Lifshitz hyperscaling violating holography

Ioannis Papadimitriou

Institúto de Física Teórica UAM/CSIC, Madrid

Kolymbari

1 September 2014

Based on: 1408.0795 [W. Chemissany, I. P.]
1405.3965 [W. Chemissany, I. P.]
1106.4826 [I. P.]
1007.4592 [I. P.]

Symplectic space of asymptotic data & holography

- Given our limited understanding of quantum gravity and of strongly coupled QFTs, a pragmatic approach to gauge gravity dualities has become popular. Namely, one is trying to *model* a given strongly coupled QFT of interest using a gravitational theory, based e.g. on symmetries or the spectrum of observables, and the hope is that the gravity description captures certain universal features of the dual strongly coupled QFT.
- Typical examples are holographic models of QCD, high temperature superconductors or other strongly correlated condensed matter systems exhibiting quantum critical transitions.
- Within this approach to gauge/gravity duality, the holographic dictionary can be derived systematically starting from the gravity side by i) providing a theory that supports the desired backgrounds, ii) identifying the radial coordinate emanating from the boundary of these geometries with the RG scale of the dual QFT iii) constructing the symplectic space of asymptotic data.

Lifshitz & hyperscaling violating Lifshitz holography

- Holographic description of quantum critical points and QFTs exhibiting hyperscaling violation
- Geometries suffer from IR pathologies – not relevant here
- These backgrounds can emerge in the IR or some intermediate energy scale starting with some other UV completion – e.g. AdS in the same or higher dimensions
- Here we will focus on the case where these geometries are considered as the UV. Otherwise we can develop the holographic dictionary in whatever UV completion these geometries emerge from

Related work on Lifshitz holography

- [Ross & Saremi '09] (Einstein-Proca)
- [Ross '11] (Einstein-Proca, vielbein formalism, counteterms derived using dilatation operator method [I.P. & Skenderis '04])
- [Griffin, Hořava, Melby-Thompson '11] (Einstein-Proca)
- [Mann & McNees '11] (Einstein-Proca)
- [Baggio, de Boer & Holsheimer '11] (Einstein-Proca)
- [Chemissany, Geissbühler, Hartong & Rollier '12] ($D = 4$ $z = 2$, Scherk-Schwarz reduction from 5d axion-dilaton model [I.P. '11])
- [Korovin, Skenderis & Taylor '13] ($z = 1 + \epsilon$)
- [Christensen, Hartong, Obers & Rollier '13] ($D = 4$ $z = 2$, Scherk-Schwarz reduction from 5d axion-dilaton model [I.P. '11])
- [Andrade & Ross '13] (Einstein-Proca, linear metric fluctuations)

Outline

Lifshitz & hyperscaling violating Lifshitz solutions

- The Lifshitz (Lif) metric is

$$ds^2_{d+2} = \ell^2 u^{-2}\left(du^2 - u^{-2(z-1)}dt^2 + dx^a dx^a\right)$$

with dynamical exponent $z \neq 1$

- This metric is invariant under the scaling transformation

$$x^a \to \lambda x^a\ ,\ t \to \lambda^z t\ ,\ u \to \lambda u$$

- The conformal boundary is located at $u = 0$
- The null energy condition

$$T_{\mu\nu}k^\mu k^\nu \geq 0, \quad k^\mu k_\mu = 0$$

requires

$$z \geq 1$$

Hyperscaling violating Lifshitz

- Hyperscaling refers to the property that the free energy and other thermodynamic quantities scale with temperature by their naive dimension. e.g. $S \sim T^{(2-\theta)/z}$, where θ is the hyperscaling violating parameter
- The Hyperscaling violating Lifshitz (hvLf) metric is [Huijse, Sachdev & Swingle '11]

$$ds^2_{d+2} = \ell^2 u^{-2(d-\theta)/d} \left(du^2 - u^{-2(z-1)} dt^2 + dx^a dx^a \right)$$

  with dynamical exponent $z \neq 1$ and hyperscaling violation exponent $\theta \neq 0$

- This metric has the scaling property that under

$$x^a \to \lambda x^a \ , \ t \to \lambda^z t \ , \ u \to \lambda u$$

  the metric transforms as

$$ds^2_{d+2} \to \lambda^{2\theta/d} ds^2_{d+2}$$

- The boundary is located at $u = 0$ for all values of θ (seen by going to the "dual frame" where the metric is asymptotically Lifshitz)

- The null energy condition requires

$$(d-\theta)(d(z-1)-\theta) \geq 0, \quad (d-\theta+z)(z-1) \geq 0$$

- The solutions of the null energy condition are:

I	$z \leq 0$	$\theta \geq d$
II	$0 < z \leq 1$	$\theta \geq d + z$
IIIa IIIb	$1 \leq z \leq 2$	$\theta \leq d(z-1)$ $d \leq \theta \leq d+z$
IVa IVb	$2 < z \leq \frac{2d}{d-1}$	$\theta \leq d$ $d(z-1) \leq \theta \leq d+z$
V	$z > \frac{2d}{d-1}$	$\theta \leq d$

- For $\theta \geq d + z$ the on-shell action does not diverge and hence there is no well defined asymptotic expansion/holographic dictionary (cf. D6 branes)
- We therefore exclude cases I and II and consider all cases with $z > 1$

- We consider a generic bottom up model of the form

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}\left(R[g] - \alpha\partial_\mu\phi\partial^\mu\phi - Z(\phi)F^2 - W(\phi)A^2 - V(\phi)\right)$$

- Preserve $U(1)$ gauge symmetry via the Stückelberg mechanism:

$$A_\mu \to B_\mu = A_\mu - \partial_\mu\omega$$

such that under a $U(1)$ transformation

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \quad \omega \to \omega + \Lambda$$

- Go to generic Weyl frame in order to accommodate both Lifshitz and hyperscaling violating backgrounds:

$$g \to e^{2\xi\phi} g$$

$$S_\xi = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}e^{d\xi\phi}\left(R[g] - \alpha_\xi\partial_\mu\phi\partial^\mu\phi - Z_\xi(\phi)F^2 - W_\xi(\phi)B^2 - V_\xi(\phi)\right)$$

$$\alpha_\xi = \alpha - d(d+1)\xi^2, \quad Z_\xi(\phi) = e^{-2\xi\phi}Z(\phi), \quad W_\xi(\phi) = W(\phi), \quad V_\xi(\phi) = e^{2\xi\phi}V(\phi)$$

Lifshitz solutions

- This model admit Lif or hvLf solutions at least asymptotically provided the potentials are of the form

$$V_\xi = V_o e^{2(\rho+\xi)\phi}, \qquad Z_\xi = Z_o e^{-2(\xi+\nu)\phi}, \qquad W_\xi = W_o e^{2\sigma\phi}$$

- The parameters are related to the parameters of the Lif solutions

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2, \quad A = \frac{\mathcal{Q}}{\epsilon Z_o}e^{\epsilon r}dt, \quad \phi = \mu r, \quad \omega = const.$$

as

$$\rho = -\xi, \quad \nu = -\xi + \frac{\epsilon - z}{\mu}, \quad \sigma = \frac{z-\epsilon}{\mu},$$
$$\epsilon = \frac{(\alpha_\xi + d^2\xi^2)\mu^2 - d\mu\xi + z(z-1)}{z-1}, \quad \mathcal{Q}^2 = \frac{1}{2}Z_o(z-1)\epsilon,$$
$$W_o = 2Z_o\epsilon(d+z+d\mu\xi-\epsilon), \quad V_o = -d(1+\mu\xi)(d+z+d\mu\xi) - (z-1)\epsilon.$$

- HvLf solutions take the form

$$ds^2 = dr^2 - r^{2\nu_z} dt^2 + r^{2\nu_1} d\vec{x}^2, \quad A = \frac{\mathcal{Q}}{\epsilon Z_o} r^{\epsilon} dt, \quad \phi = \mu \log r, \quad \omega = const.$$

where

$$\nu_z = 1 - \frac{dz}{\theta}, \quad \nu_1 = 1 - \frac{d}{\theta}, \quad u = \frac{|\theta|}{d} r^{\frac{d}{\theta}}, \quad \theta \neq 0$$

with

$$\mu(\xi + \rho) = -1, \quad \nu = -\xi - \frac{\nu_z - \epsilon}{\mu}, \quad \sigma = \frac{\nu_z - \epsilon - 1}{\mu}, \quad \mathcal{Q}^2 = \frac{1}{2} Z_o (\nu_z - \nu_1)\epsilon,$$

$$\epsilon = \frac{\left(\alpha_\xi + d^2\xi^2\right)\mu^2 - d\xi(\nu_1 + 1)\mu - \nu_1(d + \nu_z - 1) + \nu_z(\nu_z - 1)}{\nu_z - \nu_1},$$

$$W_o = 2\epsilon Z_o(d(\nu_1 + \mu\xi) + \nu_z - 1 - \epsilon),$$

$$V_o = \epsilon(\nu_1 - \nu_z) - d(\nu_1 + \mu\xi)(d(\nu_1 + \mu\xi) + \nu_z - 1).$$

- For $\xi = 0$ these are identical to the solutions of [Gouteraux, Kiritsis '12]

Relation between Lif & HvLf

- The hvLf metric is conformal to a Lif metric with the same exponent z. Namely, the coordinate transformation

$$r = e^{-\frac{\theta}{d}\bar{r}}, \quad t = \frac{|\theta|}{d}\bar{t}, \quad x^a = \frac{|\theta|}{d}\bar{x}^a$$

the hvLf solution becomes

$$ds^2 = \left(\frac{\theta}{d}\right)^2 e^{\frac{-2\theta\bar{r}}{d}} \left(d\bar{r}^2 - e^{2z\bar{r}}d\bar{t}^2 + e^{2\bar{r}}d\vec{\bar{x}}^2\right), \quad \phi = \mu_h \log r = -\frac{\theta}{d}\mu_h\bar{r} \equiv \mu_L\bar{r}$$

- It follows that the hvLf metric can be written as

$$g_h = e^{-\frac{2\theta}{d\mu_L}\phi} g_L, \quad \mu_L = -\theta\mu_h/d, \quad \ell_L = |\theta|\ell_h/d$$

- This allows us to express any hvLf solution as a Lif solution in a different Weyl frame – cf. “dual frame” for Dp branes with $p \neq 3$. Namely, if $g_h = e^{2\xi\phi}g_L$ is a hvLf metric in the Einstein frame, then g_L is a Lif metric in a Weyl frame with

$$\xi = -\frac{\theta}{d\mu_L} = \frac{1}{\mu_h}$$

Radial Hamiltonian formalism and the Hamilton-Jacobi equation

Radial Hamiltonian formalism for massive vector-scalar theory

- ADM decomposition

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

- Radial ADM Lagrangian:

$$\begin{aligned}
L \quad = \quad & \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-\gamma}Ne^{d\xi\phi}\Big\{R[\gamma] + K^2 - K^{ij}K_{ij} + \frac{2d\xi}{N}K(\dot{\phi} - N^i\partial_i\phi) \\
& -\frac{\alpha_\xi}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - \alpha_\xi\gamma^{ij}\partial_i\phi\partial_j\phi \\
& -Z_\xi(\phi)\left(\frac{2}{N^2}\gamma^{ij}(F_{ri} - N^kF_{ki})(F_{rj} - N^lF_{lj}) + \gamma^{ij}\gamma^{kl}F_{ik}F_{jl}\right) \\
& -W_\xi(\phi)\left(\frac{1}{N^2}\left(A_r - N^iA_i - \dot{\omega} + N^i\partial_i\omega\right)^2 + \gamma^{ij}B_iB_j\right) - V_\xi(\phi)\Big\}
\end{aligned}$$

- Hamiltonian:

$$\begin{aligned} H &= \int d^{d+1}x \left(\dot{\gamma}_{ij}\pi^{ij} + \dot{A}_i \pi^i + \dot{\phi}\pi_\phi + \dot{\omega}\pi_\omega \right) - L \\ &= \int d^{d+1}x \left(N\mathcal{H} + N_i \mathcal{H}^i + A_r \mathcal{F} \right) \end{aligned}$$

- where

$$\mathcal{H} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(2\pi^{ij}\pi_{ij} - \frac{2}{d}\pi^2 + \frac{1}{2\alpha}\left(\pi_\phi - 2\xi\pi\right)^2 + \frac{1}{4} Z_\xi^{-1} \pi^i \pi_i + \frac{1}{2} W_\xi^{-1} \pi_\omega^2 \right)$$
$$+ \frac{\sqrt{-\gamma}}{2\kappa^2} e^{d\xi\phi} \left(-R[\gamma] + \alpha_\xi \partial^i \phi \partial_i \phi + Z_\xi(\phi) F^{ij} F_{ij} + W_\xi(\phi) B^i B_i + V_\xi(\phi) \right)$$

$$\mathcal{H}^i = -2D_j \pi^{ji} + F^i{}_j \pi^j + \pi_\phi \partial^i \phi - B^i \pi_\omega$$

$$\mathcal{F} = -D_i \pi^i + \pi_\omega$$

Canonical momenta

- From off-shell Lagrangian:

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(K\gamma^{ij} - K^{ij} + \frac{d\xi}{N}\gamma^{ij}\left(\dot{\phi} - N^k\partial_k\phi\right)\right),$$

$$\pi^{i} = \frac{\delta L}{\delta \dot{A}_{i}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}Z_\xi(\phi)\frac{4}{N}\gamma^{ij}\left(F_{rj} - N^k F_{kj}\right),$$

$$\pi_{\phi} = \frac{\delta L}{\delta \dot{\phi}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(2d\xi K - \frac{2\alpha_\xi}{N}(\dot{\phi} - N^i\partial_i\phi)\right),$$

$$\pi_{\omega} = \frac{\delta L}{\delta \dot{\omega}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}W_\xi(\phi)\frac{2}{N}\left(\dot{\omega} - N^i\partial_i\omega - A_r + N^i A_i\right)$$

- From on-shell action:

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}, \quad \pi^{i} = \frac{\delta \mathcal{S}}{\delta A_{i}}, \quad \pi_{\phi} = \frac{\delta \mathcal{S}}{\delta \phi}, \quad \pi_{\omega} = \frac{\delta \mathcal{S}}{\delta \omega}$$

Flow equations

- Combining the two expressions for the momenta:

$$\dot{\gamma}_{ij} = -\frac{4\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\left(\gamma_{ik}\gamma_{jl} - \frac{\alpha_\xi + d^2\xi^2}{d\alpha} \gamma_{ij}\gamma_{kl} \right) \frac{\delta}{\delta\gamma_{kl}} - \frac{\xi}{2\alpha}\gamma_{ij}\frac{\delta}{\delta\phi} \right) \mathcal{S},$$

$$\dot{A}_i = -\frac{\kappa^2}{2}\frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} Z_\xi^{-1}(\phi)\gamma_{ij}\frac{\delta}{\delta A_j}\mathcal{S},$$

$$\dot{\phi} = -\frac{\kappa^2}{\alpha}\frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\frac{\delta}{\delta\phi} - 2\xi\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} \right) \mathcal{S},$$

$$\dot{\omega} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} W_\xi^{-1}(\phi)\frac{\delta}{\delta\omega}\mathcal{S}$$

Recursive solution of the Hamilton-Jacobi equation

Zero derivative solution

- The zero order solution of the HJ equation contains no transverse derivatives:

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int d^{d+1}x \sqrt{-\gamma} U(\phi, A_i A^i)$$

- Inserting this ansatz into the Hamiltonian constraint yields a PDE for $U(X, Y)$, where $X := \phi$, $Y := B_i B^i = A_i A^i$ (cf. superpotential equation)

$$\frac{1}{2\alpha} \left(U_X - \xi(d+1)U + 2\xi Y U_Y\right)^2 + Z_\xi^{-1}(X) Y U_Y^2$$
$$- \frac{1}{2d} \left((d+1)U + 2(d-1)Y U_Y\right) \left(U - 2Y U_Y\right) = \frac{1}{2} e^{2d\xi X} \big(W_\xi(X) Y + V_\xi(X)\big)$$

- This equation for the 'superpotential' $U(X, Y)$ determines the zero derivative solution of the Hamilton-Jacobi equation: It can be used to holographically renormalize any homogeneous background of the equations of motion and any exact solution of this PDE leads to exact solutions of the equations of motion via the flow equations.

Constraints from Lifshtiz asymptotics

- Imposing Lifshitz boundary conditions requires that asymptotically the gauge invariant vector field behaves as

$$B_i \sim B_{oi} = \sqrt{\frac{z-1}{2\epsilon}} Z_\xi^{-1/2}(\phi) \mathbb{n}_i$$

where $\mathbb{n}_i$ is the unit normal to the constant t surfaces

- This in turn implies that the superpotential $U(X,Y)$ must satisfy

$$U(X, Y_o(X)) \sim e^{d\xi X} \left(d(1+\mu\xi) + z - 1\right)$$
$$U_Y(X, Y_o(X)) \sim -\epsilon e^{d\xi X} Z_\xi(X)$$
$$U_X(X, Y_o(X)) \sim e^{d\xi X} \big(-\mu\alpha_\xi + d\xi(d+z)\big)$$

- Hence, the asymptotic form of the zero order solution of the HJ equation is

$$\mathcal{S}_{(0)} \sim \frac{1}{\kappa^2} \int_{\Sigma_r} d^{d+1}x \sqrt{-\gamma} e^{d\xi\phi} \left(d(1+\mu\xi) + \frac{1}{2}(z-1) - \epsilon Z_\xi(\phi) B_i B^i \right)$$

Taylor expansion of the superpotential

- Since Lifshitz boundary conditions require that $B_i \sim B_{oi}$ asymptotically, the solution of the HJ equation can be expressed as a Taylor series in $B_i - B_{oi}$
- The zero derivative solution $\mathcal{S}_{(0)}$ can be Taylor expanded in

  $$Y - Y_o = 2B_o^i(B_i - B_{oi}) + \mathcal{O}(B - B_o)^2$$

  where $Y_o \equiv B_o^i B_{oi}$, as

  $$U = e^{(d+1)\xi\phi}\left(u_0(\phi) + Y_o^{-1}u_1(\phi)(Y - Y_o(\phi)) + Y_o^{-2}u_2(\phi)(Y - Y_o(\phi))^2 + \cdots\right)$$

- Inserting this expansion in the superpotential equation for $U(X, Y)$ leads to a tower of equations for the functions $u_n(\phi)$

- An additional relation between the functions $u_0(\phi)$ and $u_1(\phi)$ is imposed by the consistency of the Taylor expansion, i.e. requiring that

$$\dot{Y} - \dot{Y}_o = \mathcal{O}(Y - Y_o)$$

- In a bottom up approach these equations can be used to *define* the potentials $V(\phi)$, $Z(\phi)$ and $W(\phi)$ in terms of $u_0(\phi)$ and $u_1(\phi)$, with all $u_n(\phi)$ for $n \geq 2$ being determined in terms of these functions.
- Lifshitz boundary conditions require

$$\boxed{\begin{aligned} & u_0(\phi) \sim (z - 1 + d(1 + \mu\xi))\, e^{-\xi\phi} \\ & u_1(\phi) \sim \frac{1}{2}(z-1)e^{-\xi\phi} \end{aligned}}$$

- The function $u_2(\phi)$ satisfies a quadratic (Riccati) equation and determines the scaling behavior of the independent mode $Y - Y_o$, while $u_n(\phi)$ with $n \geq 3$ satisfy linear equations.

Recursive solution of the HJ equation

- To summarize the above analysis, we have shown that the most general zero derivative solution of the HJ equation takes the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int_{\Sigma_r} d^{d+1}x \sqrt{-\gamma} U(\phi, B^2)$$

  where for Lifshitz boundary conditions the superpotential $U(X, Y)$ admits a Taylor expansion in $Y - Y_o$. Moreover, this zero derivative solution is the asymptotically leading one, with derivative terms entering only in asymptotically subleading orders.

- In order to systematically determine these asymptotically subleading derivative terms of the solution of the HJ equation, we expand $\mathcal{S}$ in a covariant expansion in eigenfunctions of a suitable operator.
- For backgrounds with asymptotic scaling invariance one can use the dilatation operator [I. P. & Skenderis 2004] but in the presence of an asymptotically running dilaton, meaning that asymptotic scale invariance is broken, this is not sufficient.
- Instead we need an operator such that $\mathcal{S}_{(0)}$ is an eigenfunction for any superpotential $U(\phi, B^2)$.

- In fact there are two mutually commuting such operators:

$$\widehat{\delta} := \int d^{d+1}x \left(2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right), \quad \delta_B := \int d^{d+1}x \left(2Y^{-1} B_i B_j \frac{\delta}{\delta \gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right)$$

which satisfy

$$\widehat{\delta}\mathcal{S}_{(0)} = (d+1)\mathcal{S}_{(0)}, \quad \delta_B \mathcal{S}_{(0)} = \mathcal{S}_{(0)}, \quad [\widehat{\delta}, \delta_B] = 0$$

- This allows us to seek a solution of the HJ equation in the form of a *graded* covariant expansion in simultaneous eigenfunctions of both $\widehat{\delta}$ and δ_B:

$$\mathcal{S} = \sum_{k=0}^{\infty} \mathcal{S}_{(2k)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \mathcal{S}_{(2k,2\ell)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \int d^{d+1}x \mathcal{L}_{(2k,2\ell)}$$

where

$$\widehat{\delta}\mathcal{S}_{(2k,2\ell)} = (d+1-2k)\mathcal{S}_{(2k,2\ell)}, \quad \delta_B \mathcal{S}_{(2k,2\ell)} = (1-2\ell)\mathcal{S}_{(2k,2\ell)}, \quad 0 \leq \ell \leq k+1$$

- The operator $\widehat{\delta}$ counts derivatives
- The operator δ_B annihilates the projection operator $\sigma^i_j := \delta^i_j - Y^{-1} B^i B_j$ and counts derivatives contracted with B_i, which asymptotically become time derivatives since $B_i \sim B_{oi} \propto \mathbb{n}_i$

Linear recursion equations

- Inserting the covariant expansion of $\mathcal{S}$ in simultaneous eigenfunctions of $\widehat{\delta}$ and δ_B in the Hamilton-Jacobi equation (Hamiltonian constraint) results in a system of recursive first order functional *linear* equations for the higher derivative terms:

$$\frac{1}{\alpha}\left(U_X-(d+1)\xi U+2\xi Y U_Y\right)\frac{\delta}{\delta\phi}\int\mathcal{L}_{(2k,2\ell)}+$$
$$\left((2Y+Z_\xi^{-1})U_Y+\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)\right)B_i\frac{\delta}{\delta B_i}\int\mathcal{L}_{(2k,2\ell)}-$$
$$\left(\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)(d+1-2k)+2YU_Y(1-2\ell)\right)\mathcal{L}_{(2k,2\ell)}=$$
$$e^{d\xi\phi}\mathcal{R}_{(2k,2\ell)}$$

- The inhomogeneous term $\mathcal{R}_{(2k,2\ell)}$ involves derivatives of lower order terms as well as the 2-derivative sources from the Hamiltonian constraint

Lifshitz boundary conditions

- The covariant expansion of $\mathcal{S}$ in simultaneous eigenfunctions of $\widehat{\delta}$ and δ_B, and hence the above recursion relations, is independent of the specific choice of boundary conditions
- In order to impose Lifshitz boundary conditions we must additionally expand $\mathcal{S}_{(2k,2\ell)}$ in $B_i - B_{oi}$ at each order of the covariant expansion as

$$\begin{aligned}\mathcal{L}_{(2k,2\ell)} &= \mathcal{L}^0_{(2k,2\ell)}[\gamma(x),\phi(x)] \\ &+ \int d^{d+1}x'(B_i(x')-B_{oi}(x'))\mathcal{L}^{1i}_{(2k,2\ell)}[\gamma(x),\phi(x);x'] + \mathcal{O}\left(B-B_o\right)^2\end{aligned}$$

- Inserting this Taylor expansion in the above recursion relations eliminates the derivative with respect to B_i, resulting in first order linear functional differential equations in ϕ only. Such functional differential equations appear in the relativistic case as well, e.g. for non-conformal branes or Improved Holographic QCD, and they can be solved systematically [I.P. '11].

Solution of the recursion relations up to $\mathcal{O}(B - B_o)$

- The inhomogeneous solution of these linear functional differential equations takes the form

$$\mathcal{L}^0_{(2k,2\ell)} = e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} \mathcal{R}^0_{(2k,2\ell)},$$

$$\sigma^i_j \mathcal{L}^{1j}_{(2k,2\ell)} = Z_\xi^{\frac{1}{2}} e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} Z_\xi^{-\frac{1}{2}} \sigma^i_j \mathcal{R}^{1j}_{(2k,2\ell)},$$

$$B_{oj}(x)\mathcal{L}^{1j}_{(2k,2\ell)} = \Omega^{-1} e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} \Omega B_{oj} \widehat{\mathcal{R}}^{1j}_{(2k,2\ell)}$$

where $\mathcal{C}_{k,\ell} := d + 1 - 2k + (z-1)(1-2\ell)$,

$$\mathcal{K}(\phi) := \frac{\alpha}{e^{\xi\phi}\left(u_0' + \frac{Z'}{Z}u_1\right)} \sim -\frac{1}{\mu}, \quad e^{\mathcal{A}(\phi)} = Z_\xi^{-\frac{1}{2(\epsilon - z)}} \sim e^{\phi/\mu}$$

and the $\Omega(\phi)$ can be expressed in terms of u_0, u_1 and u_2.

- If $\mu = 0$ (e.g. for Einstein-Proca theory) the corresponding solutions can be expressed *algebraically* in terms of the source terms.

Structure of the HJ solution & the holographic dictionary

- The general asymptotic solution of the HJ equation obtained via the above algorithm takes the form

$$\mathcal{S} = \sum_{k,\ell,m \mid \mathcal{C}_{k,\ell}+\theta-m\Delta_- \geq 0} \int \cdots \int (B-B_o)^m \mathcal{S}^m_{(2k,2\ell)} + \widehat{\mathcal{S}}_{ren} + \cdots$$

  where $\Delta_+ = d+z-\theta-\Delta_-$ is the scaling dimension of the scalar operator dual to the mode

$$\psi := Y_o^{-1} B_o^j (B_j - B_{oj})$$

  and $(B-B_o)^m \mathcal{S}^m_{(2k,2\ell)}$ has dilatation weight $\mathcal{C}_{k,\ell}+\theta-m\Delta_-$, while $\widehat{\mathcal{S}}_{ren}$ has dilatation weight 0.
- All terms $(B-B_o)^m \mathcal{S}^m_{(2k,2\ell)}$ with $\mathcal{C}_{k,\ell}+\theta-m\Delta_- \geq 0$ are determined by the recursion algorithm.
- For $\mathcal{C}_{k,\ell}+\theta-m\Delta_- < 0$ these terms are powerlike divergent in the UV, while terms with $\mathcal{C}_{k,\ell}+\theta-m\Delta_- = 0$ have a pole which via dimensional regularization leads to a logarithmic divergence. Such logarithmically divergence terms give rise to the conformal anomaly when $\mu = 0$, but they can be absorbed in the dilaton when $\mu \neq 0$.

- The covariant local counterterms that render the on-shell action finite and the variational problem with Lifshitz boundary conditions well posed are

$$\mathcal{S}_{ct} := -\sum_{k,\ell,m \mid \mathcal{C}_{k,\ell}+d\mu\xi-m\Delta_-\geq 0} \int\cdots\int (B-B_o)^m \mathcal{S}^m_{(2k,2\ell)}$$

- The renormalized part of the on-shell action is therefore given by the UV-finite term $\widehat{\mathcal{S}}_{ren}$, which corresponds to an independent contribution to the HJ solution and can be parameterized as

$$\widehat{\mathcal{S}}_{ren} = \int d^{d+1}x \left(\gamma_{ij}\widehat{\pi}^{ij} + B_i\widehat{\pi}^i + \phi\widehat{\pi}_\phi\right)$$

where $\widehat{\pi}^{ij}$, $\widehat{\pi}^i$ and $\widehat{\pi}_\phi$ are undetermined integration functions of the HJ equation.

Sources & VEVs

- Inserting this general asymptotic solution of the HJ equation, including the undetermined term $\widehat{\mathcal{S}}_{ren}$, in the first order flow equations one can systematically derive the generalized asymptotic Fefferman-Graham expansions for the bulk fields, including the sources and 1-point functions of the dual operators.
- The sources generically correspond to integration constants of the flow equations, while the 1-point functions are related to the integration constants of the HJ solution in $\widehat{\mathcal{S}}_{ren}$.
- Decomposing the induced fields as

$$\gamma_{ij}dx^i dx^j = -(n^2 - n_a n^a)dt^2 + 2n_a dt dx^a + \sigma_{ab}dx^a dx^b, \quad A_i dx^i = a dt + A_a dx^a,$$

where the indices a, b run from 1 to d, and introducing the linear combinations

$$\widehat{\mathcal{T}}^{ij} := -\frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\left(2\widehat{\pi}^{ij} + Y_o^{-1}B_o^i B_o^j B_{ok}\widehat{\pi}^k\right),$$

$$\widehat{\mathcal{O}}_\phi := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\left(\widehat{\pi}_\phi + (\nu + \xi)B_{oi}\widehat{\pi}^i\right),$$

$$\widehat{\mathcal{O}}_\psi := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}B_{oi}\widehat{\pi}^i, \quad \widehat{\mathcal{E}}^i := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\sqrt{-Y_o}\sigma^i_j\widehat{\pi}^j,$$

the full set of sources and VEVs is (cf. energy-momentum complex [Ross '09]):

	1-point function	source
spatial stress tensor	$\widehat{\Pi}^i_j := \mathbb{o}^i_k \mathbb{o}_{jl} \mathcal{T}^{kl} \sim e^{-(d+z-\theta)r} \Pi^i_j(x)$	$\sigma_{(0)ab}$
momentum density	$\widehat{\mathcal{P}}^i := -\mathbb{o}^i_k \mathbb{n}_l \mathcal{T}^{kl} \sim e^{-(d+2-\theta)r} \mathcal{P}^i(x)$	$n_{(0)a}$
energy density	$\widehat{\mathcal{E}} := -\mathbb{n}_k \mathbb{n}_l \mathcal{T}^{kl} \sim e^{-(d+z-\theta)r} \mathcal{E}(x)$	$n_{(0)}$
energy flux	$\widehat{\mathcal{E}}^i \sim e^{-(d+2z-\theta)r} \mathcal{E}^i(x)$	0
dilaton	$\widehat{\mathcal{O}}_\phi \sim e^{-(d+z+d\mu\xi)r} \mathcal{O}_\phi(x)$	$\phi_{(0)}$
composite scalar	$\widehat{\mathcal{O}}_\psi \sim e^{-\Delta_+ r} \mathcal{O}_\psi(x)$	ψ_-

- The momentum constraint of the radial Hamiltonian formalism leads to the diffeomorphism Ward identity

$$\mathbb{D}_j\widehat{\Pi}^i_i + \mathbb{q}_j\widehat{\Pi}^j_i + \mathbb{n}^j D_j\widehat{\mathcal{P}}_i + \mathbb{K}\widehat{\mathcal{P}}_i + \mathbb{K}^i_i\widehat{\mathcal{P}}_j + \mathbb{n}_i\mathbb{q}_j\widehat{\mathcal{P}}^j - \widehat{\mathcal{E}}\mathbb{q}_i + \widehat{\mathcal{O}}_\phi\mathbb{D}_i\phi + \widehat{\mathcal{O}}_\psi\mathbb{D}_i\psi = 0,$$

$$\mathbb{n}^i D_i\widehat{\mathcal{E}} + \mathbb{K}\widehat{\mathcal{E}} - \mathbb{K}^i_j\widehat{\Pi}^j_i + \mathbb{D}_i\widehat{\mathcal{E}}^i + \widehat{\mathcal{O}}_\phi\mathbb{n}^i D_i\phi = 0,$$

$$\mathbb{D}_i\widehat{\mathcal{P}}^i + 2\mathbb{q}_i\widehat{\mathcal{P}}^i = 0,$$

  where $\mathbb{D}_i$ is the covariant derivative w.r.t. σ_{ij}, $\mathbb{K}_{ij} = \mathbb{D}_i\mathbb{n}_j$ is the extrinsic curvature of the constant time slices, and $\mathbb{q}_i = \mathbb{n}^k D_k\mathbb{n}_i$.

- The transformation of the renormalized action under *local* anisotropic boundary Weyl transformations leads to the trace Ward identity

$$\begin{aligned} &z\widehat{\mathcal{E}} + \widehat{\Pi}^i_i + \Delta_-\psi\widehat{\mathcal{O}}_\psi - \mu\widehat{\mathcal{O}}_\phi = 0, && \mu \neq 0,\\ &z\widehat{\mathcal{E}} + \widehat{\Pi}^i_i + \Delta_-\psi\widehat{\mathcal{O}}_\psi = \mathcal{A}, && \mu = 0,\end{aligned}$$

  where $\mathcal{A}$ is the conformal anomaly, corresponding to all terms satisfying $\mathcal{C}_{k,\ell} + \theta - m\Delta_- = 0$.

Concluding remarks

Concluding remarks

- We presented a general prescription for constructing the holographic dictionary for asymptotically locally Lifshitz and hyperscaling violating Lifshitz backgrounds, with arbitrary dynamical exponents compatible with the null energy condition.
- The key to the construction of the holographic dictionary is a recursive algorithm for solving the radial Hamilton-Jacobi equation asymptotically by expanding the solution in simultaneous eigenfunctions of two commuting operators.
- The full holographic dictionary can be obtained from this asymptotic solution of the Hamilton-Jacobi equation. Crucially, we have demonstrated that there is no need for field redefinitions such as using vielbeins in an otherwise second order formalism for the bulk theory, and there is no need for using the second order equations to derive the asymptotic expansions.
- Correlation functions are also much more efficiently computed holographically using Hamilton-Jacobi techniques in order to trade the second order linear fluctuation equations for first order non-linear (Riccati) equations.