A holographic model for the fractional quantum Hall effect

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with Matthew Lippert and Anastasios Taliotis, 1409.1369

Many Thanks go to Elias Kiritsis.

Outline

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Introduction: Modular invariance in FQHE

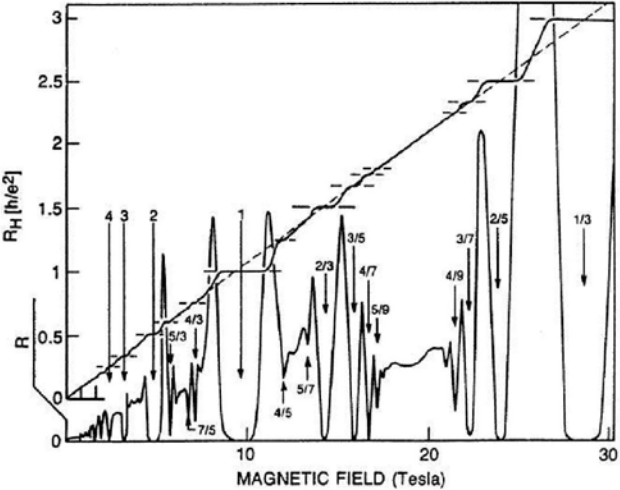
Review: Dyonic Black Holes and Modular Invariance

The SL(2,Z) invariant model

Conclusions & Further Directions

The Fractional Quantum Hall Effect

- In systems with 2D electron gases, at very low temperatures, high magnetic fields, clean samples :



Stormer (1992)

The Fractional Quantum Hall Effect

- FQHE states are gapped states with quantized Hall conductvitiy

$$\sigma_{xy} = \frac{p}{q}\left(\frac{e^2}{h}\right), \quad p, q \in \mathbb{Z}, \quad q \text{ odd}$$

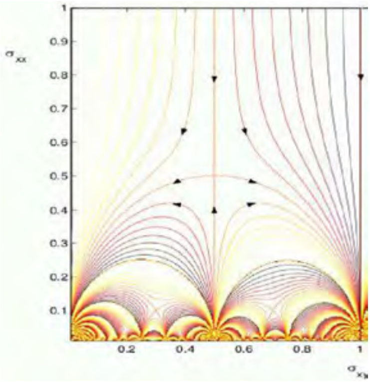
- Physics of charged quasiparticle excitations symmetric under Modular Group Action : $\sigma = \sigma_{xy} + i\sigma_{xx}$

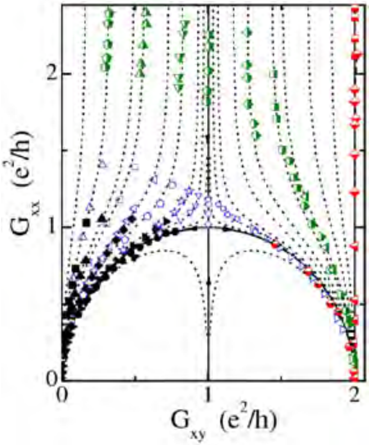
$$\sigma \mapsto \frac{a\sigma + b}{c\sigma + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2) \subset SL(2, \mathbb{Z}), \quad c \text{ even}$$

- Assumption: RG flow reduces to two-dimensional subspace
- Group action commuting with the RG flow implies that RG fixed points are $\Gamma_0(2)$ fixed points, structure imprinted on σ flows in $\sigma_{xx} - \sigma_{xy}$ plane

[Burgess+Lutken 1997, Dolan 1999, Lutken+Ross 2009, S.S. Murzin et al 2002]

The Fractional Quantum Hall Effect





The Fractional Quantum Hall Effect

- Examples:
  Selection Rule: $p'q - pq' = 1$ (e.g. $1/3 \to 2/5$) [Dolan 1998]
  Superuniversality of QH transitions

- CAN WE REPRODUCE THIS STRUCTURE (AND OTHER UNIVERSAL TRANSPORT FEATURES) IN A SINGLE HOLOGRAPHIC MODEL EMPLOYING THE MODULAR GROUP ACTION?

- Holographic model based on $SL(2, Z)$ invariance with GAPPED Quantum Hall states

Outline

$SL(2,Z)/SL(2,R)$ and Black Hole Charges

- Main idea of [1007.2490,1008.1917,1409.1369] : Use an SL(2,Z) (or SL(2,R)) invariant Einstein-Maxwell-Axio-Dilaton action to generate dyonic black branes by acting on known purely electrically charged black branes.

$$S = M_{Pl}^2 \int d^4x \sqrt{-g} \left[R - \frac{1}{2\gamma^2} \frac{\partial\tau\partial\bar{\tau}}{\tau_2^2} + V(\tau,\bar{\tau}) - \frac{1}{4}\left(\tau_2 F^2 + \tau_1 F\tilde{F}\right) \right]$$

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- Filling fraction and all other observables inherit the group action automatically.

- SL(2,Z)/SL(2,R) acts on the fields as

$\tau = a + ie^{\gamma\phi} = \tau_1 + i\tau_2\,, \quad \tau \to \frac{a\tau+b}{c\tau+d}\,, \quad ds^2 \to ds^2$ and

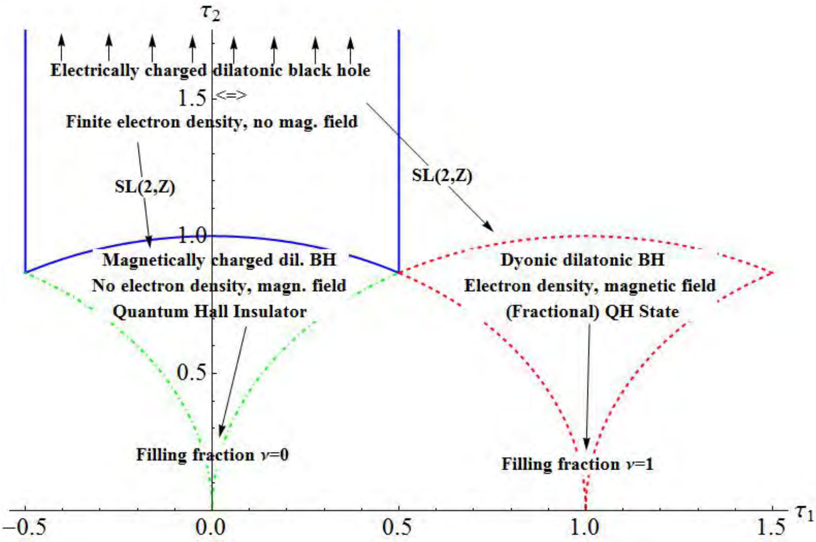
$$F \to F' = (c\tau_1 + d)F - c\tau_2\tilde{F}$$

- Dyonic black branes have charges

$$Q'_e = aQ_e\,, \quad Q'_m = cQ_e$$

Any rational (SL(2,Z)) or real (SL(2,R)) filling fraction generated in this way.

$SL(2,Z)/SL(2,R)$ and Black Hole Charges



$SL(2,Z)/SL(2,R)$ and Black Hole Charges

- Dyonic domain wall solutions flow to

$$\boxed{\tau'_{1*} = \frac{a}{c} \text{ and } \tau'_{2*} = {\tau_{2*}}^{-1} = 0}.$$

The filling fraction is hence equal to the value of the transformed axion

$$\boxed{\nu = \frac{Q'_e}{Q'_m} = \frac{a}{c} = \tau'_{1*}\,,}$$

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- [1007.2490] used a $SL(2,R)$ invariant model without scalar potential ($V(\tau,\bar{\tau}) = 2\Lambda$) and with the special value $\gamma = -1$.

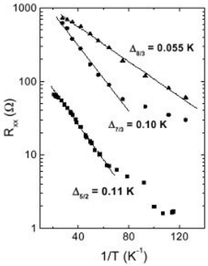
  Their QH states are unique due to the attractor mechanism , with Hall conductivity given again by a/c .

- [1008.1917] use a $SL(2,R)$ invariant DBI action on top of a $z = 5$ Lifshitz background. Besides the Hall conductivity, they reproduce the superuniversality exponents .

$SL(2,R)$ and Black Hole Charges

- Two Problems:

1. No hard gap in the charged excitations, σ_{DC} vanishes as a power law.
2. In SL(2,R) transformations between QH states $\sigma_{DC}|_{T=0} = 0$.
   This is not experimentally observed.



[Pan etal PRL 83 1999]

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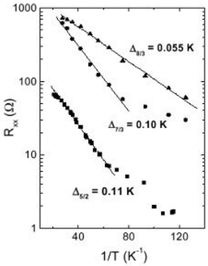

[Pan etal PRL 83 1999]

- Our model improves on the first point by using dyonic black branes in a confined phase (i.e. with a discrete and gapped spectrum), and on the second point by using $SL(2, Z)$ instead of $SL(2, R)$. In particular the latter point is expected to allow for real dynamical transitions between QH states. We also impose several well-motivated physical as well as consistency constraints.

Outline

A Gapped Holographic FQH State

- In string theory, SL(2,R) is usually broken to SL(2,Z) by nonperturbative effects. This typically will generate a SL(2,Z) invariant potential for the axio-dilaton .
- A simple choice is the real-analytic Eisenstein series

$$V(\tau,\bar{\tau}) = E_s(\tau,\bar{\tau}) = \sum_{m,n\in\mathbb{Z}^2/0,0} \left(\frac{|m+n\tau|}{\tau_2}\right)^{-s}$$

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- For large τ_2 the instanton expansion is dominated by a single exponential ($\tau_2 = e^{\gamma\phi}$),

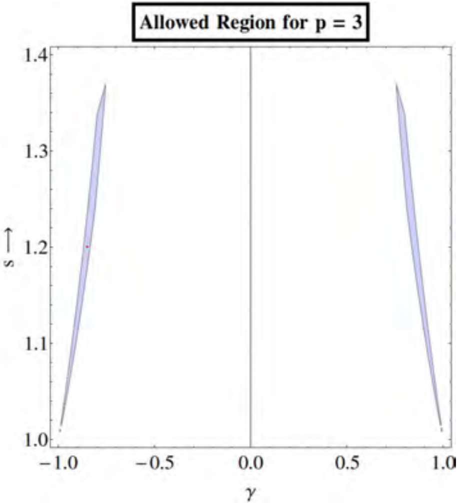
$$E_s = 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)\tau_2^{1-s} + \text{instanton contributions}$$

$\to$ [Charmousis+Gouteraux+Kim+Kiritsis+R.M. 1005.4690]

- We tune the two parameters (γ, s) such that the ground states are consistent and in particular confined. We then analyse the SL(2,Z) image of the electric state and confirm that these Quantum Hall states have a gapped and discrete spectrum.

A Gapped Holographic FQH State

- Gubser's constraint, existence of a (de)confinement transition, existence of a discrete and gapped spectrum, relevancy of UV fixed points and correct flow pattern restrict (γ, s):



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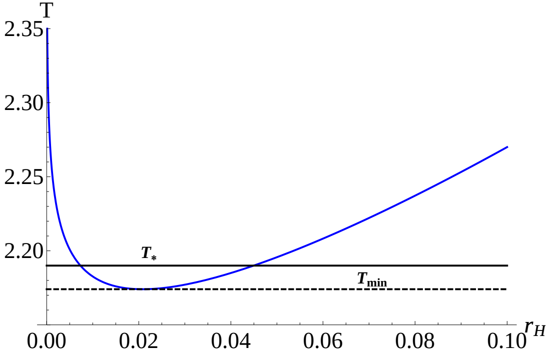

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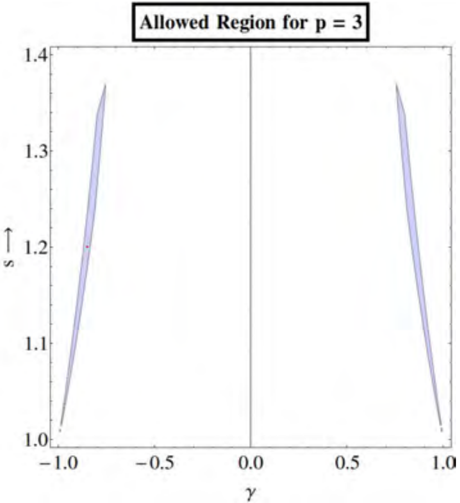


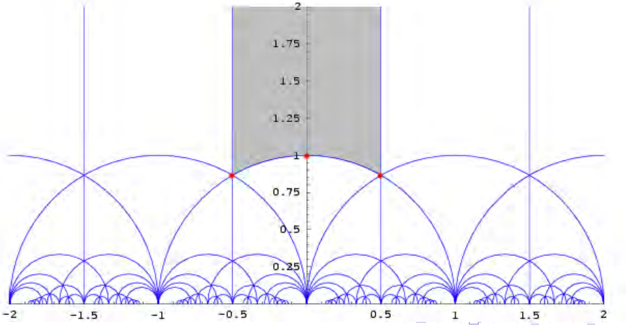
A Gapped Holographic FQH State

- QH Plateaux? Runaway minima at $\tau_1 = \frac{p}{q}$, $\tau_2 = 0$ are the images of the CDBH at $\tau_2 = \infty$, with charges

$$\frac{Q_e}{Q_m} = \frac{p}{q} = \tau_{1*} .$$

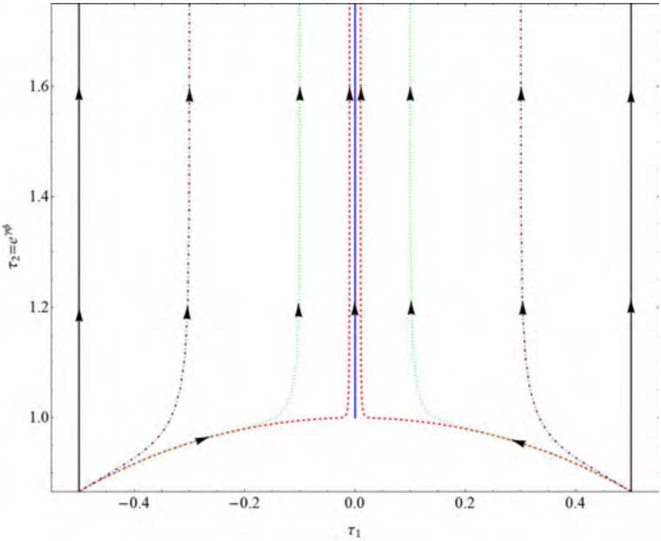
IR Geometry: Magnetically charged DBH w. $\tau_2 = e^{-\gamma\phi}$

- RG Flows: E_s is stationary in the fundamental domain at the SL(2,Z) fixed points:



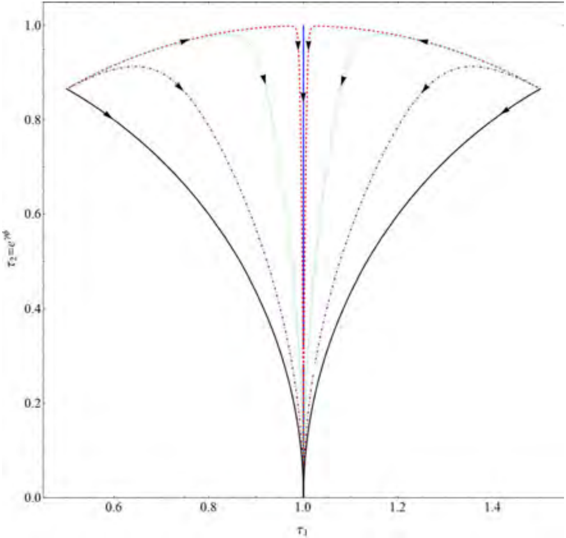
A Gapped Holographic FQH State

- Since SL(2,Z) commutes with the RG flow it suffices to construct the RG flows inside the fundamental domain:



A Gapped Holographic FQH State

- By SL(2,Z) we can generate flows to any QH plateaux $\tau_1 = p/q$. E.g. $\nu = 1$:



A Gapped Holographic FQH State

- Conductivity in el. Frame: At low enough temperatures the electric state is in a confined phase. The conductivity at small ω is dominated by the contribution from translation invariance:

$$\sigma_{xx}(\omega) \simeq \frac{iC''\mu}{\omega} + C''\delta(\omega) + ...$$

We found by numerically solving the fluctuation equations

$$\boxed{C'' = O(1)}$$

- Using $SL(2, Z)$ we find the correct Hall conductivity

$$\sigma_{xy}^{dyon} = Re\left(\frac{a\sigma + b}{c\sigma + d}\right) = \frac{(a\sigma_{xy} + b)(c\sigma_{xy} + d) + ac\sigma_{xx}^2}{(c\sigma_{xy} + d)^2 + c^2\sigma_{xx}^2} = \frac{a}{c} + \mathcal{O}(\omega^2)$$

- Consistent with direct small ω calculation of σ_{xy} in dyonic frame.
- But are the dyonic domain walls really gapped?

A Gapped Holographic FQH State

- In general dyonic solutions with running scalars the vector fluctuation equations can be decoupled into a single second order equation by

$$E_z = \omega(\delta A_x + i\delta A_y) + hg_{rr}(\delta g^x{}_t - i\delta g^y{}_t)\,.$$ [0910.0645]

$$\boxed{E_z'' + F(r,\omega)E_z' + G(r\omega)E_z = 0}$$

With $\Psi(r) = E_z(r)e^{\frac{1}{2}\int drF(z)}$ this is equivalent to

$$\boxed{-\Psi'' + V(r,\omega)\Psi = 0 \quad V(r,\omega) = \frac{1}{4}\left(F^2 - 4G + 2\partial_r F\right)}$$

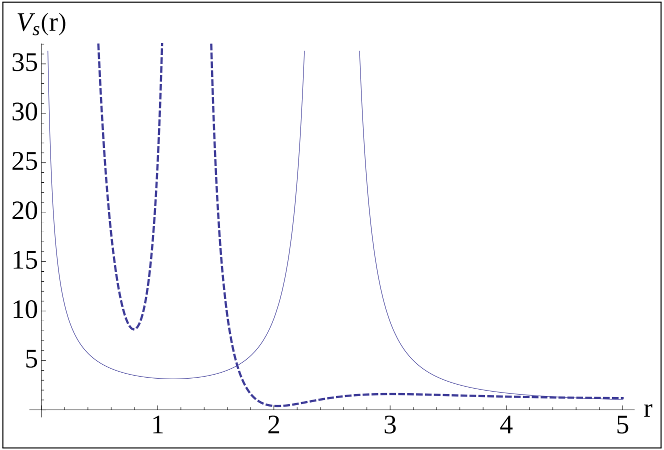
- For our choice of γ, s and at low frequencies:

$$\begin{aligned} V &\sim r^{-|\alpha|} \to +\infty \quad \text{in the IR} \\ V &\sim \frac{1}{4L^2} \quad \text{in the UV}\,. \end{aligned}$$

The spectrum is hence gapped in EVERY QH state!

A Gapped Holographic FQH State

- E.g. Flow to filling fraction one from $\tau = 1 + i$:

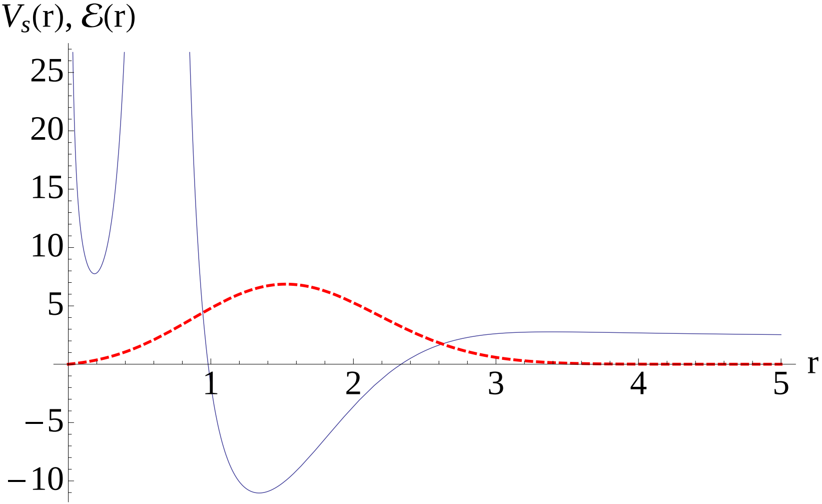


(solid: smaller *w*, dashed: bigger *w*)

- N.B.: The singularity in the potential is an accessory singularity, i.e. there is no monodromy, and the singularity is traversable by the wavefunctions. This was not appreciated in e.g. [0910.0645]]

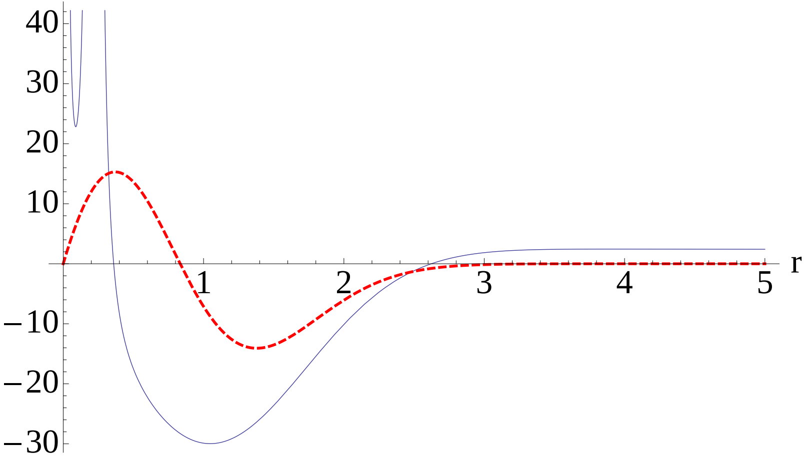
A Gapped Holographic FQH State

- For larger frequencies bound states appear
  E.g. Ground state:



A Gapped Holographic FQH State

- For larger frequencies bound states appear
  E.g. First excited state:
  $V_s(\mathrm{r}), \mathcal{E}(\mathrm{r})$



A Gapped Holographic FQH State

- The first bound state defines the charge gap.
  We find numerically that the gap and spectrum are approximately independent of the SL(2,Z) frame:

$$\frac{n}{B} = \frac{a}{c}\,, \quad B = cQ_{el.}$$

  We scanned $(a, c) \in [-20, \ldots, 20]$, i.e. filling fractions of $\left|\frac{n}{B}\right| \in [1/20, 20]$, with ω_0 varying less than 1% .
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- A better analytic understanding is clearly needed.
- N.B.: Excitations are not Landau levels, but rather "charged mesons" (as in holographic QCD).
- Our flows also show an anomalous Hall effect:

$$\boxed{\sigma_{xy}^{AHE} = \tau_1^{UV}}$$

  At least two UV completions for each QH state.
  IR physics universal.

Outline

Conclusions

- We improved on [1007.2490(,1008.1917)] by constructing holographic fractional Quantum Hall states with a gapped and discrete charge spectrum. We use a SL(2,Z) invariant Eisenstein potential , and showed that the QH states have the correct Hall conductivity , and a real gap (no $\delta(\omega)$ pole). The charged quasiparticle spectrum is independent of filling fraction and applied magnetic field.
- Future Directions/Open Questions:
  1. Interpretation of the two UV fixed points?
  2. Role of other IR fixed points (AdS_2 and running τ_1)? Transitions between QH Plateaux as a QPT? Universal Phenomenology?
  3. Why gap nearly $SL(2,Z)$ invariant?
  4. How to break SL(2,Z)? B and n/B dependent gap? How to include impurities? How to get plateaux?
  5. Subgroups – $\Gamma_0(2)/\Gamma_\theta(2)$?
  6. Top-down constructions? [G. Semenoff et.al.]
- STAY TUNED!

Backup Slides

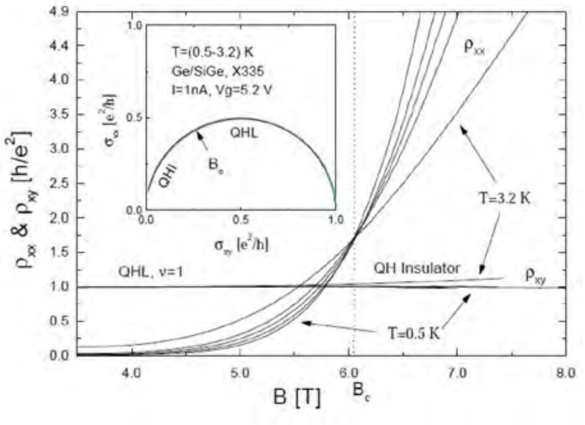
QH Plateaux Transition

Transition is a 2nd order QPT : [Fisher '90]

- Simple scaling $\Rightarrow \sigma(T, \Delta B, n, ...) = \sigma(\Delta B/T^{\kappa}, n/T^{\kappa'}, ...)$
- Superuniversality: κ and κ' are same for all transitions
- Experimentally: $\kappa = \kappa' = 0.42 \pm 0.01$ [Wanli et al 2009]

The Fractional Quantum Hall Effect

- Semicircle law: Conductivity sweeps out a semicircle in σ plane during QH transitions [e.g. Burgess etal 1008.1917]



SL(2,R) invariant probe branes [1008.1917]

- [Burgess etal 1008.1917] realized that the DC conductivity in the QH state of [1007.2490] vanishes due to the momentum-conservation pole in $\Im\sigma_{xx}$ of the purely electric solution.

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- [Burgess etal 1008.1917] realized that the DC conductivity in the QH state of [1007.2490] vanishes due to the momentum-conservation pole in $\Im \sigma_{xx}$ of the purely electric solution.
- They introduce dissipation by separating the sector that generates the gravity background of [1007.2490] from the sector of charge carriers, which they model using a SL(2,R) invariant probe brane

$$\begin{aligned} S &= M_{Pl}^2 \int d^4x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{2} \left((\partial\phi)^2 + e^{2\phi} (\partial a)^2 \right) \right] + \\ &\quad + M_{Pl}^2 S_{Lifshitz} + S_{gauge} \end{aligned}$$

The first two terms are assumed to be separately SL(2,R) invariant, and $S_{Lifshitz}$ to be chosen such as to generate the metric of the $z = 5$ Lifshitz black hole of [1007.2490], together with an appropriate axio-dilaton profile.

SL(2,R) invariant probe branes [1008.1917]

- S_{gauge} is taken to be a SL(2,R) invariant version of the DBI action, treated in the probe limit:

$$
\begin{aligned}
S_{gauge} &= -T\int d^4x\left[\sqrt{-det\left(g_{\mu\nu}+\ell^2 e^{-\phi/2}F_{\mu\nu}\right)}-\sqrt{-g}\right]\\
&\quad -\frac{1}{4}\int d^4x\sqrt{-g}aF_{\mu\nu}\tilde{F}^{\mu\nu}
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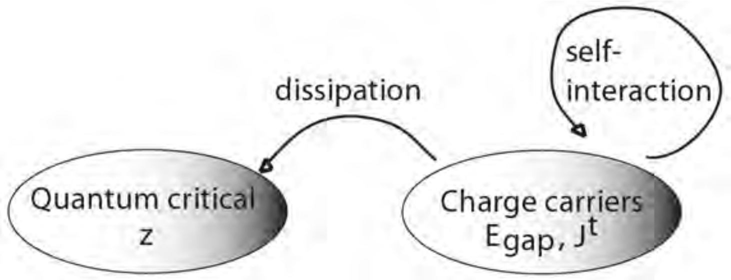
- This describes self-interacting charge carriers coupled to a large reservoir of quantum critical excitations into which they can loose energy via dissipation:



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- The QH state conductivity is then inferred by a SL(2,R) (or a subgroup such as $\Gamma_0(2)$) transformation

$$\sigma_{xx} = \frac{\sigma_0}{d^2 + c^2\sigma_0^2}\,, \quad \sigma_{xy} = \frac{ac\sigma_0^2 + bd}{d^2 + c^2\sigma_0^2}\,,$$

with $\sigma_0(T/\mu)$ the DC conductivity of the probe brane in the purely electric state (with $\sigma_{yx} = 0$). For probe branes in Lifshitz backgrounds like

$$ds_z^2 = L^2 \left[-h(r)\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2 h(r)} + \frac{dx^2 + dy^2}{r^2} \right]$$

σ_0 grows monotonically with falling temperature $\propto T^{-2/z}$, and parametrizes the RG flow of the conductivity in the QH state.

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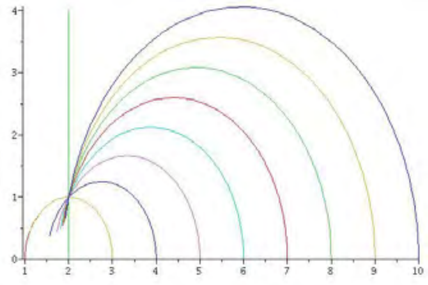
- This temperature flow commutes with SL(2,R) or any subgroup.

SL(2,R) invariant probe branes [1008.1917]

- The four parameters of the necessary SL(2,R) transformation are fixed by the data of the endpoint $(Q'_e, Q'_m, a, e^{-\phi})$. The temperature flow of the conductivities then trace out semi-circles in the σ plane, and for small T asymptote to (in linear response)

$$\sigma^{xx} \sim \frac{\rho T^{2/z}}{B^2} \to 0$$
$$\sigma^{xy} = \nu = \frac{a}{c}$$



This also predicts the superuniversality exponents $\kappa \approx \frac{2}{z} = \kappa'$ close to the measured value if $z = 5$ as in [1007.2490] .

However there is still no hard gap. .