A Simple Holographic Model of a Charged Lattice

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Summary and Results

We use holography to compute the conductivity in an inhomogeneous charged scalar background. We work in the probe limit of the four-dimensional Einstein-Maxwell theory coupled to a charged scalar. The background has zero charge density and is constructed by turning on a scalar source deformation with a striped profile. We solve for the fluctuations by making use of a Fourier series expansion. At zero temperature, the conductivity is computed analytically in a small amplitude expansion. At finite temperature, it is computed numerically by truncating the Fourier series to a relevant set of modes. In the real part of the conductivity along the direction of the stripe, we find a Drude-like peak and a delta function with a negative weight. These features are understood from the point of view of spectral weight transfer.

The Model

In the background of 4D AdS-Schwarzschild black hole,

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^2 + dy^2\right), \quad f(r) = 1 - \frac{r^3}{r_h^3},$$

we consider a Maxwell field A_μ and a charged complex scalar Φ of $m^2L^2 = -2$

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\partial_\mu\Phi - iA_\mu\Phi|^2 - m^2\Phi^2\right), \qquad F = dA\,.$$

The background is $A_\mu = 0$ and $\Phi = \phi(r,x)e^{i\theta}$ with $\theta = 0$. According to the AdS/CFT correspondence, field theory data are read off the UV asymptotics,

$$\text{as} \;\; r \to 0\,, \qquad \phi(r,x) = \phi_1(x)r + \phi_2(x)r^2 + \cdots\,.$$

$\Phi(r,x)$ is dual to a charged scalar operator $\mathcal{O}(x)$. In the standard quantization dim $\mathcal{O} = 2$, ϕ_1 is interpreted as the source and ϕ_2 as $\langle\mathcal{O}\rangle$. We then introduce a periodic source deformation of the field theory by turning on $\phi_1(x) = V\cos(Qx)$. Since the Action is quadratic, $\phi(r,x)$ can have the form $\phi(r,x) = \varphi(r)\cos(Qx)$. The eq. of motion becomes

$$\left(\partial_r^2 + \left(\frac{f'}{f} - \frac{2}{r}\right)\partial_r - \frac{m^2 + r^2Q^2}{r^2 f}\right)\varphi(r) = 0,$$

The parameter Q is the momentum associated to the striped deformation.

We refer to the $\phi(r,x)$ profile as a "charged lattice". Even though there is no background charge density, $A_\mu = 0$, and the average value of $\langle\mathcal{O}\rangle$ vanishes, the scalar field is minimally coupled to the gauge field. Applying an electric field will turn on the interactions between gauge and lattice fluctuations. Alternatively, $\phi(r,x)$ may describe the effects caused by charged impurities.

Optical Conductivity

A boundary electric field $\vec{E}$ is obtained as $E_j \equiv \lim_{r\to 0} f_{jt}$, where f is the bulk field strength of gauge field fluctuation δA and $j = x, y$. Generic bulk perturbations of our charged lattice solution have the form,

$$A_\mu \to \delta A_\mu, \qquad \Phi \to \phi/\sqrt{2} + (\delta\eta + i\delta\psi)/\sqrt{2}\,.$$

The consistent sets of perturbations in x- and y-directions are given as follows:

- When an electric field is applied in the direction transverse to the stripe,
$$\delta A_y = a_y(r,x)e^{-i\omega t}\,.$$
- When an electric field is applied in the direction longitudinal to the stripe,
$$\delta A_x = a_x(r,x)e^{-i\omega t}, \quad \delta A_t = a_t(r,x)e^{-i\omega t}, \quad \delta\psi = \psi(r,x)e^{-i\omega t}\,.$$

Comments: $\delta\psi$ produces a vibration of the lattice but plays a different role with respect to the bulk phonon in massive gravity. $\delta\eta$ decouples because $A_\mu = 0$.

Working in the gauge $\delta A_r = 0$, the current is

$$J^a = -\lim_{r\to 0}\sqrt{-g}g^{rr}g^{ab}\partial_r\delta A_b\,, \qquad a,b = t,x,y$$

and the conductivity is obtained from the definition $J^i(\vec{x}) = \sigma^{ij}(\vec{x})E_j$, where $i,j = x,y$.

We will focus on the average value of the condutivity.

The transverse channel: The Fourier series expansion of a_y is

$$a_y(r,x) = a_y^{(0)}(r) + a_y^{(2)}(r)\cos(2Qx) + a_y^{(4)}(r)\cos(4Qx) + \cdots\,.$$

We obtain the following infinite set of coupled ODEs,

$$\left(\partial_r^2 + \frac{f'}{f}\partial_r + \left(\frac{\omega^2}{f^2} - \frac{\varphi^2}{2r^2 f}\right)\right)a_y^{(0)} - \frac{\varphi^2}{4r^2 f}a_y^{(2)} = 0,$$

$$\left(\partial_r^2 + \frac{f'}{f}\partial_r + \left(\frac{\omega^2}{f^2} - \frac{4n^2Q^2}{f} - \frac{\varphi^2}{2r^2 f}\right)\right)a_y^{(2n)} - \frac{\varphi^2}{4r^2 f}\left(c_{2n-2}a_y^{(2n-2)} + a_y^{(2n+2)}\right) = 0,$$

The novelty is the spatially dependent mass term proportional to $\phi(r,x)^2$ which couples the Fourier modes with a specific patter: $a_y^{(2n)}$ directly couples only to $a_y^{(2n\pm 2)}$ whereas $a_y^{(0)}$ only couples to $a_y^{(2)}$ and not to $a_y^{(2l)}$ with $l > 2$.

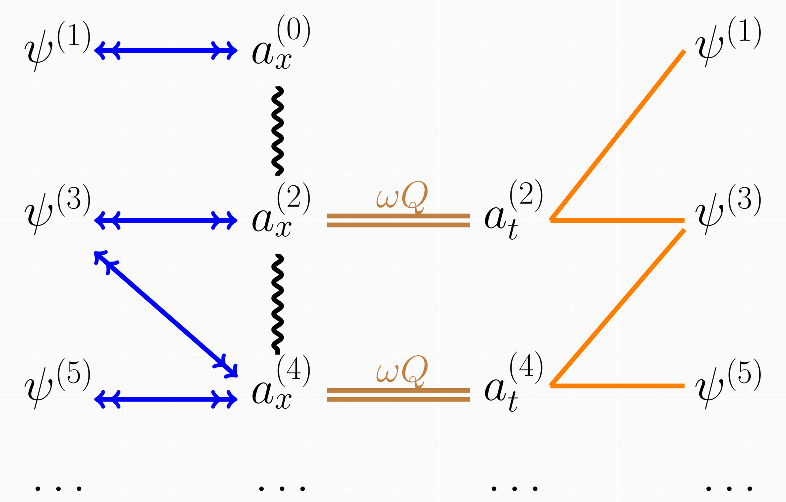
Boundary conditions imply $a_{y,0}^{(0)} = E/i\omega$ and $a_{y,0}^{(2n)} = 0$ for $n \geq 1$. The induced current $J^y(x) = \sum_{n=0}^{\infty} a_{y,1}^{(2n)}\cos(2nQx)$ is a function of x, and so is the conductivity. We focus on

$$\sigma_T(\omega) = -\frac{i}{\omega}\frac{a_{y,1}^{(0)}}{a_{y,0}^{(0)}}\,.$$

The Longitudinal Channel

$$a_x(r,x) = a_x^{(0)}(r) + a_x^{(2)}(r)\cos(2Qx) + a_x^{(4)}(r)\cos(4Qx) + \cdots\,,$$
$$a_t(r,x) = a_t^{(2)}(r)\sin(2Qx) + a_t^{(4)}(r)\sin(4Qx) + \cdots\,,$$
$$\psi(r,x) = \psi^{(1)}(r)\sin(Qx) + \psi^{(3)}(r)\sin(3Qx) + \cdots\,.$$

The pattern of interactions among the modes,



Let's see when the truncation to the $\{a_x^{(0)}, \psi^{(1)}\}$ block works:

$$\left(\partial_r^2 + \frac{f'}{f}\partial_r - \left(\frac{\varphi^2}{2r^2 f} - \frac{\omega^2}{f^2}\right)\right)a_x^{(0)} = -\frac{Q\varphi}{f r^2}\psi^{(1)} + \frac{1}{4f}\frac{\varphi^2}{r^2}a_x^{(2)}\,,$$

$$\left(\partial_r^2 + \left(\frac{f'}{f} - \frac{2}{r}\right)\partial_r - \left(\frac{m}{r^2 f} + \frac{Q^2}{f} - \frac{\omega^2}{f^2}\right)\right)\psi^{(1)} = -2\frac{Q\varphi}{f}a_x^{(0)} + \frac{i\omega\varphi}{2f^2}a_t^{(2)}\,.$$

The interaction between $a_x^{(0)}$ and $\psi^{(1)}$ dominates over that with $a_x^{(2)}$ if the condition $Q/V \gg 1$ is satisfied. The field $a_t^{(2)}$ is massive and not directly sourced by $a_x^{(0)}$. Assuming that $a^{(2)} \propto f$ at the horizon, the decoupling of $a_t^{(2)}$ occurs at small frequencies.

NOTE: A homogeneous boundary electric field is obtained by

$$a_{x,0}^{(0)} = \frac{E}{i\omega}\,, \quad \left(i\omega a_{x,0}^{(2n)} + 2n\,Q a_{t,0}^{(2n)}\right) = 0\,, \quad \forall n \geq 1 \quad \to \quad \sigma_L(\omega) = -\frac{i}{\omega}\frac{a_{x,1}^{(0)}}{a_{x,0}^{(0)}}\,.$$

Analytical and Numerical Calculations

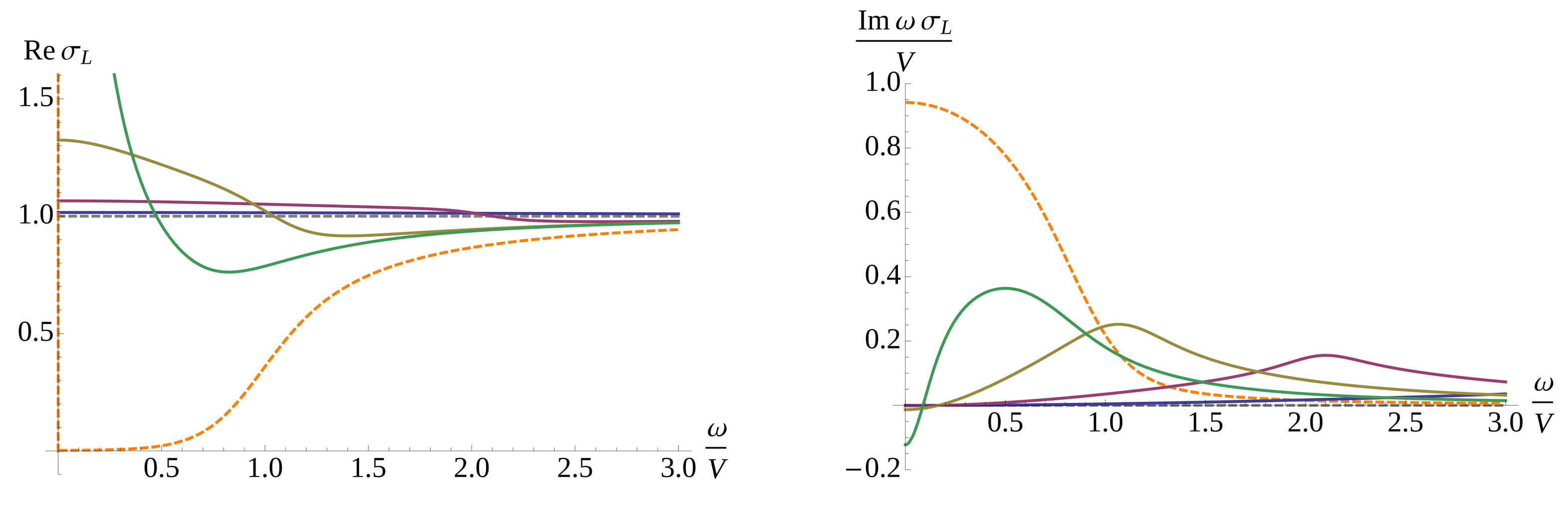
At zero temperature the charged lattice has a simple analytic form, $\varphi(r) = Vre^{-Qr}$. This makes it possible to carry out a perturbative expansion in small V/Q. Furthermore, at each order the perturbative calculation is analytic and automatically implements a truncation to a finite set of Fourier modes, i.e. heavy Fourier modes are suppressed by powers of V/Q.

- In the Transverse Channel: $\omega < Q$
$$\sigma_T(\omega) = 1 + \frac{iQ^2}{4\omega(Q - i\omega)}\frac{V^2}{Q^2} + \mathcal{O}\left(\frac{V^4}{Q^4}\right).$$
- In the Longitudinal Channel: $\omega < Q$
$$\sigma_L(\omega) = 1 + \frac{i\omega + 2(Q + \sqrt{Q^2 - \omega^2})}{4(Q - i\omega)(2Q^2 - \omega^2 + 2Q\sqrt{Q^2 - \omega^2})}V^2 + \mathcal{O}\left(\frac{V^4}{Q^4}\right)$$
$$= 1 + \left(\frac{1}{4} + \frac{5i\,\omega}{16Q} + \mathcal{O}(\omega^2)\right)\frac{V^2}{Q^2} + \left(\frac{19}{512} - \frac{i\;Q}{128\,\omega} + \mathcal{O}(\omega)\right)\frac{V^4}{Q^4} + \cdots$$

There are new features in $\sigma_L(\omega)$: Compared to the $V = 0$ case, Re σ_L is enhanced at $\omega \ll Q$ → Drude-like peak. There is delta function with negative spectral weight, yet the sum rule is satisfied → spectral weight is missing in Re σ_L: The would-be homogeneous superfluid density is reduced by lattice effects. Impurities and phase modulation induce decoherence

At finite temperature we show our numerical results. The gray/orange dashed lines are obtained for $Q/V = \infty, 0$, respectively. For the transverse conductivity, we keep $a_y^{(0)}$ and $a_y^{(2)}$. For the longitudinal conductivity, we keep only $a_x^{(0)}$ and $\psi^{(1)}$.

Optical conductivity in the longitudinal direction when $T/V = 0.25$ and for various values of Q/V. (Left panel, from bottom to top $Q/V = 4,\,2,\,1,\,0.5$). For the value $Q/V = 0.5$, $\mathrm{Re}\,\sigma_L(0^+) = 4.05$.



The behavior of $\mathrm{Re}\,\sigma_L(0^+) = 1 + (V/Q)^2/4$ at small frequencies, the same behavior found at T=0

Optical conductivity in the transverse direction when $T/V = 0.25$ and for various values of Q/V. (Left panel, from bottom to top $Q/V = 4,\,2,\,1,\,0.5$).