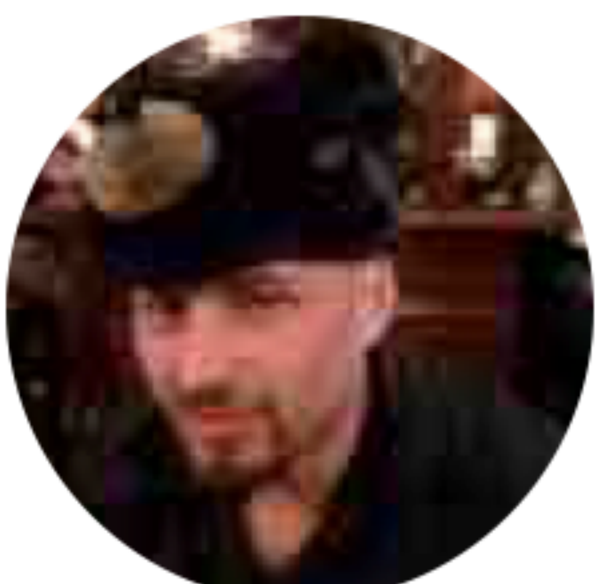
Fractional Laplacians and anomalous dimensions in the AdS/CFT correspondence

thanks to

Gabriele La Nave



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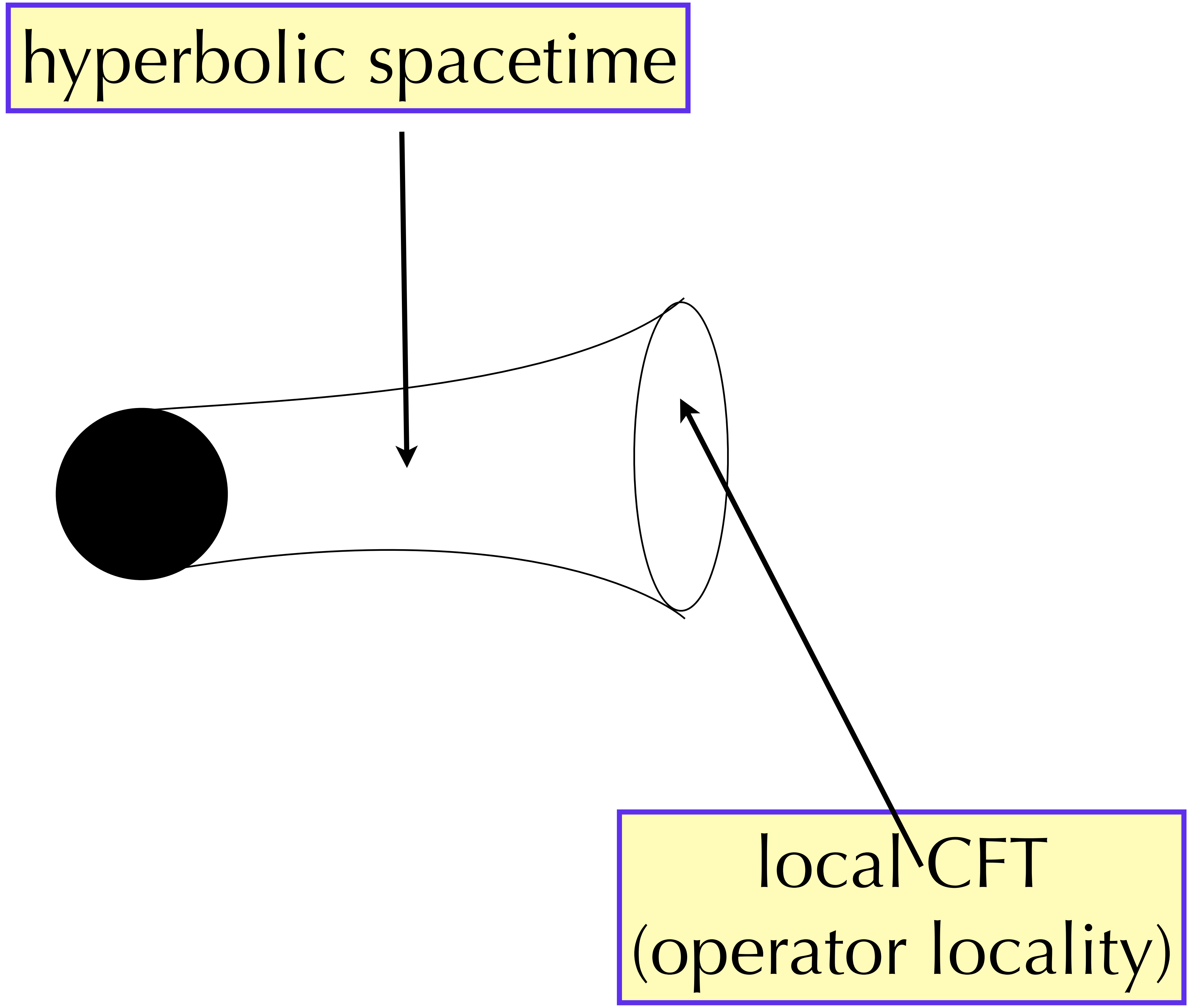
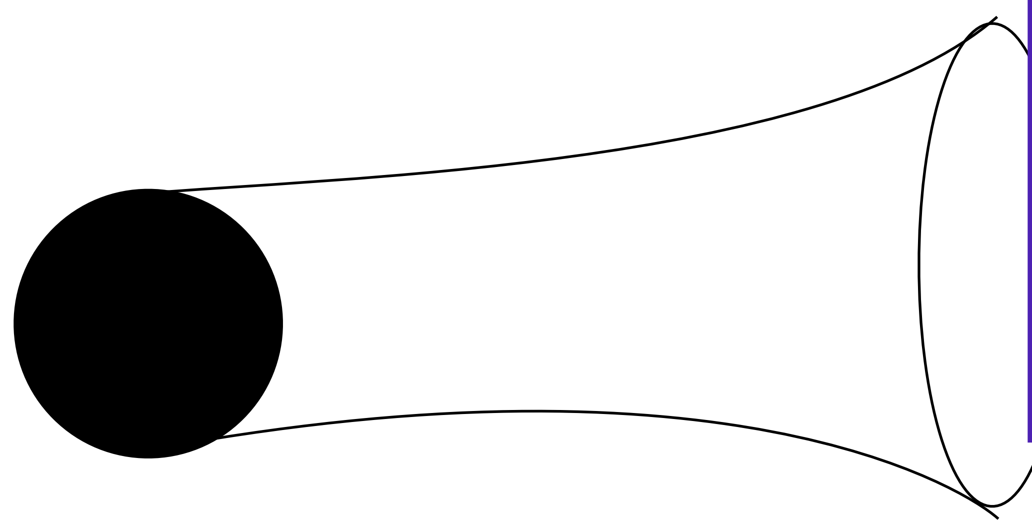


1-1 state correspondence

any theory with gravity has less observables than a theory without it!



how can a local CFT emerge at the boundary?

can exact statements
be made?

Caffarelli-Silvestre extension theorem (2007)

$$\left.\begin{aligned} u(x, y=0) &= f(x) \\ \Delta_x u + \frac{a}{y} u_y + u_{yy} &= 0 \end{aligned}\right\}$$

elliptic differential equation

$$\nabla \cdot (y^a \nabla u) = 0$$

bounded solutions

$$\lim_{y \to 0} (y^a u_y) = C_{d,\gamma} (-\Delta)^\gamma f(x) \qquad \gamma = \frac{1-a}{2}$$

fractional Laplacian

boundary is non-local!

$$(-\Delta_x)^a f(x) = C_{d,a} \int_{R^d} d\xi \frac{(f(x) - f(\xi))}{|x-\xi|^{d+2a}}$$

f must be known everywhere

Caffarelli-Silvestre extension theorem (2007)

d+1 dimensional elliptic differential equations

↓ d-dimensions

fractional Laplacian (non-local eom)

apply to holography

Local bulk operators in AdS/CFT: a boundary view of horizons and locality

Alex Hamilton[1], Daniel Kabat[1], Gilad Lifschytz[2] and David A. Lowe[3]

$$\phi_0(x) \leftrightarrow \mathcal{O}(x) \ .$$

This implies a correspondence between local fields in the bulk and *non-local* operators in the CFT.

$$\phi(z,x) \leftrightarrow \int dx' \, K(x'|z,x)\mathcal{O}(x') \ .$$

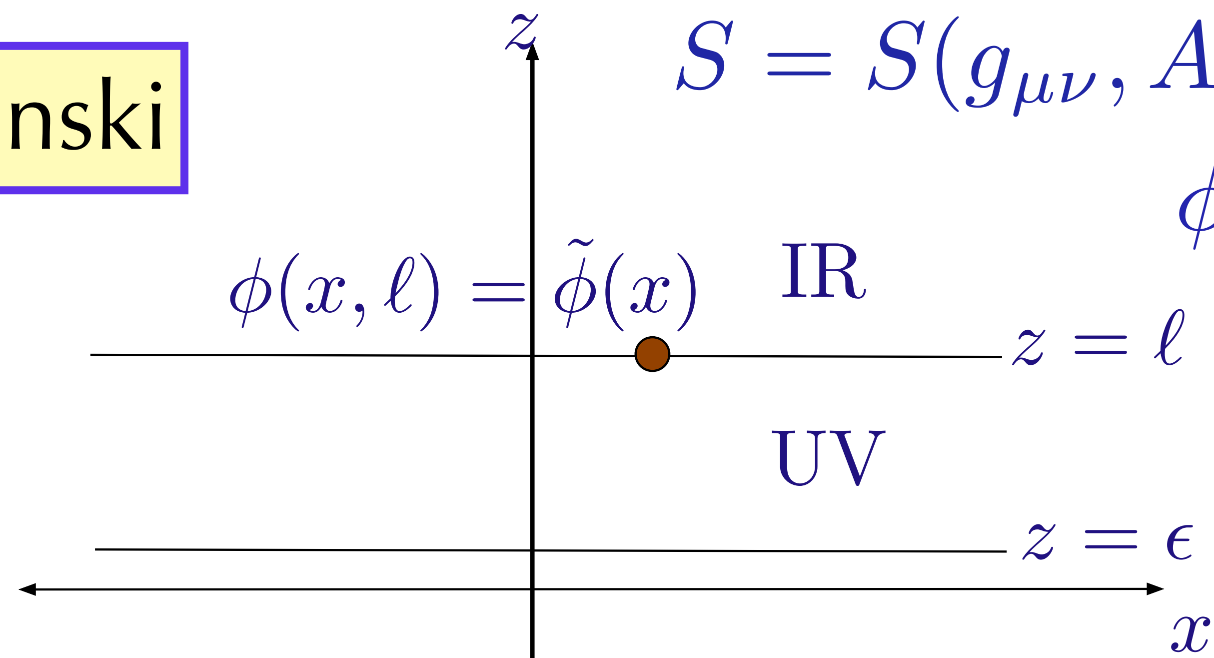
smearing function

holographic renormalization

Heemskerk/Polchinski

$$S = S(g_{\mu\nu}, A_\mu, \phi, \cdots)$$

$$\phi \approx z^{d-\Delta}\beta$$



$$Z_{\text{bulk}}[\beta] = \int \mathcal{D}\tilde{\phi} \underbrace{\int \mathcal{D}\phi|_{z>\ell} e^{-S_{z>\ell}}} \underbrace{\int \mathcal{D}\phi|_{z<\ell} e^{-S_{z<\ell}}}$$

$$Z_{\text{bulk}}[\beta] = \int \mathcal{D}\tilde{\phi}\,\Psi_{\text{IR}}[\tilde{\phi};\ell]\,\Psi_{\text{UV}}[\beta,\tilde{\phi};\epsilon,\ell]$$

$$\lim_{\ell\to 0}\int \mathcal{D}\tilde{\phi}\,\Psi_{\rm IR}\left[\frac{\delta}{\delta\beta(x_1)}\cdots\frac{\delta}{\delta\beta(x_n)}\Psi_{\rm UV}[\beta]\right]_{\beta=0}$$

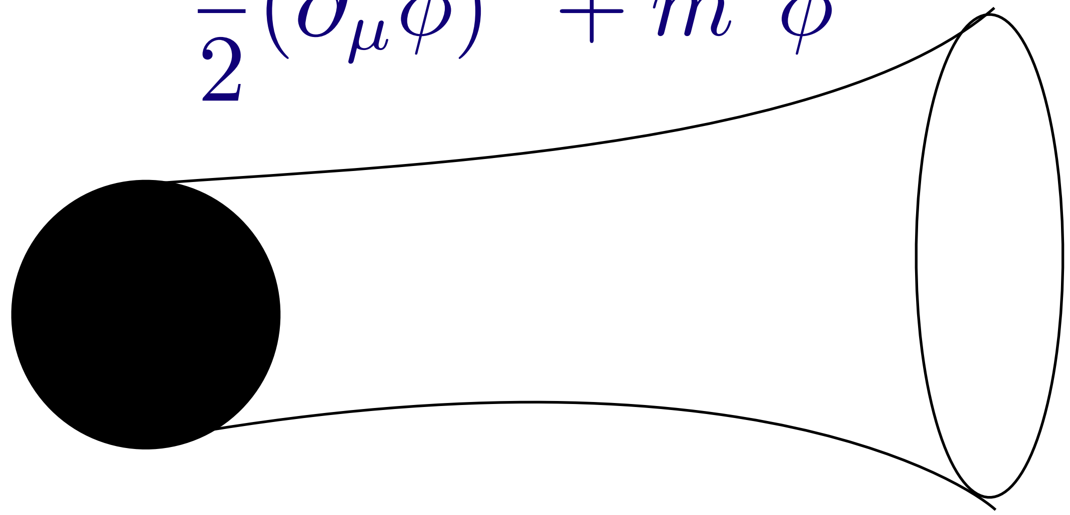
$$\lim_{\ell\to 0}\ell^{-n\Delta}\int \mathcal{D}\tilde{\phi}\,\Psi_{\rm IR}\tilde{\phi}(x_1)\cdots\tilde{\phi}(x_n)\Psi_{\rm UV}[\beta]|_{\beta=0}$$

operator identity (P):

$$\mathcal{O} = C_{\mathcal{O}}\lim_{z\to 0} z^{-\Delta}\phi(x,z)$$

use CS theorem

$$\frac{1}{2}(\partial_\mu \phi)^2 + m^2 \phi^2$$



$$\mathcal{O} = C_{\mathcal{O}} \lim_{z \to 0} z^{-\Delta} \phi(x, z)$$

ϕ — solves massive scalar problem

$g = z^{\gamma - d/2} \phi$ — solves CS extension problem

exact form of boundary operator

$$\phi = F z^{\frac{d}{2}-\gamma} + G z^{\frac{d}{2}+\gamma}, \quad F, G \in \mathcal{C}^\infty(\mathbb{H}), \quad F = \phi_0 + O(z^2), \quad G = g_0 + O(z^2)$$

$$\mathcal{O} = \lim_{z\to 0} G = g_0$$

$$g = z^{\gamma - \frac{d}{2}} \phi = F + z^{2\gamma} G$$

$$\lim_{z\to 0} g(x,z) = \lim_{z\to 0} (F + z^{2\gamma} G) = \phi_0$$

$$\lim_{z\to 0^+} z^a \frac{\partial g}{\partial z} = \lim_{z\to 0^+} z^{1-2\gamma} \frac{\partial}{\partial z}(F + z^{2\gamma} G)$$

$$= 2\gamma g_0$$

CS theorem

$$\mathcal{O} = \lim_{z\to 0} z^a \frac{\partial g}{\partial z} = \frac{(-\Delta)^\gamma \phi_0}{2\gamma}$$

independent of interactions

$$\int_{\phi(\epsilon)=\epsilon^{\Delta_-}\beta}^{\phi(\ell)=\tilde{\phi}} \mathcal{D}\phi \,|_{\epsilon<y<\ell}\; e^{-S_{bulk}[\phi]|_{\epsilon<y<\ell}}$$

$$S_{bulk}[\phi]\,|_{\epsilon<y<\ell}=\int d^dx \int_{\epsilon}^{\ell} \frac{dy}{y^{d+1}}\left(\frac{y^2}{2}\left((\partial_y\phi)^2+|\nabla_x\phi|^2\right)+V(\phi)\right)$$

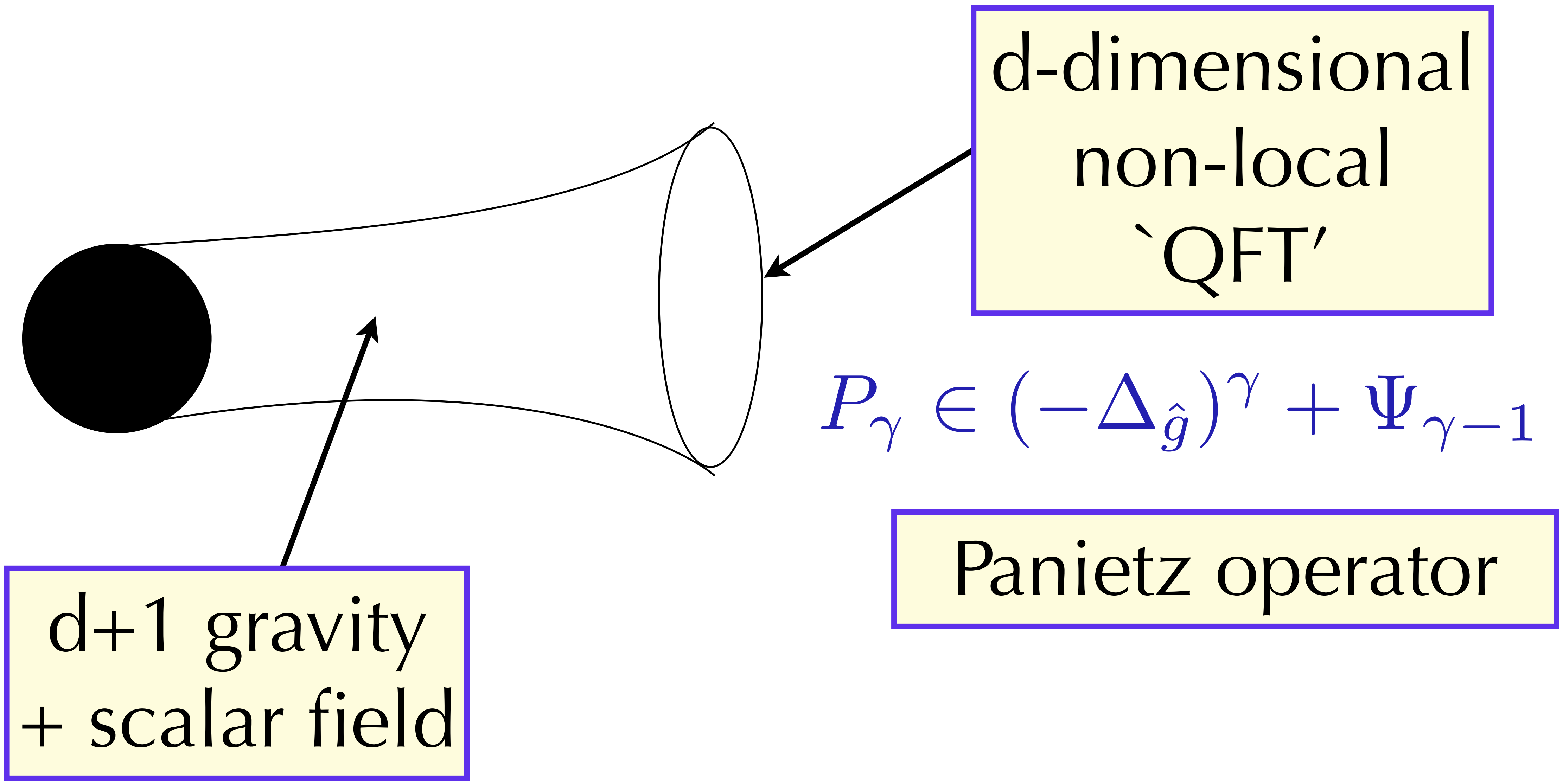
$z = y/\ell$

$$S_{bulk}[\phi]\,|_{\epsilon/\ell<z<1}=\frac{1}{\ell^d}\int d^dx \int_{\epsilon/\ell}^{1} \frac{dz}{z^{d+1}}\left(\frac{z^2}{2}(\partial_z\phi)^2+\frac{z^2\ell^2}{2}|\nabla_x\phi|^2+V(\phi)\right)$$

evaluate at saddle point:

$$\phi = Fy^{\frac{d}{2}-\gamma}+Gy^{\frac{d}{2}+\gamma}, \quad F=\phi_0+O(y^2), \quad G=(-\Delta\phi_0)^{\gamma}+O(y^2)$$

same asymptotics!



$$P_\gamma \in (-\Delta_{\hat{g}})^\gamma + \Psi_{\gamma-1}$$

application: anomalous dimensions of gauge fields

$$S = \int d^d x dy \sqrt{-g} Z(\phi) F^2 + \cdots$$

$$\downarrow \quad Z(\phi) \sim y^a$$

$$[A_\mu] \neq 1 = a$$

Karch
Gouteraux
Kiritsis

vector field at boundary
has an anomalous dimension
(without condensate)

if holography is RG then how can it lead to an anomalous dimension?

$[A_\mu] \neq 1$

Ward identities for local theories

?

$$S = \int d^d x dy \sqrt{-g} Z(\phi) F^2 + \cdots$$

eom

$$\nabla^\mu (Z(\phi) F_{\mu\nu}) = 0$$

$$Z(\phi) \sim y^a$$

$$d(y^a \star dA) = 0$$

$$y \neq 0 \longrightarrow A \to A + \partial\Lambda$$

closer look

$$\nabla \cdot (y^a \nabla u) = 0$$

scalar field
(use CS theorem)

$$d(y^a \star dA) = 0$$

holography

similar equation
1-form generalization of CS
theorem

generalize CS to p-forms

$$\lim_{y \to 0} \; y^a \frac{\partial \alpha_{i_1 \cdots i_p}}{\partial y} = C_{d,\gamma}(-\Delta)^\gamma \omega_{i_1 \cdots i_p}$$

fractional Maxwell equations
at boundary!

boundary action

$$S = \frac{1}{2}\int A_i(-\nabla)^{2\gamma}A^i,$$

boundary action has anomalous dimension (non-locality) define

$$F_{ij} = \partial_i^\gamma A_j - \partial_j^\gamma A_i \equiv d_\gamma A = d\Delta^{\frac{\gamma-1}{2}} A,$$

$$S = \int -\frac{1}{4} F_{ij} F^{ij}$$

integrate by parts

$$S = \int \frac{1}{2} A_i (-\Delta)^{2\gamma} A^i,$$

non-local boundary action

$$A \to A + d_\gamma \Lambda,$$

$$d_\gamma \equiv (\Delta)^{\frac{\gamma-1}{2}} d$$

$$[A] = \gamma$$

non-locality
circumvents the Ward
identities

$$[A] \neq 1 \to [J] \neq d$$

is this just a game?

is there a consistent algebra
for currents with fractional dimensions?

Yes

Fractional Virasoro algebra

generators

$$L_n^a = -z^{a(n+1)} \left(\frac{\partial}{\partial z}\right)^a \qquad \bar{L}_n^a := -\bar{z}^{a(n+1)} \left(\frac{\partial}{\partial \bar{z}}\right)^a$$

$$[L_n, L_m](z^{ak}) = \left(\frac{\Gamma(a(k+n)+1)}{\Gamma(a(k-1+n)+1)} - \frac{\Gamma(a(k+m)+1)}{\Gamma(a(k-1+m)+1)}\right) L_{n+m}(z^{ak})$$

$$= (A^a_{n,m}(k) \otimes L_{n+m})(z^{ak})$$

$$[L^a_m, L^a_n] = A_{m,n} L^a_{m+n} + \delta_{m,n} h(n) c Z^a$$

algebra for conformal non-local actions

$$Z^2_\star(\mathcal{W}_a, \mathcal{H}) / B^2_\star(\mathcal{W}_a, \mathcal{H})$$

physical consequences of anomalous dimension for $\mathcal{A}_\mu$

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu^{\alpha} \mathcal{G}$$

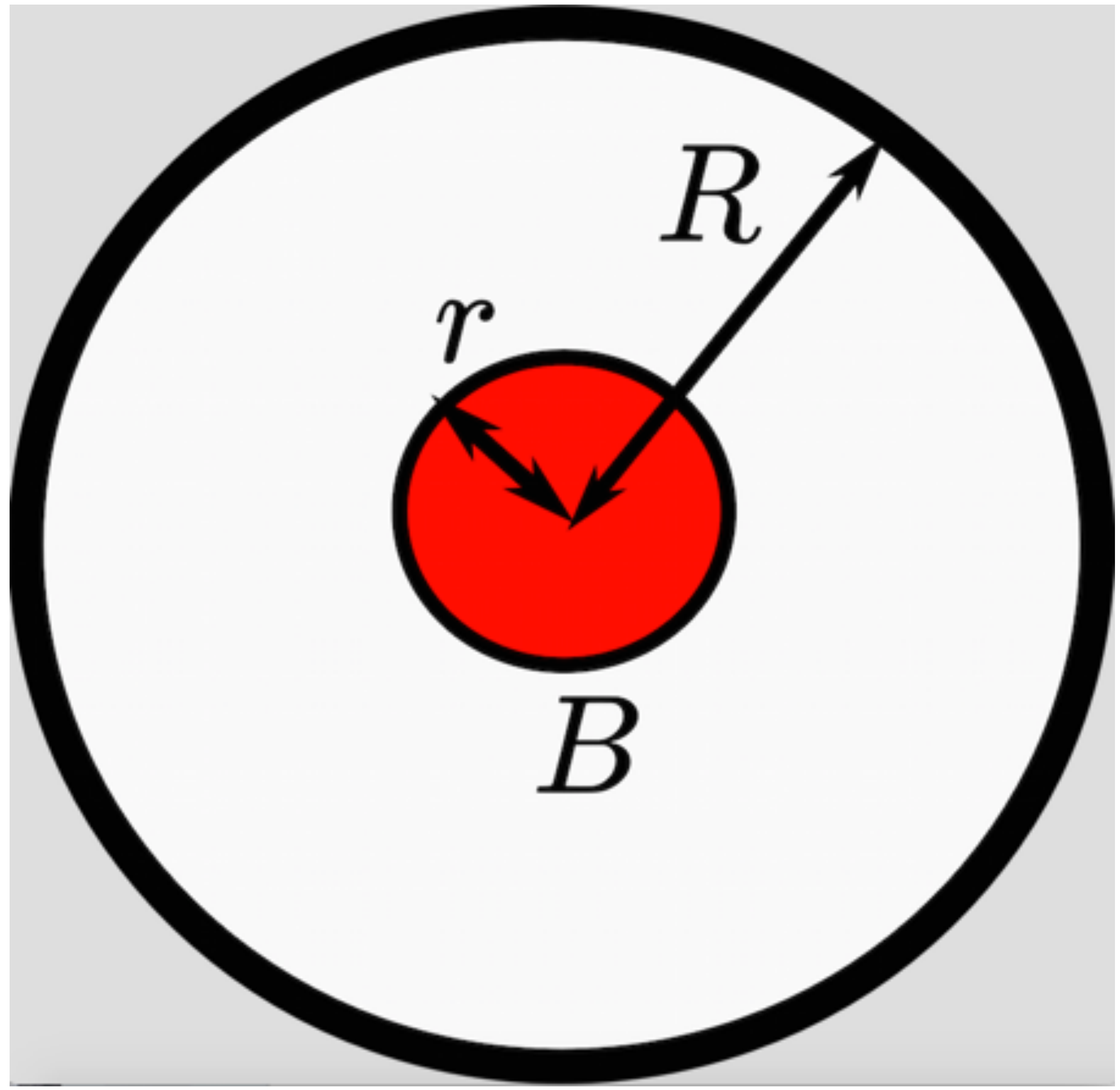
$$F_{\mu\nu} = \partial^\alpha_{[\mu} A_{\nu]}$$

$$\vec{\nabla}^\alpha \times \vec{A} = \vec{B}$$

no Stokes′ theorem

$$\oint \vec{A} \cdot d\ell \neq \int_S B \cdot d\vec{S}$$

Aharonov-Bohm Effect must change



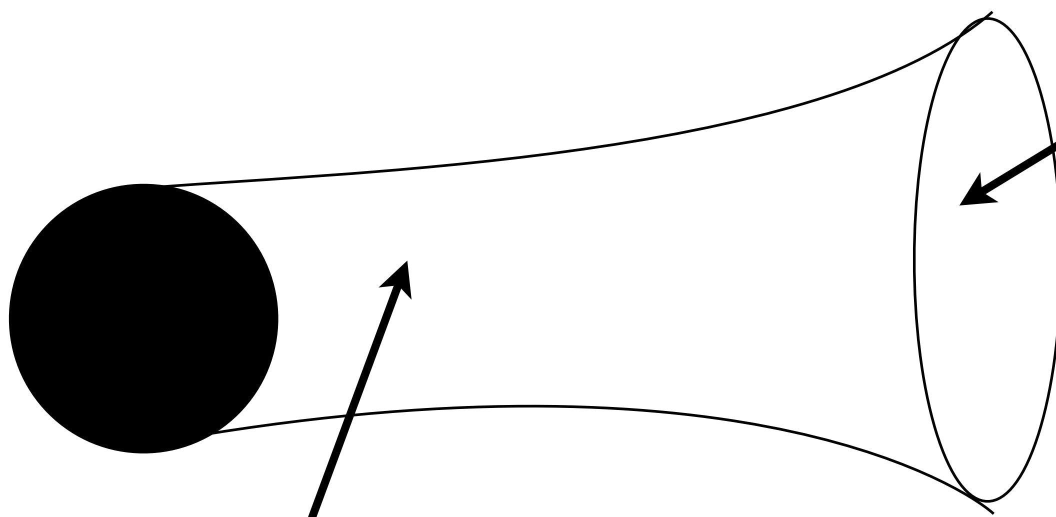
$$\Delta\phi_{\mathrm{D}} = \frac{e}{\hbar}\pi r^2 B R^{2\alpha-2}\left(\frac{\sqrt{\pi}2^{1-\alpha}\Gamma(2-\alpha)\Gamma(1-\frac{\alpha}{2})}{\Gamma(\alpha)\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\sin^2\frac{\pi\alpha}{2}\,{}_2F_1(1-\alpha,2-\alpha;2;\frac{r^2}{R^2})\right)$$

is the correction large?

$$\alpha = 1 + 2/3 = 5/3$$

$$\Delta\Phi_R = \frac{eB\ell^2}{\hbar} L^{-5/3}/(0.43)^2$$

yes!



d-dimensional
non-local
`QFT'

$[A] = \gamma$

fractional Virasoro
algebra

d+1 gravity
+ dilation

experimental detection on strange metal
in cuprates