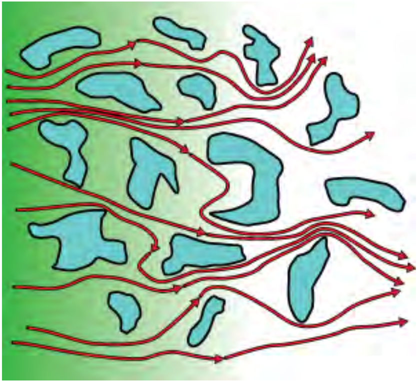
QCP Hydrodynamics

- The hydrodynamic description should take into account the effects due to the lack of boost invariance.
- The results that I will present are universal up to the value of the coefficients in the hydrodynamic expansion, which depend on the details of the critical point.

New Transport

- Our main new result is the discovery of a single new transport coefficient in the neutral case, allowed by the absence of boost invariance. The effect of the new coefficient is a production of dissipation when the fluid accelerates.
- The result applies to any system with Lifshitz scaling, but also more generally to any system where boost invariance is explicitly broken. For instance, fluids moving through a porous medium or electrons in a dirty metal.
- New transport coefficients appear in charged/superfluid Lifshitz hydrodynamics.
- We relate the flux of the spin zero perturbation across the universal horizon of Horava-Lifshitz gravity to the new dissipative transport in Lifshitz field theory hydrodynamics.

QCP Hydrodynamics



Conductivity

- We study the effects of the new coefficient on the conductivity of a strange metal using the Drude model and find a non-linear dependence on the electric field.
- Interestingly, we also find that scaling arguments fix the resistivity to be linear in the temperature, under the reasonable assumption that the dependence on the mass density is linear. This behaviour is universal: it is independent of the number of dimensions and the value of the dynamical exponent.

Equation of State

- In a field theory the scaling symmetry is manifested as a Ward identity involving the components of the energy-momentum tensor

$$zT^0_{\ 0} + \delta^j_{\ i} T^i_{\ j} = 0 \tag{22}$$

- At finite temperature $T^0_{\ 0} = -\varepsilon$, $T^i_{\ j} = p\delta^i_{\ j}$, leading to the equation of state

$$z\varepsilon = dp \tag{23}$$

- This fixes the temperature dependence of energy and pressure. Taking the dimension of spatial momentum to be one, the scaling dimensions are

$$[T] = z\ , \quad [\varepsilon] = [p] = z + d \tag{24}$$

Equation of State

- The Lifshitz algebra can be generalized for constant velocities u^μ, $u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$ $(\mu, \nu = 0, 1, \cdots, d)$, with scaling dimension $[u^\mu] = 0$.
- We define the generators

$$P^{\parallel} = u^\mu \partial_\mu, \quad P^{\perp}_\mu = P_\mu^{\ \nu} \partial_\nu, \quad D = z x^\mu u_\mu P^{\parallel} - x^\mu P^{\perp}_\mu. \tag{25}$$

Where $P_\mu^{\ \nu} = \delta_\mu^{\ \nu} + u_\mu u^\nu$. Then, the momentum operators commute among themselves and

$$[D, P^{\parallel}] = z P^{\parallel} , \quad [D, P^{\perp}_\mu] = P^{\perp}_\mu . \tag{26}$$

- The Ward identity associated to D becomes

$$z T^\mu_{\ \nu} u_\mu u^\nu - T^\mu_{\ \nu} P_\mu^{\ \nu} = 0 . \tag{27}$$

It coincides with (22) only when $z = 1$, but leads to the equation of state (23) for any velocity.

Hydrodynamics

- The conservation of the energy-momentum tensor determines the hydrodynamic equations $\partial_\mu T^{\mu\nu} = 0$.
- Lorentz symmetry forces the energy-momentum tensor to be symmetric. If boost or rotational symmetries are broken this condition can be relaxed.
- This allows many new terms in the hydrodynamic energy-momentum tensor, but as usual there are ambiguities in the definition of the hydrodynamic variables in the constitutive relations. In order to fix them, we impose the Landau frame condition

$$T^{\mu\nu} u_\nu = -\varepsilon u^\mu \ . \tag{28}$$

Hydrodynamics

- Then, the generalized form of the energy-momentum tensor is

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} + \pi_S^{(\mu\nu)} + \pi_A^{[\mu\nu]} + (u^\mu \pi_A^{[\nu\sigma]} + u^\nu \pi_A^{[\mu\sigma]})u_\sigma \ . \quad (29)$$

- The first line is the ideal part of the energy-momentum tensor, π_S contains symmetric dissipative contributions and must satisfy the constraint $\pi_S^{(\mu\nu)} u_\nu = 0$. π_A contains all possible antisymmetric terms.
- In a theory with rotational invariance $\pi_A^{[ij]} = 0$.

First Viscous Order

- To first dissipative order

$$\pi_S^{(\mu\nu)} = -\eta^{\mu\nu\alpha\beta}\partial_\alpha u_\beta = -\eta P^{\mu\alpha}P^{\nu\beta}\Delta_{\alpha\beta} - \frac{\zeta}{d}P^{\mu\nu}\partial_\alpha u^\alpha \quad (30)$$

- η and ζ are the shear and bulk viscosities respectively. The shear tensor is defined as

$$\Delta_{\alpha\beta} = 2\partial_{(\alpha}u_{\beta)} - \frac{2}{d}P_{\alpha\beta}(\partial_\sigma u^\sigma) \quad (31)$$

The Entropy Current

- The new terms from $\pi_A^{[\mu\nu]}$ should be compatible with the laws of thermodynamics, in particular with the second law. Its local form in terms of the divergence of the entropy current is $\partial_\mu j_s^\mu \geq 0$.
- The divergence of the entropy current can be derived from the conservation equation

$$0 = \partial_\mu T^{\mu\nu} u_\nu = -T\partial_\mu(su^\mu) - \pi_A^{[\mu\sigma]}(\partial_{[\mu} u_{\sigma]} - u_{[\mu} u^\alpha \partial_\alpha u_{\sigma]}) + \cdots . \qquad (32)$$

- In the Landau frame we can define the entropy current as $j_s^\mu = su^\mu$ to first dissipative order.

The Local Second Law

- The dots denote contributions originating in symmetric terms in the energy-momentum tensor. To first order in derivative corrections they will simply be the shear and bulk viscosity contributions, which are manifestly positive for positive values of the transport coefficients.
- The new terms are possible only if

$$\pi_A^{[\mu\nu]} = -\alpha^{\mu\nu\sigma\rho}(\partial_{[\sigma} u_{\rho]} - u_{[\sigma} u^{\alpha} \partial_{\alpha} u_{\rho]}) \tag{33}$$

where $\alpha^{\mu\nu\sigma\rho}$ contains all possible transport coefficients to first dissipative order and must satisfy the condition that, for an arbitrary real tensor $\tau_{\mu\nu}$,

$$\tau_{\mu\nu}\alpha^{\mu\nu\sigma\rho}\tau_{\sigma\rho} \geq 0 \tag{34}$$

Transport Coefficients

- If only boost invariance is broken, there is a single possible transport coefficient $\alpha \geq 0$

$$\pi_A^{[\mu\nu]} = -\alpha u^{[\mu} u^{\alpha} \partial_{\alpha} u^{\nu]} \tag{35}$$

- For a theory with Lifshitz symmetry the scaling dimension of the transport coefficients is $[\eta] = [\zeta] = [\alpha] = d$, which determines their temperature dependence to be

$$\eta \sim \zeta \sim \alpha \sim T^{\frac{d}{z}} \tag{36}$$

Charged Lifshitz Hydrodynamics

- There can be other new transport coefficients in a theory with conserved charges.
- With a single conserved global current have

$$T^{[\mu\nu]} = \pi_A^{[\mu\nu]} = -\alpha u^{[\mu} a^{\nu]} - \alpha' T u^{[\mu} P^{\nu]\sigma} \partial_\sigma \left(\frac{\mu}{T} \right) \tag{37}$$

and

$$J^\mu = \rho u^\mu - \sigma P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) - \alpha' a^\mu \tag{38}$$

- We imposed Onsager relation.

Kubo Formulas

- The first order transport coefficient is given by the Kubo formulas

$$\alpha = \lim_{\omega\to 0} \frac{1}{i\omega B} \left\langle T_{0i} T^{i}{}_{0} \right\rangle (\omega, \mathbf{k}=0), \tag{39}$$

$$\alpha = \lim_{\omega\to 0} \frac{1}{i\omega A} \left\langle T_{0i} T^{0}_{i} \right\rangle (\omega, \mathbf{k}=0), \tag{40}$$

$$\alpha' = -\lim_{\omega\to 0} \frac{1}{iB} \frac{\partial}{\partial\omega} \left\langle j^{i} T^{i}{}_{0} \right\rangle (\omega, \mathbf{k}=0), \tag{41}$$

$$\alpha' = -\lim_{\omega\to 0} \frac{1}{iA} \frac{\partial}{\partial\omega} \left\langle j^{i} T^{0}_{i} \right\rangle (\omega, \mathbf{k}=0). \tag{42}$$

Where we have defined the coefficients

$$A = \frac{1}{2} - \frac{t^{00}}{\varepsilon_0 + p_0}, \quad B = \frac{t_{00}}{\varepsilon_0 + p_0}, \tag{43}$$

and t^{00}, t_{00} are the zero frequency two-point functions

$$\left\langle T^{0}_{i} T^{0}_{j} \right\rangle = t^{00}\delta_{ij}, \ \left\langle T^{i}{}_{0} T^{0}_{j} \right\rangle = t^{0}_{0}\delta^{i}_{j}, \ \left\langle T^{i}{}_{0} T^{j}{}_{0} \right\rangle = t_{00}\delta^{ij}. \tag{44}$$

Novel Transport in Horava-Lifshitz Gravity

- The new transport contributes to the divergence of the entropy current $s^\mu = su^\mu$

$$\partial_\mu s^\mu = \frac{2\eta}{T}\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{\xi}{T}(\partial_\mu u^\mu)^2 + \frac{\alpha}{T}a_\mu a^\mu \qquad (45)$$

- In gravity this is the focusing equation

$$(G_{AB} - T^{ae}_{AB} + v_A E_B)\ell^A \ell^B = 0 \qquad (46)$$

where E_A is the variation of the action with respect to v_A and ℓ^A is the normal to the horizon. It measures the flux of matter-energy across the horizon.

- Both the spin 2 and spin 0 helicity fluxes contribute to η and ζ. The spin 0 helicity flux contributes to α.

Lifshitz Transport in Einstein Gravity With Matter

- We computed the bulk viscosity in holographic models dual to theories with Lifshitz scaling (and/or hyperscaling violation), using a generalization of the bulk viscosity formula (Eling-Oz) derived from the null focusing equation.
- We found that models with massive vector fields have a vanishing bulk viscosity.
- For other holographic models with scalars and/or massless vector fields we find a universal formula in terms of the dynamical exponent and the hyperscaling violation exponent

$$\frac{\zeta}{\eta} = -2\frac{\theta}{d(d-\theta)} + 2\frac{z-1}{d-\theta} \qquad (47)$$

- $z\varepsilon = (d-\theta)p$.

Non-relativistic Limit

- We now study fluids with broken Galilean boost invariance. In the relativistic fluid the maximal velocity c appears in $u^\mu = (1, \beta^i)/\sqrt{1-\beta^2}$, where $\beta^i = v^i/c$.
- In the non-relativistic limit $c \to \infty$, the pressure is not affected while the relativistic energy is expanded in terms of the mass density ρ and the internal energy U as

$$\varepsilon = c^2\rho - \frac{\rho v^2}{2} + U \, . \tag{48}$$

Non-relativistic Limit

- The relativistic hydrodynamic equations reduce to the non-relativistic form

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \,, \tag{49}$$

$$\partial_t U + \partial_i \left(U v^i \right) + p \partial_i v^i = \frac{\eta}{2} \sigma^{ij} \sigma_{ij} + \frac{\zeta}{d} (\partial_i v^i)^2 + \alpha (V_A^i)^2 \,, \tag{50}$$

$$\partial_t(\rho v^i) + \partial_j(\rho v^j v^i) + \partial^i p = \partial_j \left(\eta \sigma^{ij} + \frac{\zeta}{d} \delta^{ij} \partial_k v^k \right) + \partial_t(\alpha V_A^i) + \partial_j \left(\alpha v^j V_A^i \right) \,. \tag{51}$$

- The shear tensor is $\sigma_{ij} = \partial_i v_j + \partial_j v_i - (2/d) \delta_{ij} \partial_k v^k$.

Non-relativistic Limit

- While taking the limit, we have absorbed factors of $1/c$ in the shear and bulk viscosities η and ζ and a factor $1/c^2$ in α.
- The vector V^i_A is the acceleration of the fluid

$$V^i_A = D_t v^i = (\partial_t + v^k \partial_k) v^i \tag{52}$$

- Similarly to the viscosities, the coefficient α determines the dissipation that is produced in the fluid when the motion is not inertial.

Non-relativistic Lifshitz Scaling

- In a fluid with Lifshitz symmetry the scaling dimensions of the hydrodynamic variables are

$$[v^i] = z - 1, \quad [p] = [U] = z + d, \quad [\rho] = d + 2 - z, \quad (53)$$

  while the temperature has scaling dimension $[T] = z$.
- We can determine the scaling dimensions of the transport coefficients by imposing that all the terms in the hydrodynamic equations have the same scaling. We find

$$[\eta] = [\zeta] = d\ , \quad [\alpha] = d - 2(z - 1)\ . \quad (54)$$

Drude Model of a Strange Metal

- We model the collective motion of electrons in the strange metal as a charged fluid moving through a static medium, that produces a drag on the fluid.
- We are interested in describing a steady state where the fluid has been accelerated by the electric field, increasing the current until the drag force is large enough to compensate for it.
- In order to simplify the calculation we will consider an incompressible fluid $\partial_i v^i = 0$.

Drude Model of a Strange Metal

- The fluid motion is described by the Navier-Stokes equations

$$\rho v^k \partial_k v^i + \partial^i p = \rho E^i - \lambda \rho v^i + \eta \nabla^2 v^i + \alpha \partial_j \left(v^j v^k \partial_k v^i \right) \ . \tag{55}$$

- We have added two new terms: the force produced by the electric field E^i, and a drag term, whose coefficient λ has scaling dimension $[\lambda] = z$.
- We can solve this equation order by order in derivatives, keeping the pressure constant $\partial^i p = 0$. To leading order the current satisfies Ohm's law

$$J^i = \rho v^i \simeq \frac{\rho}{\lambda} E^i \ , \tag{56}$$

and the conductivity is simply $\boldsymbol{\sigma}_{ij} = \rho/\lambda \delta_{ij}$.

Conductivity

- At higher orders in derivatives the conductivity depends on the electric field and its gradients.
- When the electric field takes the form $E_x = E_0 \cos(y/L)$, the contribution of α to the conductivity is y dependent

$$\boldsymbol{\sigma}_{xx}(E_x, E_y) = \frac{\rho}{\lambda}\left[1 - \frac{1}{\rho\lambda}\left(\eta + \left[\frac{\alpha}{\lambda} + \frac{\rho}{\lambda^3}\right] E_y^2\right)\frac{1}{L^2} - \frac{1}{\lambda^2}\frac{E_y \partial_y E_x}{E_x}\right] \tag{57}$$

Conductivity

- If we average on the y direction, we find that the conductivity decreases with the magnitude of the transverse electric field

$$\boldsymbol{\sigma}_{xx}(E_x, E_y) = \frac{\rho}{\lambda}\left[1 - \frac{1}{\rho\lambda}\left(\eta + \left[\frac{\alpha}{\lambda} + \frac{\rho}{\lambda^3}\right] E_y^2\right)\frac{1}{L^2}\right] \tag{58}$$

- In the case where the electric field is linear in y, $E_x = E_0 y/L$, the conductivity is simplified to

$$\boldsymbol{\sigma}_{xx} = \frac{\rho}{\lambda}\left[1 + \frac{\alpha E_0^2}{\rho L^2 \lambda^3}\right] \tag{59}$$

The contribution from the shear viscosity drops. This gives a way to identify the new transport coefficient α.

Lifshitz Scaling

- In contrast with a relativistic fluid, the density is approximately independent of the temperature. This introduces an additional scale, and in general the transport coefficients can be non-trivial functions of the ratio $\tau = T^{\frac{d+2-z}{z}}/\rho$.
- The conductivity will have the following temperature dependence

$$\boldsymbol{\sigma}_{xx} = T^{\frac{d-2(z-1)}{z}}\hat{\boldsymbol{\sigma}}(\tau) \simeq \frac{\rho}{T}\ , \tag{60}$$

  where we assumed a linear dependence on the density as obtained from the calculation with the drag term.
- This predicts a resistivity linear in the temperature and *independent* of the dynamical exponent and the number of dimensions.

Dissipative Effects

- Consider the heat production due to the introduction of external forces and the drag (Kiritsis). An electric field or temperature gradient will induce an acceleration

$$a^i = -\partial^i p/\rho + E^i = (s/\rho)\partial^i T + E^i \tag{61}$$

- We impose $\partial_t a^i = 0$, $\partial_j a^i = 0$. The Navier-Stokes equations for homogeneous configurations takes the form

$$\partial_t v^i - (\alpha/\rho)\partial_t^2 v^i + \lambda v^i = a^i \tag{62}$$

Heat Production

- If the forces are suddenly switched on at $t = 0$, the evolution of the velocity is determined by this equation with the initial conditions $v^i(t = 0) = 0$, $\partial_t v^i(t = 0) = \frac{\rho a^i}{2\alpha\lambda}\left(\sqrt{\frac{4\alpha\lambda}{\rho} + 1} - 1\right)$.

$$\partial_t v^i(t = 0) = \frac{\rho a^i}{2\alpha\lambda}\left(\sqrt{\frac{4\alpha\lambda}{\rho} + 1} - 1\right) \tag{63}$$

- At late times the system evolves to a steady state configuration with constant velocity, so the heat production rate becomes constant $v^i = a^i/\lambda$. Subtracting this contribution for all times, the total heat produced is

$$\Delta Q = -\frac{\rho a^2}{2\lambda^2}\left(\sqrt{4\lambda\alpha/\rho + 1} + 2\right) \tag{64}$$

Geometrical Structure

- Since time scales differently than space, one has to consider the time direction separately, by foliating spacetime into equal-time slices.
- When considering a theory defined over a general curved manifold, this structure is generalized to a codimension-one foliation defined over the manifold.
- The foliation structure over a manifold can be locally represented by a 1-form t_α.
- A vector V^α is tangent to the foliation if and only if $t_\alpha V^\alpha = 0$. By the Frobenius theorem, such a 1-form (locally) defines a codimension-1 foliation if and only if it satisfies the condition:

$$t_{[\alpha}\partial_\beta t_{\gamma]} = 0 \tag{65}$$

Geometrical Structure

- The background fields of a Lifshitz field theory can be taken to be the metric $g_{\mu\nu}$ and the foliation 1-form t_α.
- On a curved manifold the lack of boost invariance translates into foliation preserving diffeomorphisms $\mathcal{L}_\xi t_\alpha \sim t_\alpha$, or

$$t \to f(t), \qquad x \to g(x,t) \tag{66}$$

- In addition there is an anisotropic Weyl symmetry.

Scale Anomalies

- It is given by the relative cohomology of the scaling operator modulo foliation preserving diffeomorphisms.
- The cohomology depends on z and d.
- The $z = 1$ cohomology differs from the trace anomaly.
- Anomalies are B-type : scale invariant objects.

Open Problems

- Theoretical : As listed in the beginning of the talk and more
- Experimental verifications