Exact results for Kardar-Parisi-Zhang growth equation from replica Bethe ansatz and sine-Gordon field theory

P. Le Doussal
(LPTENS) Paris

KPZ is a NL stochastic growth equation Kardar-Parisi-Zhang (1986)

1985-1990 exact scaling exponents d=1

burgers, random directed polymer,..

2000-2010 many solvable (discrete) models in physics and mathematics found to exhibit same large scale universality called KPZ class (d=1) related to random matrix theory:

Tracy-Widom distributions of largest eigenvalue of GUE, GOE..

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2010-now

in physics

- exact solutions directly continuum KPZ eq./DP (at all times)
  methods integrability (Bethe Ansatz) + disorder (replica)
- new very controled precise experiments
- new systems in KPZ class   big and growing!

in math:

- rigorous definition KPZ and stochastic eqs.
  Hairer's Fields medal
- broader mathematical picture of "stochastic integrability"
  rigorous results, "rigorous replica"

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outline:

- growth of 1D interfaces
  KPZ equation, KPZ universality class

- random matrix theory
  largest eigenvalues (Tracy Widom universal distributions)

- solving KPZ using (imaginary time) quantum mechanics
  attractive bose gas (integrable) => TW distribution for KPZ
    - droplet initial condition
    - flat initial condition

- KPZ from sine-Gordon QFT

Universality, symmetries (RMT) initial conditions (KPZ) with : Pasquale Calabrese (Univ. Pise)

also: Alberto Rosso (LPTMS Orsay)
Thomas Gueudre (LPTENS)

P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
P. Calabrese, P. Le Doussal, Phys. Rev. Letters 106 250603 (2011) and J. Stat. Mech. P06001 (2012) T. Gueudre, P. Le Doussal, EPL 100 26006 (2012).

also works by: V. Dotsenko
H. Spohn, T. Sasamoto

also: (math) A. Borodin, I. Corwin, J. Quastel, N. O'Connell..

also G. Schehr, Reymenik, Ferrari ..

reviews KPZ: Corwin arXiv 1106.1596, H. Spohn..

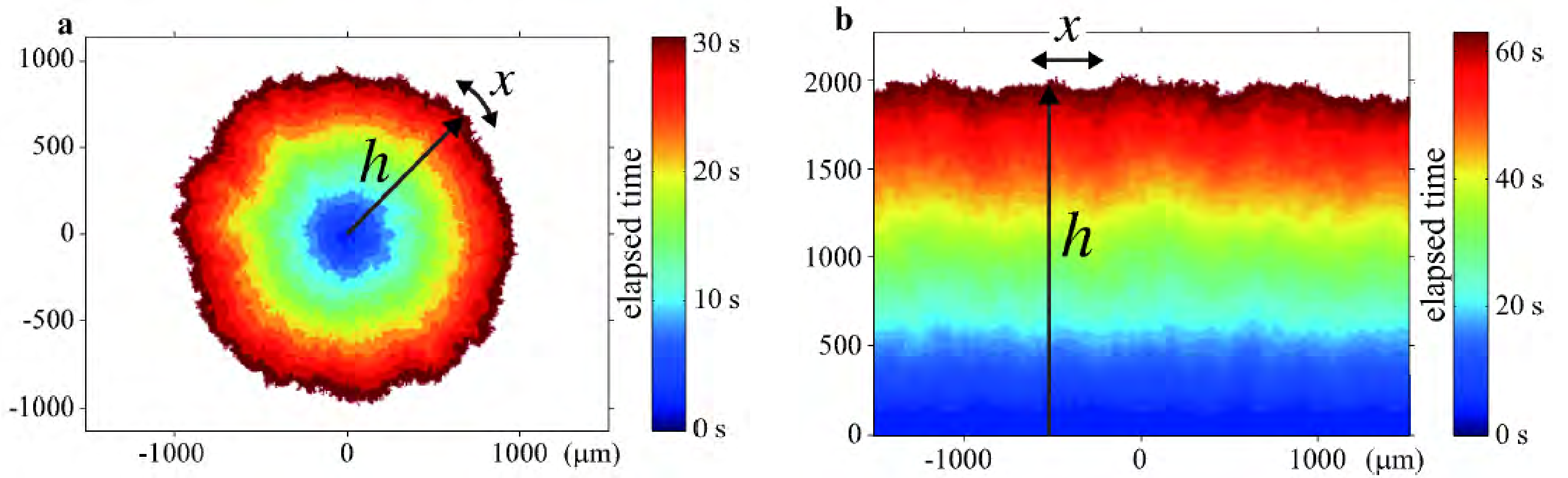
RMT: Mehta's book, Edelman, Fyodorov, Majumdar,..

Universal Fluctuations of Growing Interfaces: Evidence in Turbulent Liquid Crystals

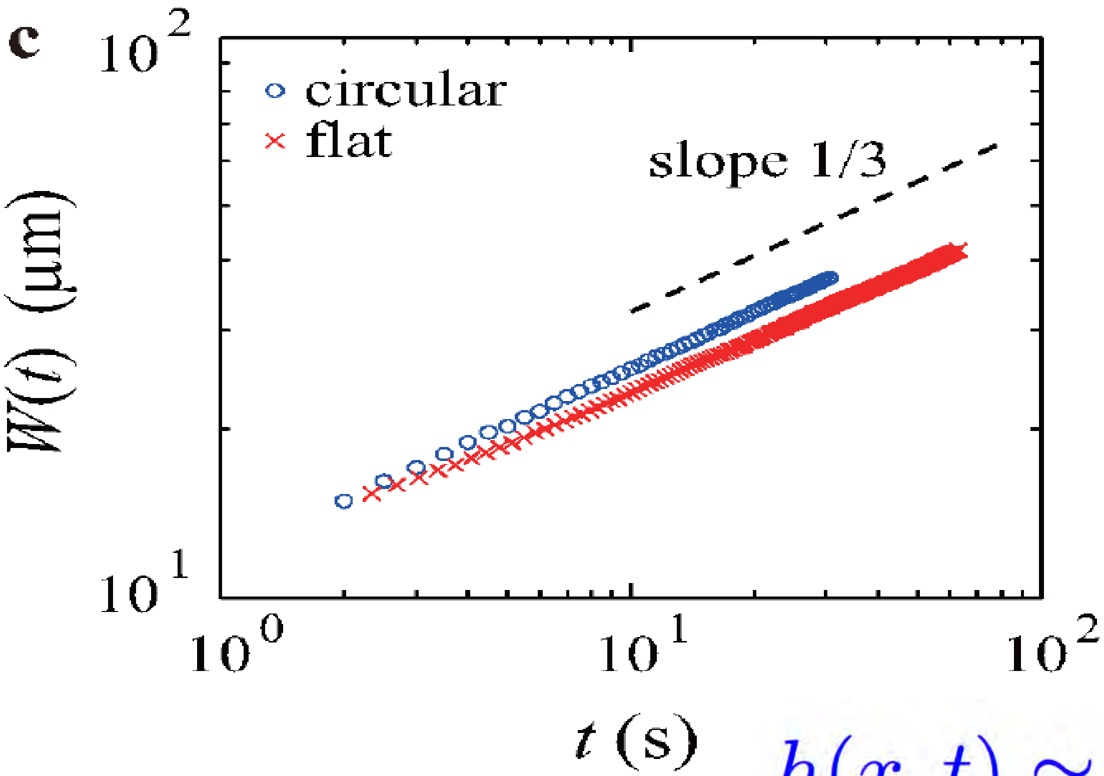
Kazumasa A. Takeuchi* and Masaki Sano
Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
(Received 28 January 2010; published 11 June 2010)

measure

h(x,t)



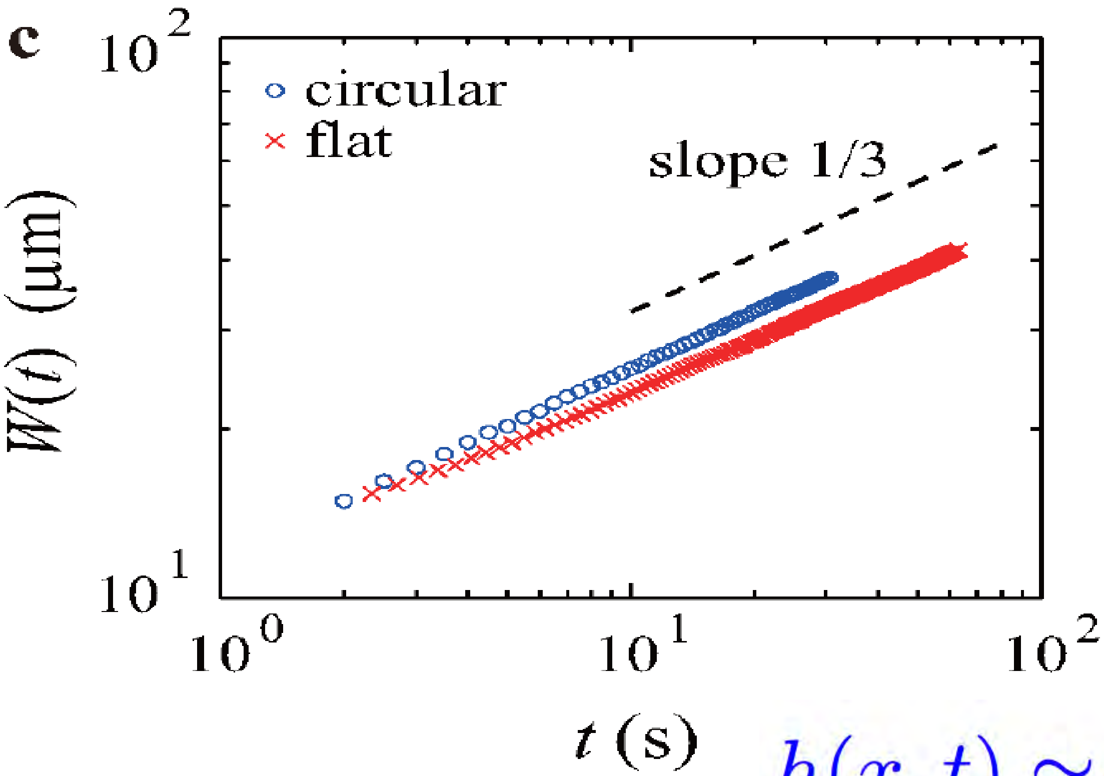
Experimental evidence. We study the convection of nematic liquid crystal, confined in a thin container and driven by an electric field[19,20], and focus on the interface between two turbulent states, called dynamic scattering modes 1 and 2 (DSM1 and DSM2)[20,21]. The latter consists of a large quantity of topological defects and can be created by nucleating a defect with a ultraviolet laser pulse. Whereas



$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x,t) \simeq_{t \to +\infty} v_\infty t + \chi \, t^{1/3}$$

χ is a random variable



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$$w(l,t) \equiv \left\langle \sqrt{\left\langle \left[h(x,t) - \langle h \rangle_l\right]^2 \right\rangle_l} \right\rangle$$

$$\simeq t^{1/3} F[\ell / t^{2/3}]$$

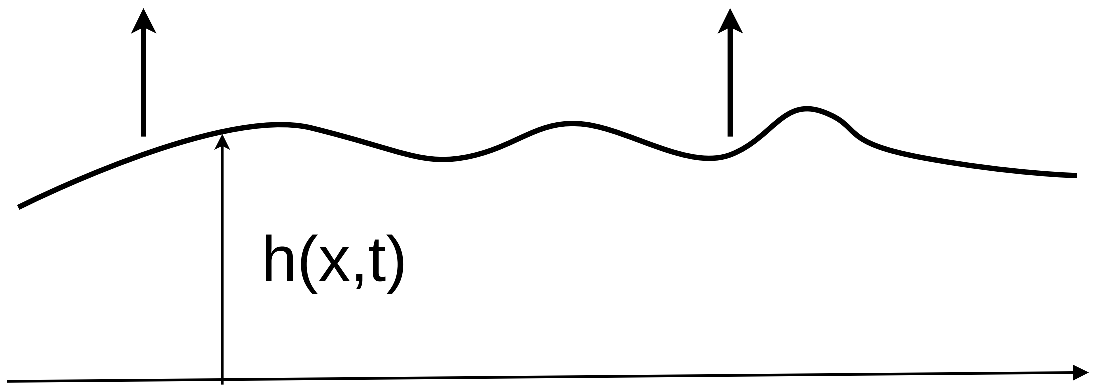
$$h \sim t^{1/3} \sim x^{1/2}$$

$$x \sim t^{2/3}$$

z=3/2

interface is random self-affine

how to model a growing interface ?



neglect overhangs

large scale effective model

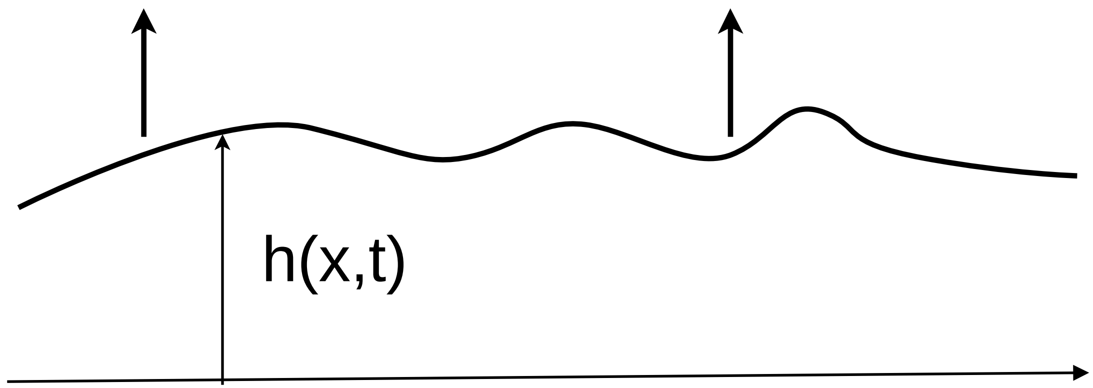
Edwards-Wilkinson

$$\partial_t h = \nu \partial_x^2 h + \eta(x,t) + v$$

surface tension noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

how to model a growing interface ?



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$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

$$h_{\omega,q} = \frac{\eta_{\omega,q}}{\nu q^2 + i\omega}$$

$$\overline{hh}(q,\omega) = \frac{D}{\nu^2 q^4 + \omega^2}$$

$$h \sim t^{1/4} \sim x^{1/2}$$

$$x \sim t^{1/2}$$

P(h) is gaussian, simple diffusive dynamics

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \eta(x,t)$$

diffusion

noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

- 1D scaling exponents $h \sim t^{1/3} \sim x^{1/2}$ $x \sim t^{2/3}$

- P(h=h(x,t)) non gaussian

depends on some details of initial condition

flat	h(x,0) =0
wedge (droplet)	h(x,0) = - w |x|

$\lambda_0 = 0$ Edwards Wilkinson P(h) gaussian

$$h \sim t^{1/3} \sim x^{1/2}$$

Also reported in:

slow combustion of paper

J. Maunuksela et al. PRL 79 1515 (1997)

bacterial colony growth

Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)

fronts of chemical reactions

S. Atis (2012)

formation of coffee rings via evaporation

Yunker et al. PRL (2012)

but.. quenched disorder, LR interactions..

KPZ needs local growth mechanism

Random matrix theory

Large N by N random matrices H, with Gaussian independent entries

eigenvalues λ_i $i = 1, ..N$

$$P[\lambda] = c_{N,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4} \sum_{k=1}^N \lambda_k^2}$$

H is:

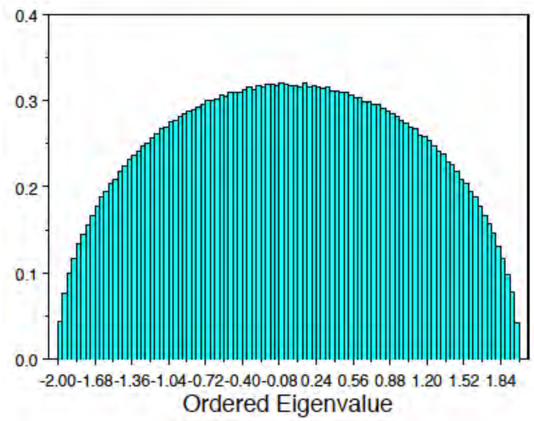
$\beta =$ 1 (GOE) real symmetric

2 (GUE) hermitian

4 (GSE) symplectic

Universality large N :

- DOS: semi-circle law



histogram of eigenvalues N=25000

- distribution of the largest eigenvalue

$H \to NH$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

Tracy Widom (1994)

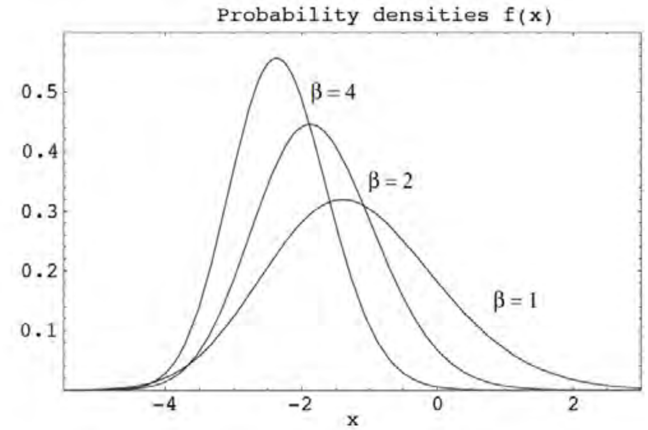
Tracy-Widom distributions (largest eigenvalue of RM)

GOE $$F_1(s) = Det[I - K_1]$$

Fredholm determinants

$$K_1(x,y) = \theta(x) Ai(x+y+s) \theta(y)$$

$$(I-K)\phi(x) = \phi(x) - \int_y K(x,y)\phi(y)$$



GUE $$F_2(s) = Det[I - K_2]$$

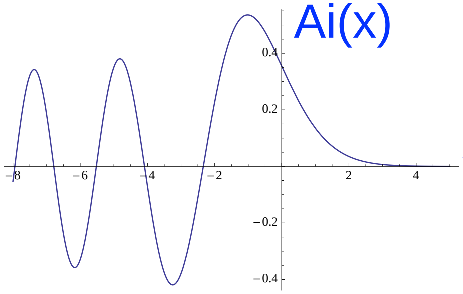
$$K_2(x,y) = K_{Ai}(x+s, y+s)$$

tails

$$K_{Ai}(x,y) = \int_{v>0} Ai(x+v) Ai(y+v)$$

Ai(x-E)

is eigenfunction E
particle linear potential

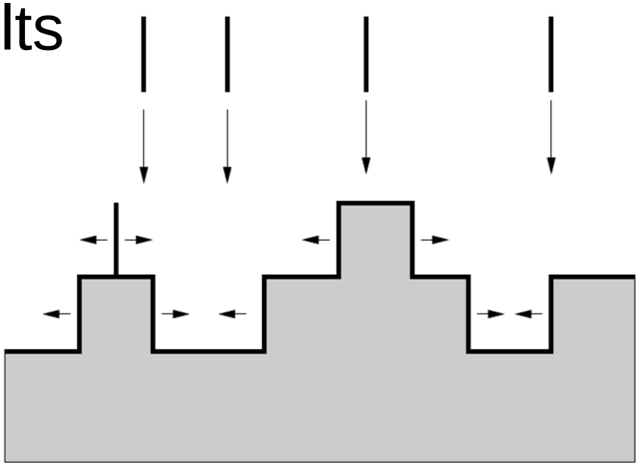


Exact solutions of discrete models

discrete models in KPZ class/exact results

- polynuclear growth model (PNG)

  Prahofer, Spohn, Baik, Rains (2000)



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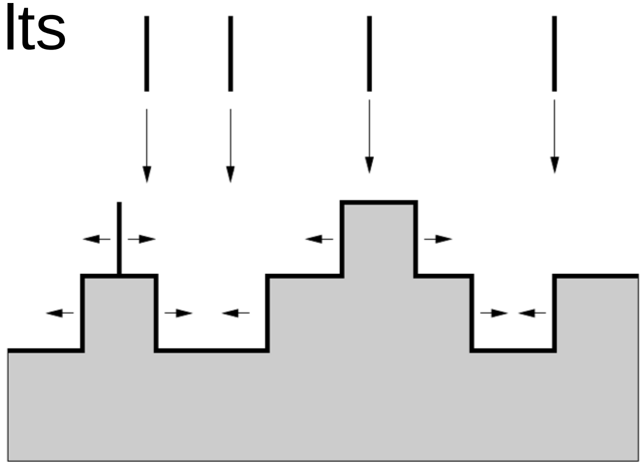
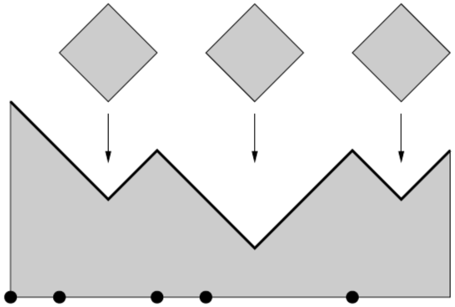
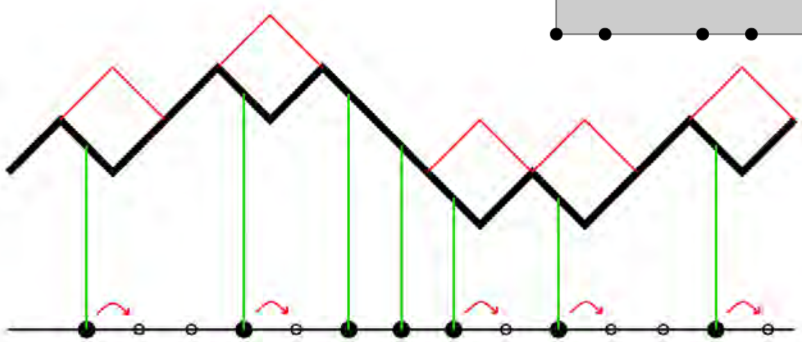
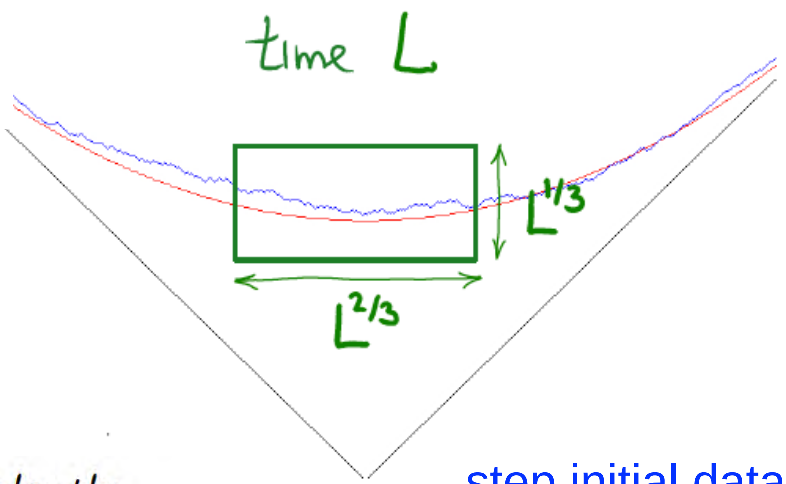


- totally asymmetric exclusion process (TASEP)





Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as exp(−x)dx.



step initial data

Johansson (1999)

discrete models in KPZ class/exact results

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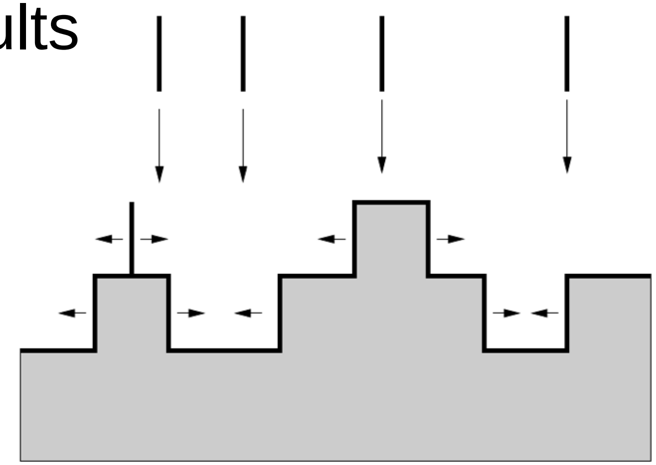

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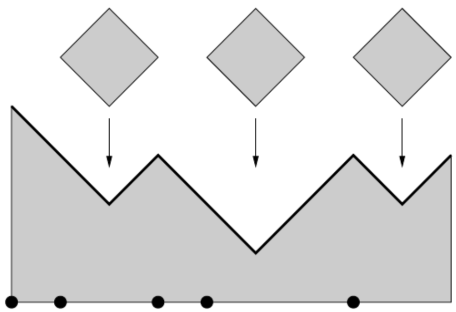

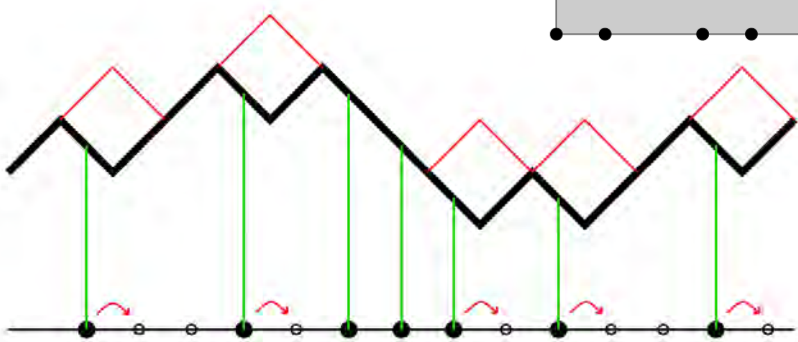

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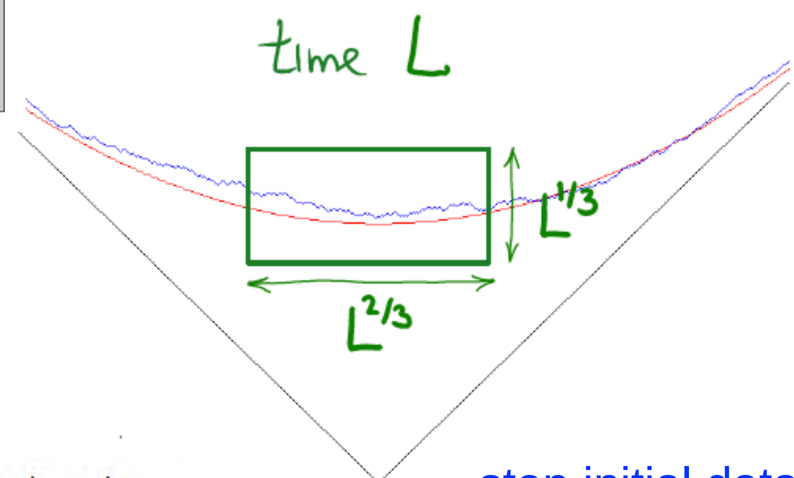

step initial data

Johansson (1999)

q-TASEP rate $1-q^{gap}$ $0<q<1$

Exact results for height distributions for some discrete models in KPZ class

- PNG model

Baik, Deft, Johansson (1999)

$$h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$$

droplet IC
GUE

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

flat IC $\chi = \chi_1$ GOE

- similar results for TASEP Johansson (1999), ...

(2000-2010) multi-point correlations
Airy processes

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3} A_n(y)$$

$A_2(y)$ GUE

$A_1(y)$ GOE

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$A_2(y)$ GUE

$A_1(y)$ GOE

- Longest increasing subsequence of a random permutation

$$\{1,2,3,\dots,N\}$$

$N = 10$ $\{8, 2, 7, \underline{1}, \underline{3}, \underline{4}, 10, \underline{6}, \underline{9}, 5\}$

$$l_N = 5$$

$$l_N \to 2\sqrt{N} + N^{1/6}\chi$$

GUE

Baik, Deft, Johansson (1999)

directed paths

$N \sim t^2$

also in tilings etc..

Continuum KPZ via replica

Question: is KPZ equation in KPZ class ?

Cole Hopf mapping

KPZ equation ⟷ Continuum
Directed paths (polymers)
in a random potential

Quantum mechanics
of bosons
(imaginary time)

Kardar 87

Continuum DP fixed endpoint/KPZ Narrow wedge (droplet)

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)

- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010
Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010)
Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).

- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

Continuum DP one free endpoint/KPZ Flat (RBA)

P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Quastel, J. Ortmann and D. Remenik in preparation

Cole Hopf mapping

KPZ equation:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \eta(x,t)$$

define:

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

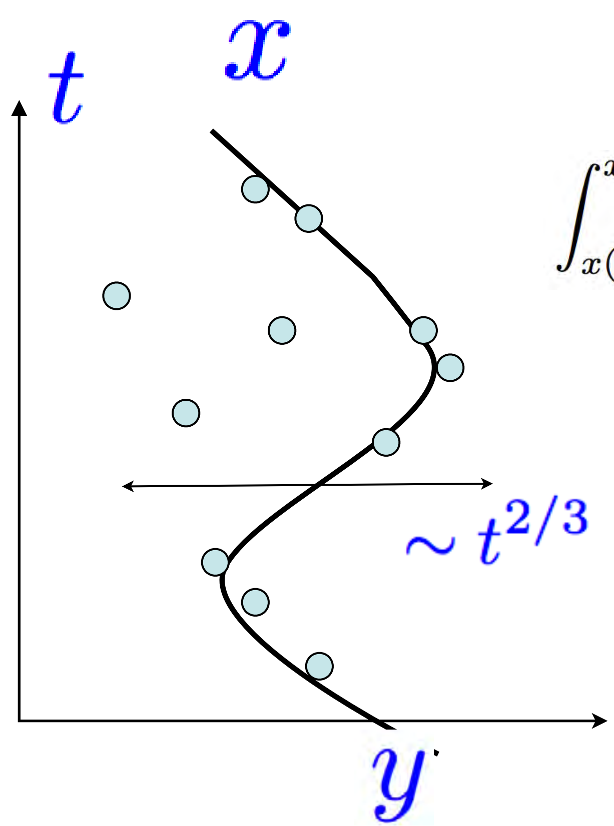
$$T = 2\nu$$

it satisfies:

$$\partial_t Z = \frac{T}{2} \partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\lambda_0 \eta(x,t) = -V(x,t)$$

describes directed paths in random potential V(x,t)



$$Z(x,t|y,0) =$$

$$\int_{x(0)=y}^{x(t)=x} Dx(\tau)\, e^{-\frac{1}{T}\int_0^t d\tau\, \frac{\kappa}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau),\tau)}$$

$$\overline{V(x,t)V(x',t')} = \bar{c} \quad \delta(t-t')\delta(x-x')$$

Feynman Kac

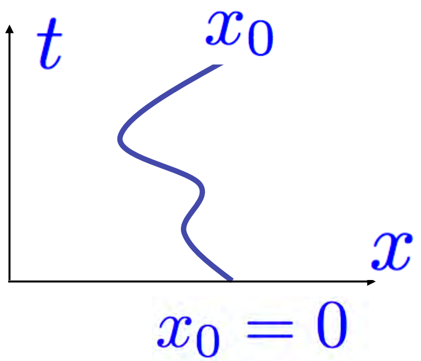
$$Z(x,y,t=0) = \delta(x-y)$$

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu} h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu} h(y,t=0)}$$

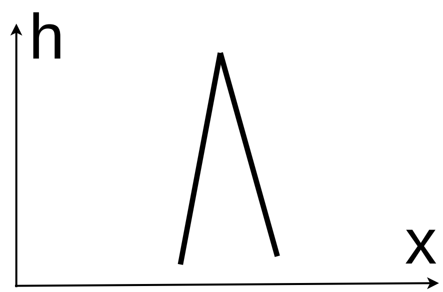
1) DP both fixed endpoints $Z(x_0,t|x_0,0)$



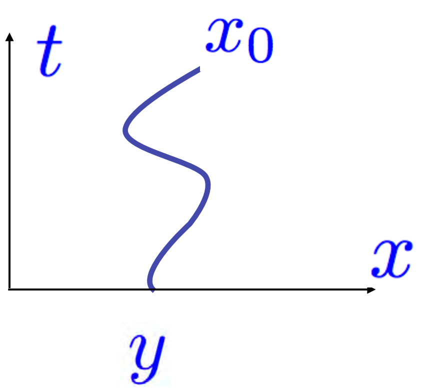
KPZ: narrow wedge <=> droplet initial condition

$$h(x,t=0) = -w|x|$$

$$w \to \infty$$



2) DP one fixed one free endpoint $\int dy Z(x_0,t|y,0)$



KPZ: flat initial condition

$$h(x,t=0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

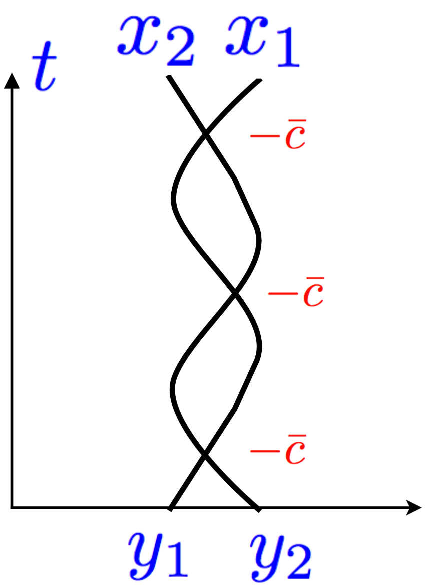
calculate $\overline{Z^n} = \int dZ\, Z^n P(Z)$ $\quad n \in \mathbb{N}$

“guess” the probability distribution from its integer moments:

$$P(Z) \to P(\ln Z) \to P(h)$$

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1, t|y_1, 0)..Z(x_n, t|y_n 0)} = \langle x_1, ..x_n| e^{-tH_n} |y_1, ..y_n\rangle$$



$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$

$$x = T^3 \kappa^{-1} \tilde{x} \quad , \quad t = 2T^5 \kappa^{-1} \tilde{t}$$

drop the tilde..

$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1 \le i < j \le n} \delta(x_i - x_j)$$

Attractive Lieb-Lineger (LL) model (1963) what do we need from quantum mechanics ?

- KPZ with droplet initial condition

  = fixed endpoint DP partition sum

μ eigenstates

E_μ eigen-energies

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} =< x_0 ... x_0 | e^{-tH_n} | x_0, .. x_0 >$$

symmetric states = bosons

$$= \sum_\mu \Psi_\mu^*(x_0..x_0) \Psi_\mu(x_0..x_0) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

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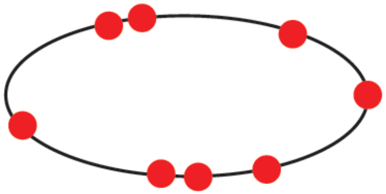
symmetric states = bosons

$$= \sum_\mu \Psi^*_\mu(x_0 .. x_0) \Psi_\mu(x_0 .. x_0) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

- flat initial condition

$$\overline{(\int_y Z(x_0 t|y0))^n} = \sum_\mu \Psi^*_\mu(x_0, .x_0) \int_{y_1,.y_n} \Psi_\mu(y_1, .y_n) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

LL model: n bosons on a ring with local delta attraction



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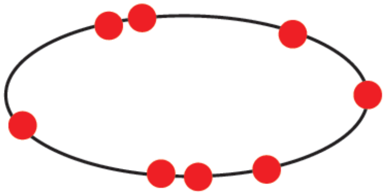

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Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^{n} e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^{n} \lambda_j^2$$

$$A_P = \prod_{n\geq\ell>k\geq 1}\left(1 - \frac{ic\ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

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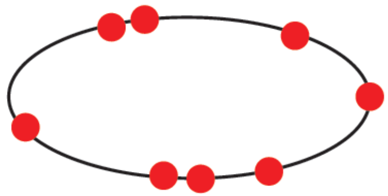

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They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell \ne j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1,..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|)$$

$$E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

exponent 1/3

can it be continued in n?

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1,..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

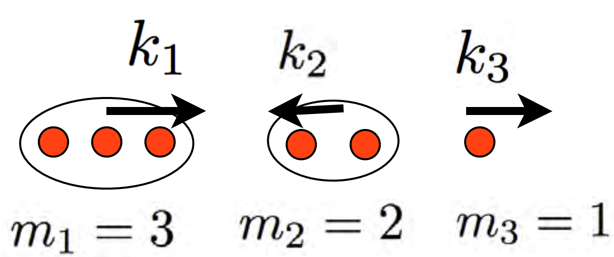
exponent 1/3

can it be continued in n?

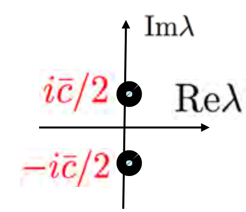
need to sum over all eigenstates !

NO!!

- all eigenstates are: All possible partitions of n into ns “strings” each with mj particles and momentum kj



$$n = \sum_{j=1}^{n_s} m_j$$



$$E_\mu = \sum_{j=1}^{n_s}(m_j k_j^2 - \frac{\bar{c}^2}{12}m_j(m_j^2-1))$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_\mu \frac{|\Psi_\mu(0..0)|^2}{||\mu||^2} e^{-E_\mu t}$$

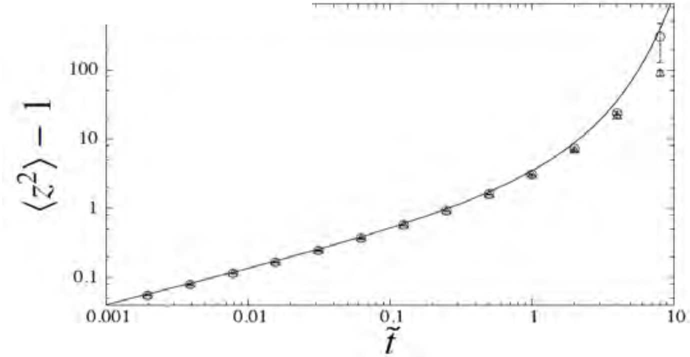
$$\Psi_\mu(0..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\dots m_{n_s})_n} \qquad n = \sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 c^2/4}{(k_i - k_j)^2 + (m_i + m_j)^2 c^2/4}$$



how to get P(ln Z) i.e. P(h) ?

$$\ln Z = -\lambda f \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3} \qquad f$$

random variable
expected O(1)

introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

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what we aim to calculate= Laplace transform of P(Z)

what we actually study

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1,\ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\le i<j\le n_s} \frac{(k_i-k_j)^2+(m_i-m_j)^2\lambda^3}{(k_i-k_j)^2+(m_i+m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1,\ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

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double Cauchy formula

$$det[\frac{1}{i(k_i - k_j)\lambda^{-3/2} + (m_i + m_j)}]$$

$$= \prod_{i<j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

$$\frac{1}{X} = \int_0^{\infty} dv e^{-vX}$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s,x) = \prod_{j=1}^{n_s} \int_{v_j>0} dv_j \, det[K(v_j, v_\ell)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1,v_2) = -\int \frac{dk}{2\pi} dy Ai(y+k^2-x+v_1+v_2) e^{-ik(v_1-v_2)} \frac{e^{\lambda y}}{1+e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s,x) = Det[I+K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1,v_2) \equiv \theta(v_1) K(v_1,v_2) \theta(v_2)$$

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2) large time limit $\lambda = +\infty$ $\quad \frac{e^{\lambda y}}{1 + e^{\lambda y}} \longrightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2 + v + v') e^{ik(v - v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

g(x)= $Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$

$$K_{Ai}(v, v') = \int_{y>0} Ai(v + y) Ai(v' + y)$$

GUE-Tracy-Widom distribution

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) g(s=-x) is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j + q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i}\delta(k_i+k_j)(-1)^{m_i}\delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2\delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)}\mathrm{sgn}(m_i-m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i+m_i-2ik_j-m_j}{2ik_i+m_i+2ik_j+m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j>0} \mathrm{Pf}[\mathbf{K}(v_i,v_j)]_{2n_s,2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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2) large time limit $\lambda = +\infty$

$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s]$$

GOE Tracy Widom

$$\mathcal{B}_s = \theta(x) Ai(x+y+s)\theta(y)$$

Fredholm Pfaffian Kernel at any time t

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1 + v_i + s + 4k^2) Ai(y_2 + v_j + s + 4k^2) [\frac{e^{-2i(v_i - v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)})$$

$$+ \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2} \int_y Ai(y + s + v_i)(e^{-2e^{\lambda y}} - 1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i - v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\, {}_1F_2\,(1; 2 - 2ik, 2 + 2ik; -z)}{\sinh\,(2\pi k)\,\Gamma\,(2 - 2ik)\,\Gamma\,(2 + 2ik)}\,, \qquad (19)$$

$$F(z_i, z_j) = \sinh(z_2 - z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$

$$\times J_0(2\sqrt{z_1 z_2 (1-u)})[z_1 \sinh(z_1 u) - z_2 \sinh(z_2 u)]\,.$$

$$g_\lambda(s) = \sqrt{Det(1 - 2K_{10})}(1 + \langle\tilde{K}|(1 - 2K_{10})^{-1}|\delta\rangle)$$

$$K_{10}(v_1, v_2) = \partial_{v_1} K_{11}(v_1, v_2) \qquad K_{12}(v_1, v_2) = \tilde{K}(v_1)\delta(v_2)$$

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large time limit

$$\lim_{\lambda\to+\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

$$\lim_{\lambda\to+\infty} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) =$$

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Summary: we found

for droplet initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3}(\frac{t}{t^*})^{1/3}\chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions

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similar (more involved)

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$$t^* = \frac{8(2\nu)^5}{D^2 \lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

large for weak noise, large diffusivity

GSE ?

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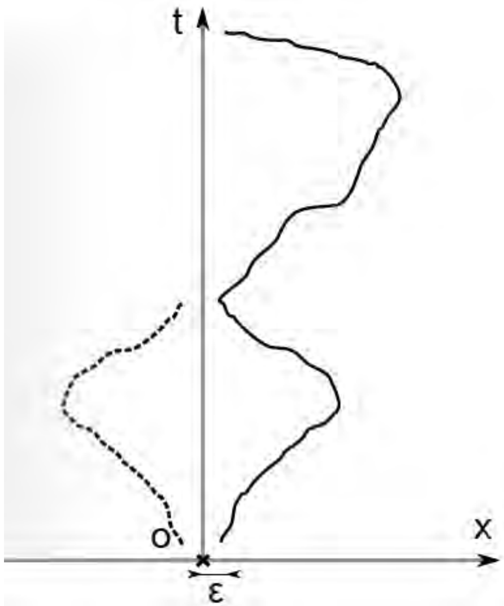
GSE ? KPZ in half-space

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal,
EPL 100 26006 (2012)



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$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

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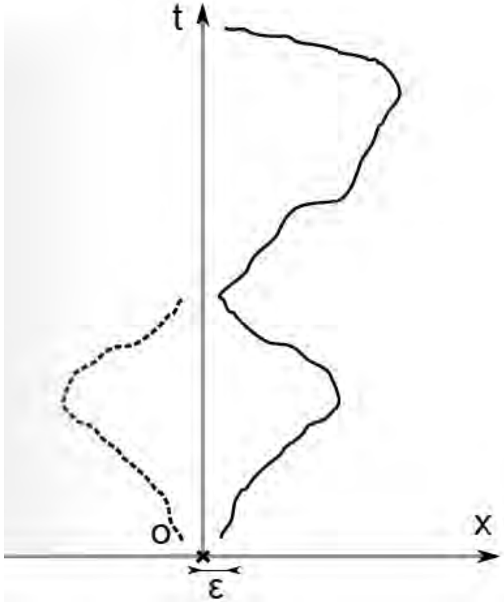

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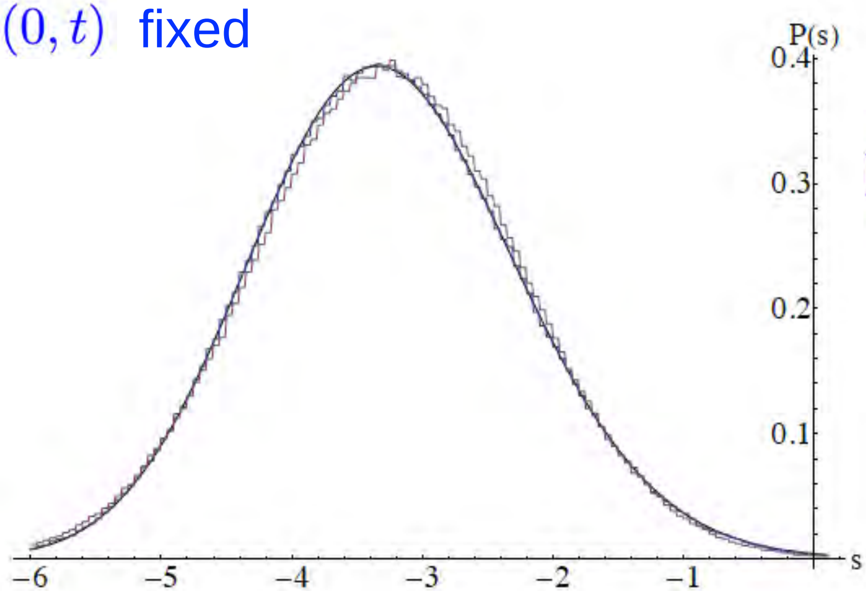
$$\lim_{\lambda \to \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

$$\lambda = (\frac{\bar{c}^2 t}{8T^5})^{1/3} = (\frac{D\lambda_0^2 t}{8(2\nu)^5})^{1/3}$$



$$\ln Z = \frac{\lambda_0}{2\nu} \tilde{h}(0, t) = v_\infty t + 2^{2/3} \lambda \chi_4$$

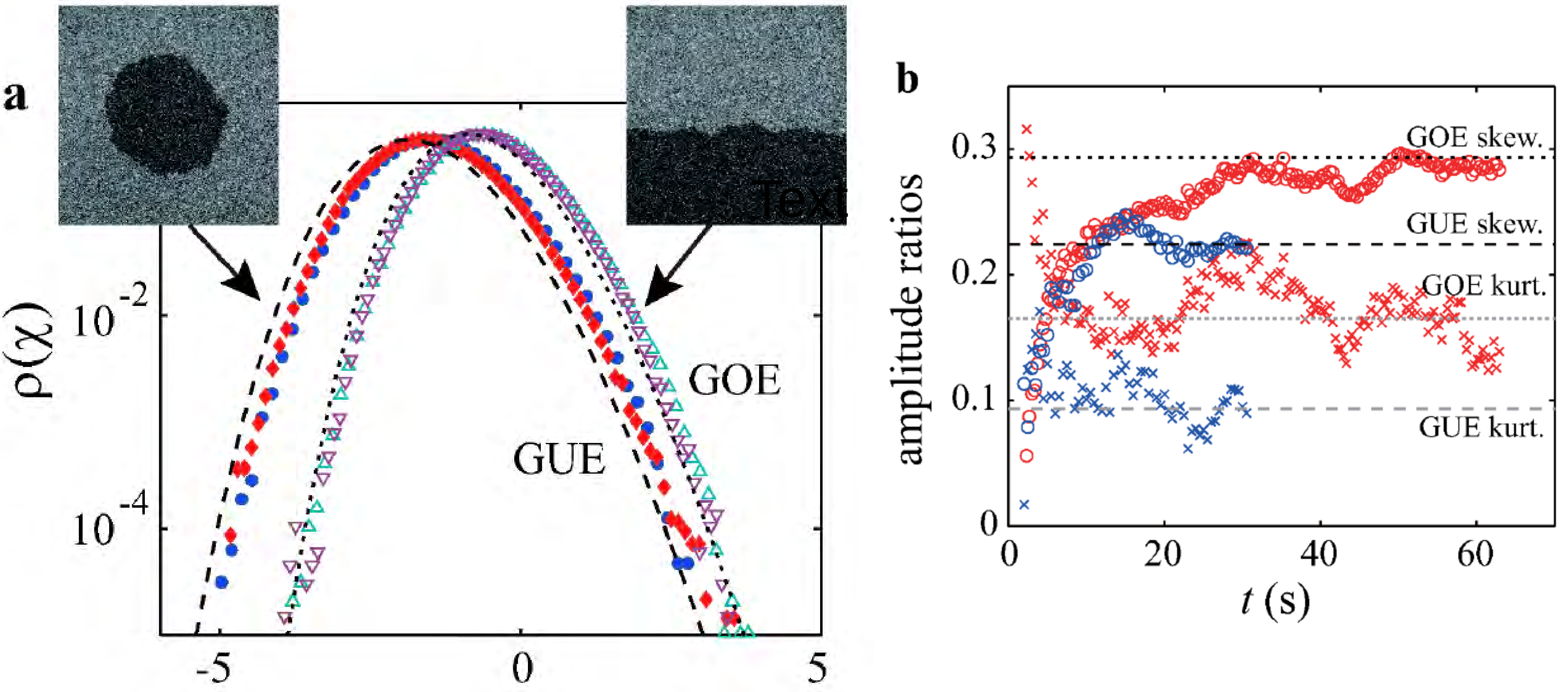
χ_4 distributed as $F_4(s)$

Gaussian Symplectic Ensemble

$$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi,$$

skewness =

$$\frac{<(h-<h>)^3>}{<(h-<h>)^2>^{3/2}}$$



Other systems believed to show KPZ class

- classical fluid of interacting particles finite T
  sound peak described by KPZ

H. Spohn, Van Beijeren 2012

$$S^{(\pm)}_{\text{phonon}}(k,\omega) \propto \frac{1}{\Gamma_k} f_{\text{PS}}\left(\frac{\omega \pm c|k|}{\Gamma_k}\right) \qquad \Gamma_k \propto |k|^{3/2}$$

in quantum case ?

Kulkarni, Lamacraft, 2013

Gross Pitaevski Bose gas

- in localized phase of 2D quantum particles

  fluctuations of log-conductance are Tracy-Widom

  interferences of (complex) directed paths => KPZ class

Kardar, Ortuno, Somoza

- connections to quantum quench

From the sine Gordon field theory to KPZ

P. Calabrese, M. Kormos, PLD
arXiv/1405.2582, EPL (2014)

1. integrable quantum field theory $\phi(x,t)$

imaginary time

$$\int D\phi \; e^{-\int dx dt \mathcal{L}_{\rm sG}[\phi]}$$

$$\mathcal{L}_{\rm sG}[\phi] = \frac{1}{2c_l^2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 - \frac{m_0^2 c_l^2}{\beta^2}(\cos(\beta\phi) - 1)$$

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2. excitation spectrum

$$H = \int dx \Pi(x)^2 + \frac{1}{2}(\partial_x \phi)^2 + ..$$

- solitons mass -> infinity decouple

- breathers B_1 "particle" sinh-Gordon

B_m m-breather

"bound state"

$$E_m(\theta) = M_m c_l^2 \cosh\theta$$

$$P_m(\theta) = M_m c_l \sinh(\theta)$$

$$M_m = M \frac{\sin m\pi\alpha/2}{\sin \pi\alpha/2}$$

$$m_{max} = [1/\alpha]$$

$$\alpha = c_l \beta^2/(8\pi - c_l \beta^2)$$

$$\mathcal{L}_{\mathrm{sG}}[\phi] = \frac{1}{2c_l^2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 - \frac{m_0^2 c_l^2}{\beta^2}(\cos(\beta\phi) - 1)$$

3. non-relativistic limit (NRL): $\begin{cases} c_l \to +\infty \\ \beta \to 0 \end{cases}$ $\beta c_l = 4\sqrt{\bar{c}}$

→ non-linear Schrodinger

$$\alpha \approx c_l \beta^2/(8\pi) \to 0$$

$$\phi(x,t) = e^{-m_0 c_l^2 t}\Psi(x,t) + e^{m_0 c_l^2 t}\Psi^+(x,t)$$

$$m_{max} = [1/\alpha] \to \infty$$

ShG → repulsive Lieb-Liniger

Mussardo, Kormos et al. (2014)

SG → attractive LL

B_m → m-string

$$E_m(\theta) = Mmc_l^2 + \frac{\bar{c}^2}{24M}(m - m^3) + m\frac{p^2}{2M}$$

$$M \approx m_0$$

$$\mathcal{L}_{\mathrm{sG}}[\phi] = \frac{1}{2c_l^2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 - \frac{m_0^2 c_l^2}{\beta^2}(\cos(\beta\phi) - 1)$$

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$M \approx m_0$

4. SG is integrable QFT

Form factors known explicitly

satisfy functional recursion relations, analyticity,..

Smirnov, Mussardo, ..

$$\langle 0|e^{ik\beta\phi(0,0)}|\theta_1, .., \theta_n\rangle = F_n^k(\theta_1, ..\theta_n)$$

$$\downarrow$$

$$|B_{m_1}(\theta_1), .., B_{m_{n_s}}(\theta_{n_s})\rangle$$

$$e^{ik\beta\phi(0,0)} = e^{i\tilde{k}\phi(0,0)}$$

$$k = \tilde{k}/\beta$$

2-point correlation function in SG

$$G(\tilde{k},t) = \langle 0|e^{i\tilde{k}\phi(0,t)}e^{-i\tilde{k}\phi(0,0)}|0\rangle$$

Lehman formula:

$$G(\tilde{k},t) \simeq \sum_{n_s=0}^{\infty} \frac{1}{n_s!} \prod_{j=1}^{n_s} \sum_{m_j=1}^{m_{max}} \int \frac{\mathrm{d}\theta_1}{2\pi} \dots \frac{\mathrm{d}\theta_{n_s}}{2\pi} |\langle 0|e^{i\tilde{k}\phi(0,0)}|B_{m_1}(\theta_1)\dots B_{m_{n_s}}(\theta_{n_s})\rangle|^2 e^{-\sum_{j=1}^{n_s} E_{m_j}(\theta_j)|t|}$$

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↓ non-relativistic limit

$$G(\tilde{k},t) \simeq |\langle e^{i\tilde{k}\phi}\rangle|^2 \sum_{n_s=0}^{\infty} \frac{\bar{c}^{n-n_s}}{n_s!} \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} \left[\frac{2}{\sqrt{\bar{c}}} \sin\left(\frac{\sqrt{\bar{c}}}{2}\tilde{k}\right)\right]^{2m_j}$$

$$\times \prod_{j=1}^{n_s} \int \frac{\mathrm{d}p_j}{2\pi m_j} e^{-m_j M c_l^2 t - \frac{\bar{c}^2}{12}(m_j - m_j^3)t - m_j p_j^2 t} \Phi[p,m]$$

$$\Phi[p,m] = \prod_{1\le j<l\le n_s} \frac{4(p_i-p_j)^2 + \bar{c}^2(m_i-m_j)^2}{4(p_i-p_j)^2 + \bar{c}^2(m_i+m_j)^2}$$

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$$g(u) = \sum_{n=0}^{+\infty} \frac{(-u)^n}{n!} \overline{Z(t)^n}|_{KPZ}$$

$$\Phi[p,m] = \prod_{1\le j<l\le n_s} \frac{4(p_i-p_j)^2 + \bar{c}^2(m_i-m_j)^2}{4(p_i-p_j)^2 + \bar{c}^2(m_i+m_j)^2}$$

$$\langle e^{i\tilde{k}(\phi(0,0)-\phi(0,t))}\rangle / \langle e^{i\tilde{k}\phi(0,0)}\rangle^2 \to_{NRL} g(u) \qquad u = -\left(\frac{2}{\sqrt{\bar{c}}}\sin\left(\frac{\sqrt{\bar{c}}}{2}\right)\right)^2 e^{-Mc_l^2 t}$$

$$\langle e^{\tilde{k}(\phi(0,0)-\phi(0,t))}\rangle / \langle e^{\tilde{k}\phi(0,0)}\rangle^2 \to_{NRL} g(u) \qquad u = \left(\frac{2}{\sqrt{\bar{c}}}\sinh\left(\frac{\sqrt{\bar{c}}}{2}\right)\right)^2 e^{-Mc_l^2 t}$$

2-point correlation in SG ⟶ point to point (droplet) KPZ moments

Perspectives/other works

- replica BA method

all times: stationary KPZ Sasamoto Inamura (2013) Airy process $A_2(y)$

only infinite time: $t \to \infty$ - 2 space points $Prob(h(x_1,t),h(x_2,t))$ Prohlac-Spohn (2011) Dotsenko (2013)

- endpoint distribution of DP Dotsenko (2013) Schehr, Quastel et al (2011)

2 times (still open) $Prob(h(0,t),h(0,t'))$ Dotsenko (2012)

- sine-Gordon FT

- rigorous replica..

Borodin, Corwin, Quastel, O Neil, ..

avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

Martin Hairer

(Warwick)

Fields medal 2014 intern. math. union:

In a spectacular achievement, Hairer overcame these difficulties by describing a new approach to the KPZ equation that allows one to give a mathematically precise meaning to the equation and its solutions. What is more, in subsequent work he used the ideas he developed for the KPZ equation to build a general theory, *the theory of regularity structures*, that can be applied to a broad class of stochastic PDEs. In particular, Hairer's theory can be used in higher dimensions.

Ultraviolet KPZ
counterterms

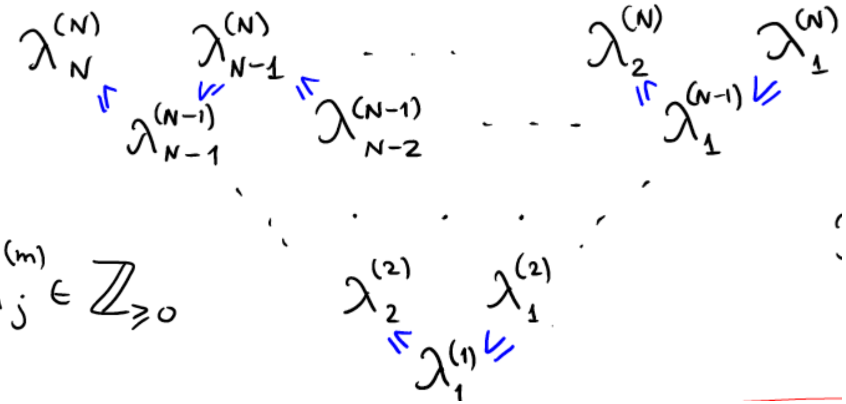
Alexei Borodin

(MIT)

Infrared KPZ
integrability

1) Macdonald processes


(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand-Tsetlin patterns)



$\lambda_j^{(m)} \in \mathbb{Z}_{\geq 0}$

Borodin, Corwin, 2013

rows distributed as
random matrix eigenvalues

2) side of triangle=particles in q-TASEP

1) Macdonald processes

Borodin, Corwin, 2013

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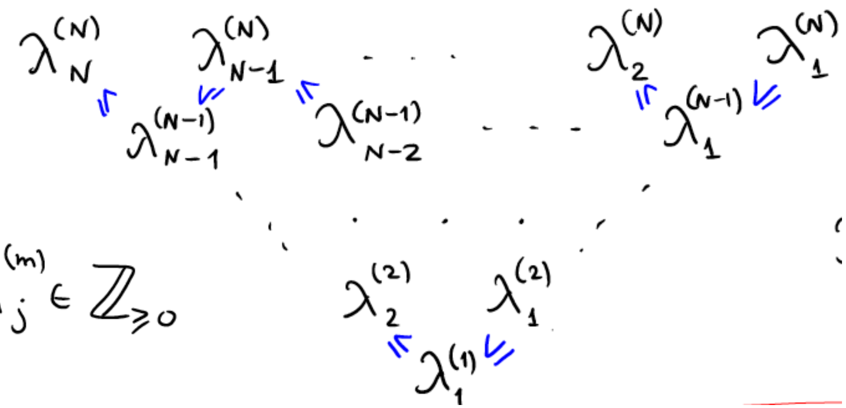


$\lambda_j^{(m)} \in \mathbb{Z}_{\geq 0}$

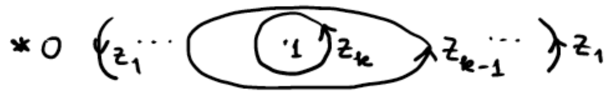
rows distributed as random matrix eigenvalues

2) side of triangle=particles in q-TASEP

moments of particle positions in q-TASEP as nested contour integrals

$$\mathbb{E}\, q^{(x_{N_1}(t)+N_1)+\dots+(x_{N_k}(t)+N_k)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \dots \geq N_k)$



$0 < q < 1$ moments define distribution uniquely

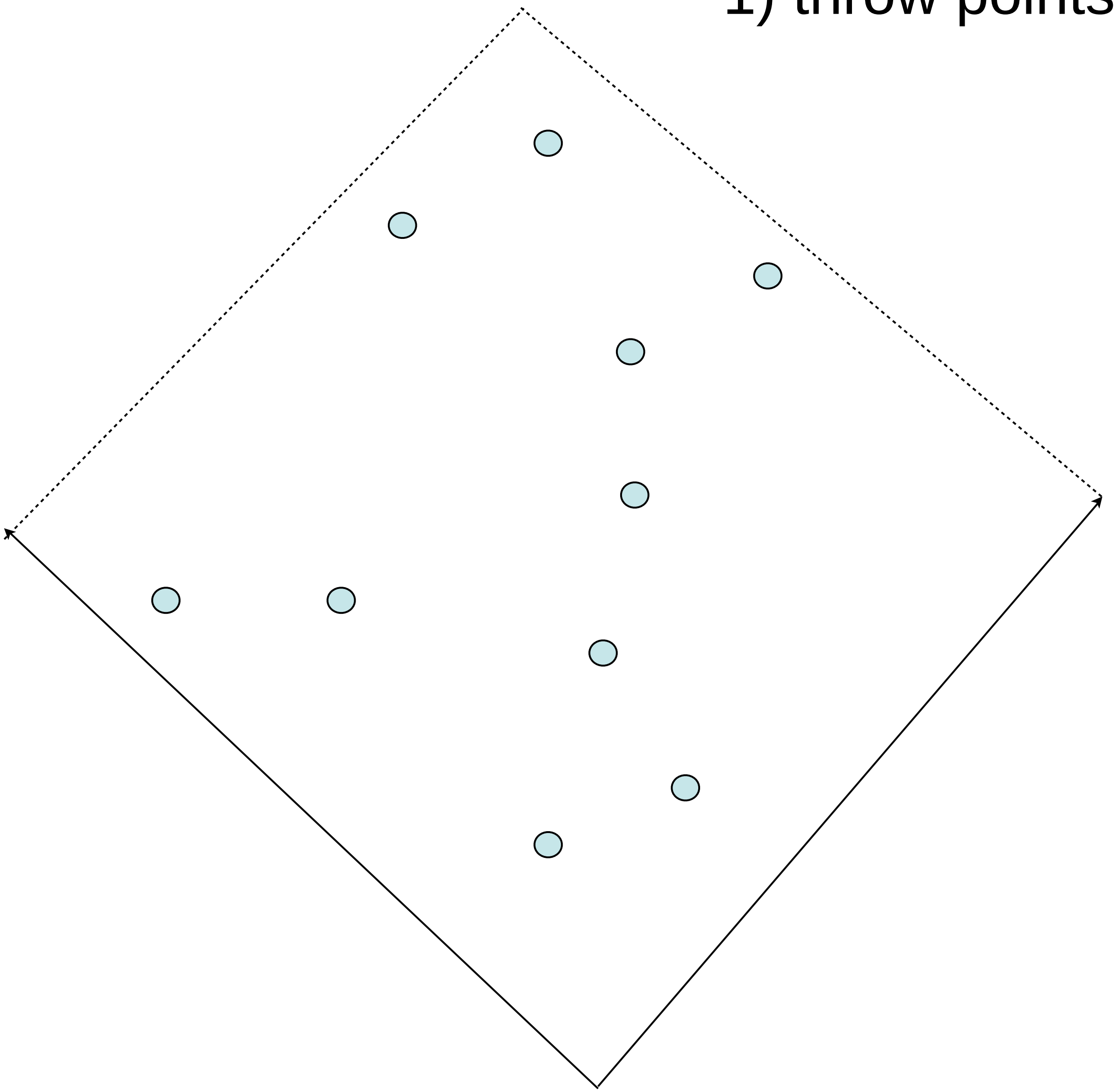
3) $q \to 1$ Bose gas, KPZ moments !

mapping LIS to optimal directed path

1) throw points randomly in square

(Poisson)

N=10

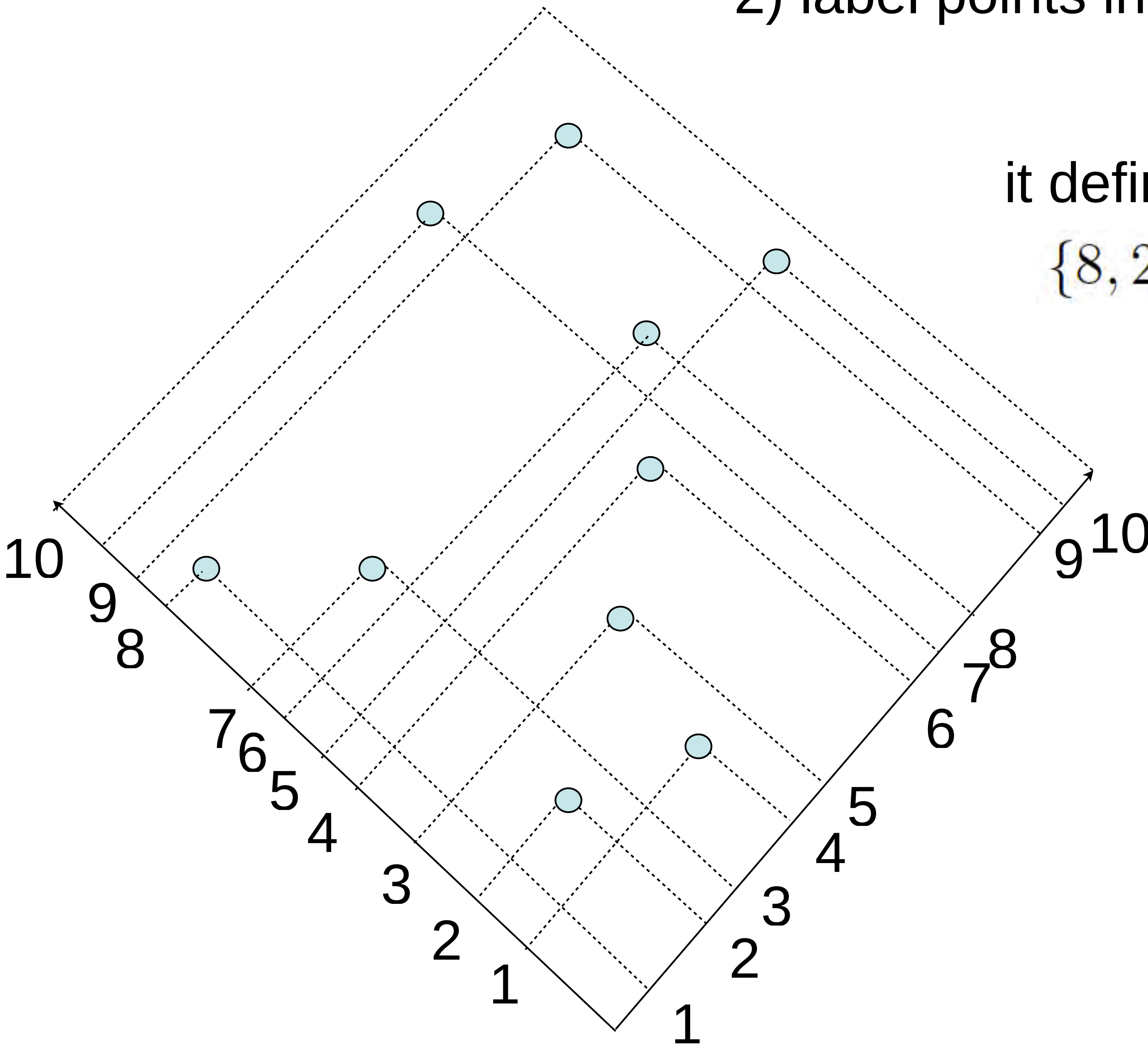


mapping LIS to optimal directed path

2) label points in increasing order along each axis

it defines a permutation

$$\{8, 2, 7, \underline{1}, \underline{3}, \underline{4}, 10, \underline{6}, \underline{9}, 5\}$$



mapping LIS to optimal directed path

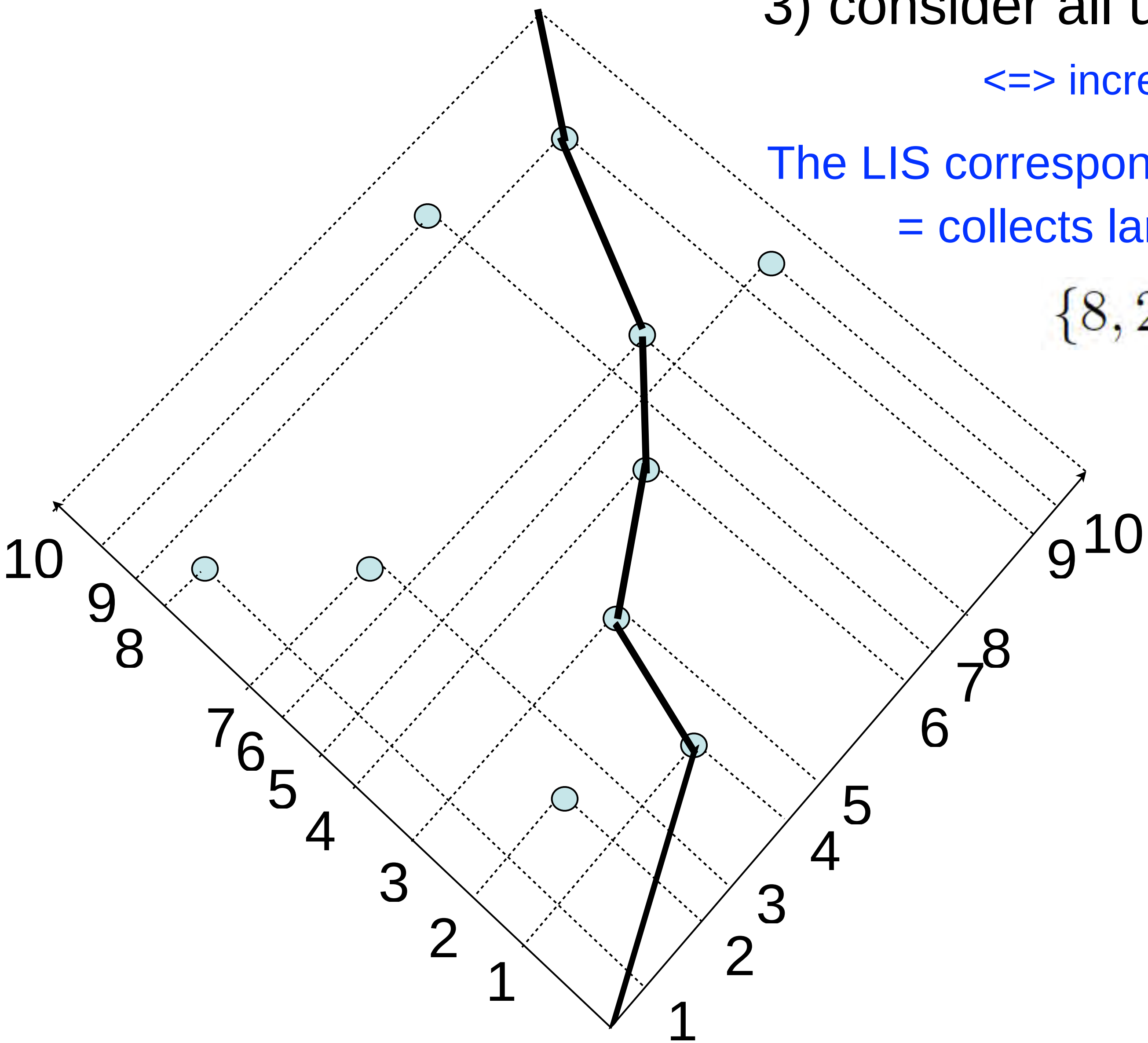
3) consider all up- going paths

<=> increasing subsequence

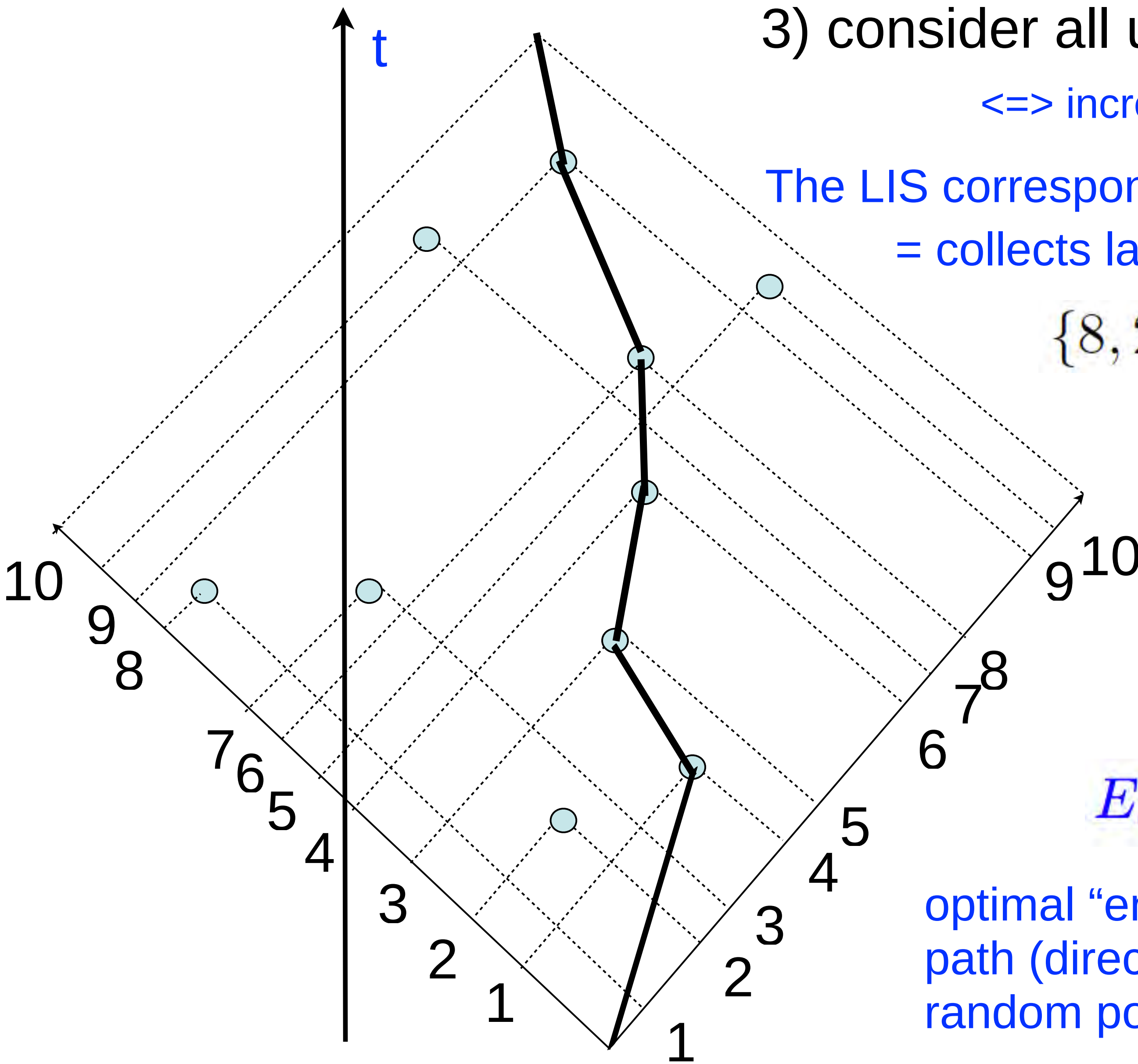
The LIS corresponds to the optimal path

= collects largest number of points

$$\{8, 2, 7, \underline{1}, \underline{3}, \underline{4}, 10, \underline{6}, \underline{9}, 5\}$$



mapping LIS to optimal directed path



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$$l_N \to 2\sqrt{N} + N^{1/6}\chi$$

$$N \sim t^2$$

$$E_{opt} = \sqrt{2}t + t^{1/3}\chi$$

optimal “energy” of directed path (directed polymer in random potential)