Large coordination number expansion for any quantum lattice systems

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Abstract:

Quantum lattice systems are encountered in many field of physics: solid states, ultracold gases, metamaterials and even quantum chromodynamics. For this class of systems, we establish a set of hierarchy equations describing the non equilibrium time evolution of the n-site spatial correlation reduced density matrix and solve it through a $1/Z$ expansion where Z is the coordination number i.e. number of interaction of a site with its neighbors [1, 2]. We apply this quite general method specifically to Bose and Fermi-Hubbard gases or Heisenberg spin magnets. In some cases, an analytical solution is found and allows to determine the quasi-excitation spectra and the quantum correlations [1, 2, 3, 4]. In the more general situation, a numerical solution yields the time-dependence of the correlations and provides an alternative to other techniques such as the density matrix renormalization group (or matrix-product state) method, especially for higher dimensions. We illustrate the powerfulness of these general concepts for a typical example of prethermalisation after a quench.

References

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Quantum lattice systems

Model	Hamiltonian: $\hat{H}=\sum_\mu \hat{H}_\mu+\frac{1}{Z}\sum_{\mu\nu}\hat{H}_{\mu\nu}$
Heisenberg	$\hat{H}=-\frac{J}{Z}\sum_{\mu\nu}T_{\mu\nu}\hat{\boldsymbol{S}}_\mu\cdot\hat{\boldsymbol{S}}_\nu$
Quantum Ising	$\hat{H}=-\frac{J}{Z}\sum_{\mu\nu}T_{\mu\nu}\hat{S}_{z,\mu}\hat{S}_{z,\nu}-\sum_\mu B\hat{S}_{x,\mu}$
Bose-Hubbard	$\hat{H}=-\frac{J}{Z}\sum_{\mu\nu}T_{\mu\nu}\hat{b}^\dagger_\mu\hat{b}_\nu+\frac{U}{2}\sum_\mu\hat{n}_\mu(\hat{n}_\mu-1)$
Fermi-Hubbard	$\hat{H}=-\frac{J}{Z}\sum_{\mu\nu,s}T_{\mu\nu}\hat{c}^\dagger_{\mu,s}\hat{c}_{\nu,s}+U\sum_\mu\hat{n}^\uparrow_\mu\hat{n}^\downarrow_\mu$
Jaynes-Cummings-Hubbard	$\hat{H}_\mu=\omega_c\hat{a}^\dagger_\mu\hat{a}_\mu+\omega_c\lvert e\rangle_\mu\langle e\rvert+g(\lvert e\rangle_\mu\langle g\rvert\hat{a}_\mu+\lvert g\rangle_\mu\langle e\rvert\hat{a}^\dagger_\mu)$ $\hat{H}_{\mu\nu}=-JT_{\mu\nu}\hat{a}^\dagger_\mu\hat{a}_\nu$

Large coordination number expansion

Density Matrix: $\hat{\rho}\equiv\hat{\rho}_{\mu_1\mu_2\ldots\mu_N}$: $\quad i\partial_t\hat{\rho}=\left[\hat{H},\hat{\rho}\right]=\frac{1}{Z}\sum_{\mu\nu}\mathcal{L}_{\mu\nu}\hat{\rho}+\sum_\mu\mathcal{L}_\mu\hat{\rho}$

Reduced density matrix: $\hat{\rho}_\mu=\mathrm{Tr}_{\not\mu}(\hat{\rho})\quad\hat{\rho}_{\mu\nu}=\mathrm{Tr}_{\not\mu\not\nu}(\hat{\rho})\quad\hat{\rho}_{\mu\nu\kappa}=\mathrm{Tr}_{\not\mu\not\nu\not\kappa}(\hat{\rho})\quad\ldots$

Scaling on correlated parts: $Z\to\infty\Rightarrow$ closed dynamical equations

$$\hat{\rho}_\mu\sim\mathcal{O}(Z^0)\quad\to\text{on}-\text{site correlations}$$

$$\hat{\rho}^c_{\mu\nu}=\hat{\rho}_{\mu\nu}-\hat{\rho}_\mu\hat{\rho}_\nu\sim\mathcal{O}(Z^{-1})\quad\to\text{off}-\text{site correlations}$$

$$\hat{\rho}^c_{\mu\nu\kappa}=\hat{\rho}_{\mu\nu\kappa}-\hat{\rho}^c_{\mu\nu}\hat{\rho}_\kappa-\hat{\rho}^c_{\mu\kappa}\hat{\rho}_\nu-\hat{\rho}^c_{\nu\kappa}\hat{\rho}_\mu-\hat{\rho}_\nu\hat{\rho}_\kappa\hat{\rho}_\mu\sim\mathcal{O}(Z^{-2})$$

$\ldots$

Leading order Z^0: on-site correlations

$$i\partial_t\hat{\rho}_\mu=\mathcal{L}_\mu\hat{\rho}_\mu+\sum_\nu\mathrm{Tr}_\nu[\frac{\mathcal{L}^S_{\mu\nu}}{Z}(\hat{\rho}_\mu\hat{\rho}_\nu+\hat{\rho}^c_{\mu\nu})]\quad\text{with}\quad\hat{\rho}^c_{\mu\nu}\sim\mathcal{O}(Z^{-1})\to 0$$

Linearization: $\hat{\rho}_\mu=\hat{\rho}^0_\mu+\delta\hat{\rho}_\mu$

$$i\partial_t\delta\hat{\rho}_\mu=\mathcal{L}_\mu\delta\hat{\rho}_\mu+\frac{1}{Z}\sum_\kappa\mathrm{Tr}_\kappa\left\{\mathcal{L}^S_{\mu\kappa}\left(\hat{\rho}^0_\kappa\delta\hat{\rho}_\mu+\delta\hat{\rho}_\kappa\hat{\rho}^0_\mu\right)\right\}$$

$\delta\hat{\rho}_\mu(t)=\sum_\nu\mathrm{Tr}_\nu\left\{\hat{W}^\nu_\mu(t,t_0)\delta\hat{\rho}_\nu(t_0)\right\}\to$ NON UNITARY EVOLUTION !

Mapping: $\hat{A}_\nu(t_0)\to\hat{A}_\nu(t)=\sum_\mu\mathrm{Tr}_\mu\left\{\hat{W}^\nu_\mu(t,t_0)\hat{A}_\mu(t_0)\right\}$

- Eigenvalues $\to$ Excitation spectrum
- Imaginary spectrum $\to$ Phase transition

Example 1: Heisenberg model

Antiferromagnetism $J<0$: $\quad\hat{\rho}^0_\mu=\begin{cases}\lvert\downarrow\rangle\langle\downarrow\rvert\ \mu\in\mathcal{A}\\ \lvert\uparrow\rangle\langle\uparrow\rvert\ \mu\in\mathcal{B}\end{cases}$

$$(i\partial_t\pm J)\hat{S}^\pm_\mu\pm\sum_\nu\frac{JT_{\mu\nu}}{Z}\hat{S}^\pm_\nu=0\qquad\mu\in\mathcal{A},\ \nu\in\mathcal{B}$$

$$(i\partial_t\mp J)\hat{S}^\pm_\nu\mp\sum_\mu\frac{JT_{\mu\nu}}{Z}\hat{S}^\pm_\mu=0\qquad i\partial_t\hat{S}_{z,\nu}=0$$

Magnon spectrum: $\omega_{\mathbf{k},\pm}=\pm J\sqrt{1-T^2_{\mathbf{k}}}\overset{\mathbf{k}\to 0}{\sim}\pm\mathbf{k}$

$1/Z$ expansion ($Z=2D$) equivalent to $1/S$ expansion ($S=1/2$)

Next order Z^{-1}: off-site correlations

$$i\partial_t\hat{\rho}^{\rm corr}_{\mu\nu}=\mathcal{L}_\mu\hat{\rho}^{\rm corr}_{\mu\nu}+\frac{1}{Z}\mathcal{L}_{\mu\nu}\hat{\rho}^0_\mu\hat{\rho}^0_\nu-\frac{\hat{\rho}^0_\mu}{Z}\mathrm{Tr}_\mu\left\{\mathcal{L}^S_{\mu\nu}\hat{\rho}^0_\mu\hat{\rho}^0_\nu\right\}+\frac{1}{Z}\sum_\kappa\mathrm{Tr}_\kappa\left\{\mathcal{L}^S_{\mu\kappa}(\hat{\rho}^{\rm corr}_{\mu\nu}\hat{\rho}^0_\kappa+\hat{\rho}^{\rm corr}_{\nu\kappa}\hat{\rho}^0_\mu)\right\}+(\mu\leftrightarrow\nu)+\mathcal{O}(1/Z^2)$$

$$\hat{\rho}^{\rm corr}_{\mu\nu}(t)=\sum_{\alpha\beta}\mathrm{Tr}_{\alpha\beta}\left\{\hat{W}^\alpha_\mu(t,t_0)\hat{W}^\beta_\nu(t,t_0)\hat{\rho}^{\rm corr}_{\alpha\beta}(t_0)\right\}+\int_{t_0}^t dt'\sum_{\alpha\beta}\mathrm{Tr}_{\alpha\beta}\left\{\hat{W}^\alpha_\mu(t,t')\hat{W}^\beta_\nu(t,t')\hat{Q}_{\alpha\beta}(t')\right\}$$

$$\hat{Q}_{\alpha\beta}=i\frac{\hat{\rho}^0_\alpha}{Z}\mathrm{Tr}_\alpha\left\{\mathcal{L}^S_{\alpha\beta}\hat{\rho}^0_\alpha\hat{\rho}^0_\beta\right\}+i\frac{\hat{\rho}^0_\beta}{Z}\mathrm{Tr}_\beta\left\{\mathcal{L}^S_{\alpha\beta}\hat{\rho}^0_\alpha\hat{\rho}^0_\beta\right\}-\frac{i}{Z}\mathcal{L}^S_{\alpha\beta}\hat{\rho}^0_\alpha\hat{\rho}^0_\beta$$

Antiferromagnetism: Virtual magnons excitations in pair

$\to$ Quantum fluctuations e.g. $\langle\hat{S}^+_\mu\hat{S}^-_\nu\rangle$

Example 2: Bose-Hubbard model

Mott phase: $\hat{\rho}^0_\mu=\lvert 1\rangle_\mu\langle 1\rvert$

$\to$ particle and hole operators: $\hat{h}_\mu=\lvert 0\rangle_\mu\langle 1\rvert$ and $\hat{p}_\mu=\lvert 1\rangle_\mu\langle 2\rvert$

$$i\partial_t\hat{h}_\mu=\frac{J}{Z}\sum_\nu T_{\mu\nu}\left[\hat{h}_\nu+\sqrt{2}\,\hat{p}_\nu\right]$$

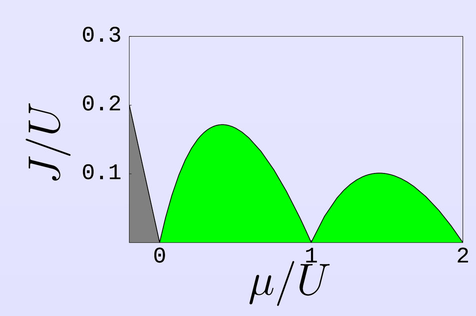
$$\left[i\partial_t-U\right]\hat{p}_\mu=-\frac{J}{Z}\sum_\nu T_{\mu\nu}\left[2\,\hat{p}_\nu+\sqrt{2}\,\hat{h}_\nu\right]$$

Spectrum:

$$\omega_{\mathbf{k},\pm}=\frac{U-JT_{\mathbf{k}}\pm\sqrt{U^2-6JUT_{\mathbf{k}}+J^2T^2_{\mathbf{k}}}}{2}$$

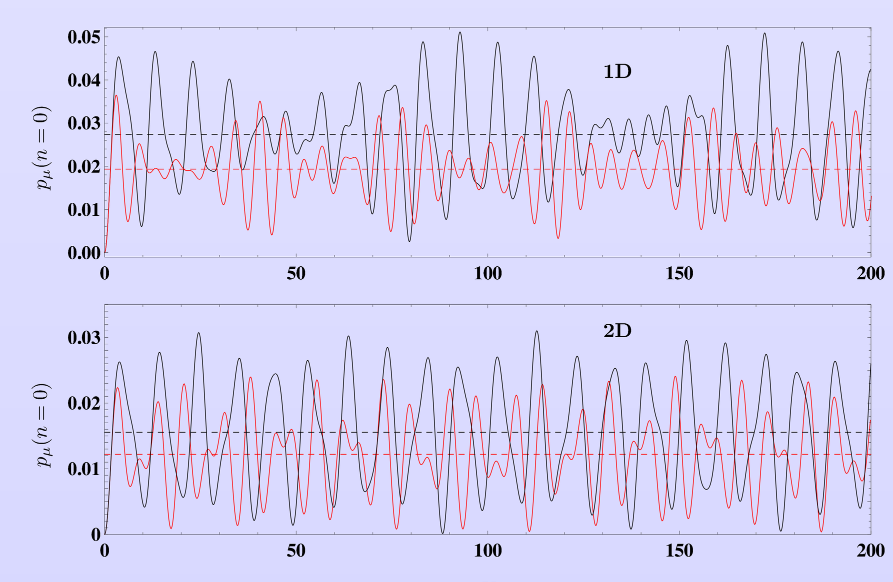
Imaginary for $J^{crit}/U=3-2\sqrt{2}$

$\to$ onset of superfluidity



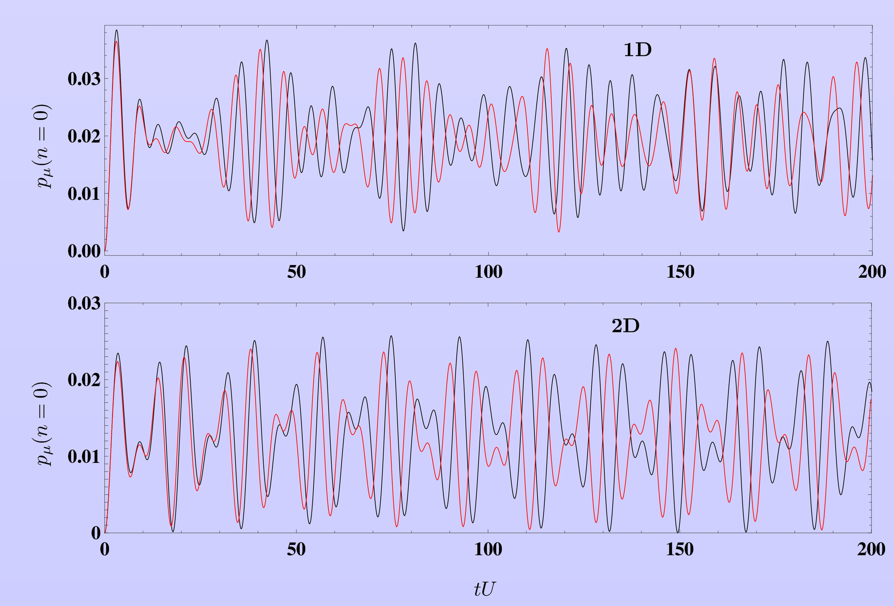
Accuracy test: hole probability to order $1/Z$

Quench: $J/U=0\to 0.1$ 1D: 11 sites 2D: 3×3 sites red: exact



Accuracy test: hole probability to order $1/Z^2$

Quench: $J/U=0\to 0.1$ 1D: 11 sites 2D: 3×3 sites red: exact



Convergence to the exact solution !