The Geometry of Quantum Hall Effect: An Effective Action for all Dimensions

DIMITRA KARABALI

Lehman College of the CUNY

9th Regional Meeting in String Theory

Kolymbari, Greece

July 12, 2017

- (2+1) dim systems of (nonrelativistic) electrons in strong magnetic field
- Hall conductivity is quantized

$$
\begin{aligned}
J^i &= \sigma_H \epsilon^{ij} E_j \\
\sigma_H &= \frac{\nu}{2\pi} \frac{e^2}{\hbar}
\end{aligned}
$$

  where $\nu =$ filling fraction

  $\nu = 1, 2, \cdots$ for IQHE and $\nu = 1/3, 1/5, \cdots$ for FQHE.

- Framework for interesting ideas
  - topological field theories (Chern-Simons effective actions)
  - bulk-edge dynamics
  - non-commutative geometries, fuzzy spaces

Charged particle moving on 2d plane (or S^2) in strong external magnetic field (Landau problem)

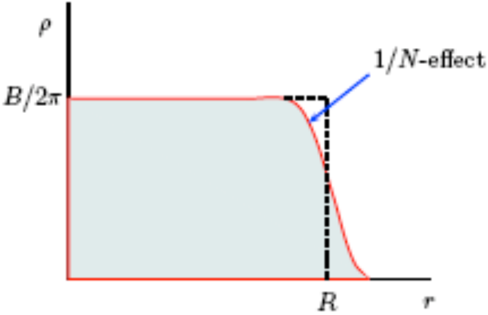
- Distinct Landau levels, separated by energy gap ($\sim B$)
- Each Landau level is degenerate
- Lowest Landau level (LLL) :

$$\psi_n \sim z^n e^{-|z|^2/2}$$

$$z = x + iy$$

Many-body problem $\Longrightarrow$ quantum Hall droplets

- Degeneracy of each LL is lifted by confining potential $(V = \frac{1}{2}ur^2)$
- Exclusion principle $\rightarrow$ N-body ground state = incompressible droplet
- Large $N \rightarrow$ sharp boundary



Low energy excitations of droplets $\Longleftrightarrow$ area preserving boundary fluctuations (edge excitations)

Edge dynamics is collectively described by 1d chiral boson ϕ (WEN, STONE,..)

$$S_{\text{edge}} = \int_{\partial D} \Big(\partial_t \phi + u\, \partial_\theta \phi\Big) \partial_\theta \phi, \qquad\qquad u \sim \frac{\partial V}{\partial r^2}\Big]_{\text{boundary}}$$

In the presence of electromagnetic interactions

$$A_\mu = \underbrace{a_\mu}_{\text{Constant } B} + \underbrace{\delta A_\mu}_{\text{Perturbation}}$$

- The bulk dynamics is described by an effective action

$$S_{\text{bulk}} = S_{CS} = \frac{\nu}{4\pi}\int_D \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

S_{CS} is not gauge invariant in presence of boundaries.

- The edge dynamics is described by

$$S_{\text{edge}} \sim \text{gauged chiral action}$$

Anomaly cancellation between bulk and edge actions,

$$\delta S_{\text{bulk}} + \delta S_{\text{edge}} = 0$$

- The effective action S_{CS} captures the response of the system to electromagnetic fluctuations

$$\begin{aligned}\frac{\delta S_{CS}}{\delta A_0} &= \rho = \frac{\nu}{2\pi} \\ \frac{\delta S_{CS}}{\delta A_i} &= J^i = \frac{\nu}{2\pi}\epsilon^{ij}E_j\end{aligned}$$

  Coefficient in front of S_{CS} is related to Hall conductivity.

- What about other transport coefficients?
  - How does the system respond to stress and strain?
  - Calculate stress tensor $\Longleftrightarrow$ couple theory to gravity

$$S[\psi,\psi^\dagger;A_\mu,g_{ij}] \;=\; \int dt d^2x\sqrt{g}\;\left[i\,\psi^\dagger\,D_t\,\psi - \frac{1}{2m}g^{ij}\,(D_i\psi)^\dagger(D_j\psi)\right]$$

$$\exp\;[iS_{eff}] \;=\; \int d\psi\,d\psi^\dagger \exp\;\left[iS[\psi,\psi^\dagger;A_\mu,g_{ij}]\right]$$

$$S_{eff} = \frac{1}{4\pi}\sum_{s=0}^{N-1}\int\left[[A+(s+\tfrac{1}{2})\omega]\,d[A+(s+\tfrac{1}{2})\omega] - \frac{1}{12}\omega d\omega\right] + \cdots$$

$\omega =$ spin connection $\qquad s=0\to LLL\;,\; s=1\to$ 1st LL, $\cdots$

$$\frac{\delta S_{eff}}{\delta\omega_0} \sim n_H = \text{Hall viscosity}$$

ABANOV AND GROMOV, 2014

- QHE on S^4 (HU AND ZHANG)

  Generalization to arbitrary even (spatial) dimensions

- QHE on $\mathbb{CP}^k$ (*with* V.P. NAIR)

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)} \qquad (2k \text{ dim space})$$

  - $U(k) \sim U(1) \times SU(k) \implies$ We can have both $U(1)$ and $SU(k)$ background magnetic fields
  - Landau wavefunctions are functions on $SU(k+1)$ with particular transformation properties under $U(k)$.
  - Each Landau level forms an irreducible $SU(k+1)$ representation, whose degeneracy is easy to calculate.

Wavefunctions are functions on $SU(k+1)$; expressed in terms of Wigner $\mathcal{D}$ functions

$$\Psi \sim \mathcal{D}^{(J)}_{L,R}(g) = \langle\ L\ \ |\,\hat{g}\,|\ R\ \rangle$$

quantum numbers of states in J rep.

Right/left transformations: $\quad \hat{R}_A\,\hat{g} = \hat{g}\,T_A, \qquad \hat{L}_A\,\hat{g} = T_A\,\hat{g}$

- $\hat{R}_a,\ \hat{R}_{k^2+2k} \rightarrow$ gauge transformations $\quad (U(k))$
- $\hat{R}_{+i}, \hat{R}_{-i} \rightarrow$ covariant derivatives $\quad (i = 1, \cdots, k) \quad [\hat{R}_{+i}, \hat{R}_{-j}] \in U(k)$
- $\hat{L}_A \rightarrow$ magnetic translations $\qquad (A \in SU(k+1))$
- How Ψ transforms under $U(k)_R$ depends on choice of background fields
- Choose "constant" $U(1)$ or $U(k)$ background magnetic fields.

$$U(1): \qquad \bar{F} = d\bar{a} = n\Omega, \quad \Omega = \text{Kahler } 2-\text{form}$$

$$U(k): \qquad \bar{F}^a \sim \bar{R}^a \sim f^{a\alpha\beta} e^{\alpha} e^{\beta}$$

$$\Psi^{J}_{m;\alpha} \sim \langle m \mid \hat{g} \mid \underbrace{R} \;\rangle$$

particular $SU(k)_R$ repr. $\tilde{J}$ with fixed $U(1)_R$ charge $\sim n$

$m = 1, \cdots \dim J \Longrightarrow$ counts degeneracy of LL

$\alpha =$ internal gauge index $= 1, \cdots, N' = \dim \tilde{J}$

- Hamiltonian

$$H = \frac{1}{2Mr^2} \sum_{i=1}^{k} \hat{R}_{+i} \hat{R}_{-i} + \text{constant}$$

Lowest Landau level: $\hat{R}_{-i} \Psi = 0$ Holomorphicity condition

($\mid R \,\rangle$ is lowest weight state)

- QHE on a compact space $M \Longrightarrow$ LLL defines an N-dim Hilbert space
  In the presence of confining potential $\Longrightarrow$ incompressible QH droplet

- Density matrix for ground state droplet : $\hat{\rho}_0$

$$\hat{\rho}_0 = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix} \begin{array}{l} K \\ \\ N-K \end{array}$$

  K filled states

- Under time evolution: $\hat{\rho}_0 \to \hat{\rho} = \hat{U}\, \hat{\rho}_0\, \hat{U}^\dagger$
  $\hat{U} = N \times N$ unitary matrix ; "collective" variable describing excitations within the LLL

The action for $\hat{U}$ is

$$S_0 = \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right]$$

which leads to the evolution equation for density matrix

$$i\frac{d\hat{\rho}}{dt} = [\hat{V}, \hat{\rho}]$$

S_0 has no explicit dependence on properties of space on which QHE is defined, abelian or nonabelian nature of fermions, etc.

S_0 = action of a noncommutative field theory

$$S_0 = \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right]$$
$$= N \int d\mu \, dt \; \left[i(\rho_0 * U^\dagger * \partial_t U) \; - \; (\rho_0 * U^\dagger * V * U) \right]$$

$$\underbrace{\hat{\rho}_0, \hat{U}, \hat{V}}_{(N \times N) \text{ matrices}} \qquad \Longrightarrow \qquad \underbrace{\rho_0(\vec{x}), U(\vec{x}, t), V(\vec{x})}_{\text{symbols}}$$

- $O(\vec{x}, t) = \frac{1}{N} \sum_{m,l} \Psi_m(\vec{x}) \hat{O}_{ml}(t) \Psi_l^*(\vec{x})$
- Matrix multiplication $\Longrightarrow$ $*$ product of symbols
- Tr $\Longrightarrow$ $N \int d\mu$

S_0 = exact bosonic action describing the dynamics of LL fermions

SAKITA: 2 dim. context

DAS, DHAR, MANDAL, WADIA,...

Extend this to include fluctuating gauge fields by gauging S_0

$$\partial_t \to \hat{D}_t = \partial_t + i\hat{\mathcal{A}}$$

$$S = \int dt \, \mathrm{Tr}\left[i\hat{\rho}_0 \hat{U}^\dagger \hat{\partial}_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} - \underbrace{\hat{\rho}_0 \, \hat{U}^\dagger \hat{\mathcal{A}} \hat{U}}_{\text{gauge interactions}} \right]$$

In terms of bosonic fields

$$S = N \int dt \, d\mu \; \mathrm{tr} \; \left[i\rho_0 * U^\dagger * \partial_t U \; - \; \rho_0 * U^\dagger * (V + \mathcal{A}) * U \right]$$

QUESTION: How is $\mathcal{A}$ related to the gauge fields coupled to the original fermions?

S is invariant under

$$\begin{aligned} \delta U &= -i\lambda * U \\ \delta\mathcal{A}(\vec{x},t) &= \partial_t\lambda(\vec{x},t) - i\left(\lambda * (V+\mathcal{A}) - (V+\mathcal{A}) * \lambda\right) \end{aligned} \tag{1}$$

Since S describes gauge interactions it has to be invariant under usual gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda + i[\underbrace{\bar{A}_\mu}_{\text{Background}} + \underbrace{A_\mu}_{\text{Perturbation}}, \Lambda]\,, \qquad \delta\bar{A}_\mu = 0 \tag{2}$$

we should choose

$$\begin{aligned} \mathcal{A} &= \text{function}(A_\mu, \bar{A}_\mu, V) \\ \lambda &= \text{function}(\Lambda, A_\mu, \bar{A}_\mu) \end{aligned}$$

such that the gauge transformation (2) induces $\delta\mathcal{A}$ in (1) (generalized Seiberg-Witten map)

Calculate S_0 for large N, K with $N \gg K \gg 1$ ($n \to \infty$ limit)

$$S_0 = N \int d\mu \, dt \, \left[i(\rho_0 * U^\dagger * \partial_t U) \; - \; (\rho_0 * U^\dagger * V * U) \right]$$

A. Abelian background magnetic field $U(1)$

- $\left([\hat{X}, \; \hat{Y}]\right)_{symbol} \to \frac{i}{n}\{X(\vec{x},t), \; Y(\vec{x},t)\}_{\rm PB} = \frac{i}{n}(\Omega^{-1})^{ij} \, \partial_i X(\vec{x},t) \, \partial_j Y(\vec{x},t)$
- $\rho_0 \to \Theta\Big(R_D^2 - r^2\Big),$ $\quad R_D$= droplet radius

$$S_0 \sim \int_{\partial D} (\partial_t \phi + u \, \mathcal{L}\phi) \mathcal{L}\phi$$

$(2k-1)$ (space) dim chiral action defined on droplet boundary

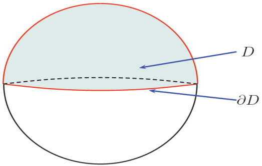
$$\mathcal{L}\phi = (\Omega^{-1})^{ij} \hat{r}_j \partial_i \phi, \qquad \mathcal{L} = \begin{cases} \text{derivative along boundary of droplet} \\ \quad \to \partial_\theta \text{ in 2 dim.} \end{cases}$$

B. Nonabelian background magnetic field $U(k)$

Symbol = $(N' \times N')$ matrix valued function $\longrightarrow$ action in terms of $G \in U(N')$

$$
\begin{aligned}
S_0 =& \frac{1}{4\pi}\int_{\partial D} \mathrm{tr}\left[\left(G^\dagger \dot{G} + u\, G^\dagger \mathcal{L} G\right) G^\dagger \mathcal{L} G\right] \\
&+ \frac{1}{4\pi}\int_D \mathrm{tr}\Big[-d\left(i\bar{A}dGG^\dagger + i\bar{A}G^\dagger dG\right) + \underbrace{\frac{1}{3}\left(G^\dagger dG\right)^3\Big]\left(\frac{\Omega}{2\pi}\right)^{k-1}\frac{1}{(k-1)!}}_{\text{WZW-term in } 2k+1 \text{ dim}} \\
\equiv& S_{\mathrm{WZW}}(A^L = A^R = \bar{A})
\end{aligned}
$$

$\mathcal{L} = (\Omega^{-1})^{ij}\hat{r}_j D_i$ = covariant derivative along the boundary of droplet



- In the presence of gauge fluctuations (DK)

$$\begin{aligned} S &= N \int dt\, d\mu\, \mathrm{tr}\, \left[i\rho_0 * U^\dagger * \partial_t U \;-\; \rho_0 * U^\dagger * (V + \mathcal{A}) * U \right] \\ &= S_{\text{edge}} + S_{\text{bulk}} \end{aligned}$$

$S_{\text{edge}} \sim S_{\text{WZW}}\left(A^L = A + \bar{A}\,, A^R = \bar{A}\right)$ = Chirally gauged WZW action in $2k$ dim

$S_{\text{bulk}} \sim S_{CS}^{2k+1}(\tilde{A}) + \cdots$ = $(2k+1)$ dim CS action

$\tilde{A} = (A_0 + V, \bar{a}_i + \bar{A}_i + A_i)$ = background + fluctuations

- Gauge Invariance $\Longrightarrow$ Anomaly Cancellation

$$\delta S_{\text{edge}} \neq 0, \qquad \delta S_{\text{bulk}} \neq 0$$

$$\delta S_{\text{edge}} + \delta S_{\text{bulk}} = 0$$

- Universal matrix action describing dynamics within each Landau level
- single-particle spectrum + large N limit $\Longrightarrow$
  (bulk + edge) effective actions
- gauge invariance is automatically built in
- Questions:
  - How important is precise knowledge of single particle spectrum?
  - Can we deviate from $\mathbb{CP}^k$?
  - How do we introduce metric perturbations?

- The lowest Landau level obeys the holomorphicity condition

$$\hat{R}_{-i}\Psi = 0$$

- The number of normalizable solutions is given by the Dolbeault index.
- The Dolbeault index is given by

$$\mathrm{Index}(\bar{\partial}_V) = \int_M \mathrm{td}(T_C M) \wedge \mathrm{ch}(V)$$

  - $\mathrm{td}(T_c M) =$ Todd class (for complex tangent space) = given in terms of traces of curvatures
  - ch (V) = Chern character = Tr $\left(e^{iF/2\pi}\right)$

- For a fully filled LLL (each particle carries $e = 1$):

  degeneracy = charge $\Longrightarrow$ index density = charge density

- So we can use

$$\frac{\delta S_{\mathit{eff}}}{\delta A_0} = J_0 = \text{Dolbeault Index Density}$$

and integrate up to get S_{eff}.

- Consider QHE on $\mathbb{CP}^1 = SU(2)/U(1) \sim S^2$

$$\text{Index}(\bar{\partial}_V) = \int_M \frac{iF}{2\pi} + \frac{iR}{4\pi}$$

The background values for the gauge fields and curvatures are

$$\bar{F} = -in\Omega, \quad \bar{R}|_{T_M K} = -i2\Omega, \quad \Omega = i\left[\frac{dz d\bar{z}}{1+z\bar{z}} - \frac{\bar{z}dz\, zd\bar{z}}{(1+z\bar{z})^2}\right]$$

Index$(\bar{\partial}_V)$ = degeneracy of LLL = $n+1$ as expected HALDANE

- Charge density including fluctuations

$$J_0 = \frac{iF}{2\pi} + \frac{iR}{4\pi}$$

- This leads to the effective action

$$S_{3d}^{LLL} = \frac{i^2}{4\pi} \int A\Big[F + R\Big] + S_{\text{grav}} + \cdots$$

- For higher Landau levels there is no holomorphicity condition, Dolbeault index is problematic
- The wave functions in the s-th LL for $\mathbb{CP}^1 = SU(2)/U(1) \sim S^2$ are

$$\Psi_m(g) \sim \langle J, m|g|J, -n/2\rangle, \qquad J = n/2 + s$$

  They satisfy $R_3\Psi = -n/2\ \Psi$, but do not satisfy holomorphicity $R_-\Psi \neq 0$
- The lowest weight state in the same representation

$$\tilde{\Psi}_m(g) \sim \langle J, m|g|J, -n/2 - s\rangle$$

  which satisfy the holomorphicity condition $R_-\tilde{\Psi} = 0 \Longrightarrow$ LLL

- It couples to a $U(1)$ background field

$$\bar{\mathcal{F}} = -i\big(n + 2s\big)\Omega = \bar{F} + s\bar{R} = \bar{F} + \bar{\mathcal{R}}_s$$

- So we have a mapping

  Particle with spin-0 in s-th LL $\Longrightarrow$ Particle with spin-s in LLL

- counting of states same $\Longrightarrow$ use Dolbeault index for degeneracy
- Chern character : $\mathrm{Tr}\left(e^{iF/2\pi}\right) \to \mathrm{Tr}\left(e^{i(\mathcal{R}_s+F)/2\pi}\right)$

- For s-th Landau level on $\mathbb{CP}^1 = SU(2)/U(1) \sim S^2$ we find

$$\mathrm{Index}(\bar{\partial}_V) = \int \left[\frac{iF}{2\pi} + (s+\tfrac{1}{2})\frac{iR}{2\pi}\right]$$

- S_{eff} can be determined from this

$$S^{(s)}_{3d} = \frac{i^2}{4\pi}\int A\,[F + (2s+1)R] + S_{\mathrm{grav}} + \cdots$$

- The purely gravitational term can be obtained from the gravitational part of the index density in $2k+2$ dim following the descent approach

$$[\mathrm{Index\ Density}]_{2k+2} = d\,[\cdots]$$

- Derive the full topological effective action

$$S^{(s)}_{3d} = \frac{i^2}{4\pi}\int \left\{ \left[A + (s+\tfrac{1}{2})\,\omega\right] d \left[A + (s+\tfrac{1}{2})\,\omega\right] - \tfrac{1}{12}\omega\, d\omega \right\}$$

ABANOV,GROMOV; KLETSOV, MA, MARINESCU, WIEGMANN; BRADLYN, READ; CAN, LASKIN, WIEGMANN

- In $2k+1$ dimensions (fluctuations around $\mathbb{CP}^k = SU(k+1)/U(k)$ background fields)
  - We have curvatures in the algebra of $U(k)$

$$R = -i\left[R^0\mathbf{1} + \tilde{R}^a t_a\right], \qquad R^0 = d\omega^0, \qquad \tilde{R} = d\tilde{\omega} + \tilde{\omega}\wedge\tilde{\omega}$$

  - Abelian and/or nonabelian gauge fields
  - Nonzero spin to include higher Landau levels

$$\mathcal{R}_s = -i\left[sR^0\mathbf{1} + \tilde{R}^a T_a\right]$$

The full topological bulk effective action capturing both gauge and metric fluctuations is

$$S^{(s)}_{2k+1} = \int \Big[\mathrm{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(\omega_s + A)\Big]_{2k+1} + 2\pi \int \Omega^{\mathrm{grav}}_{2k+1}$$

where

$$[\mathrm{td}(T_c K) \wedge \mathrm{ch}(S)]_{2k+2} = d\,\Omega^{\mathrm{grav}}_{2k+1} + \frac{1}{2\pi}\, d \Big[\mathrm{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(\omega_s)\Big]_{2k+1}$$

Gives correct expressions for degeneracies in all cases we know QHE wavefunctions: $\mathbb{CP}^k$ (abelian and nonabelian gauge fields), $S^2 \times S^2$, etc

- 2+1 dim, s-th Landau level

$$S_{3d}^{(s)} = \frac{i^2}{4\pi} \int \left\{ \left[A + (s + \frac{1}{2})\, \omega\right] d \left[A + (s + \frac{1}{2})\, \omega\right] - \frac{1}{12} \omega\, d\omega \right\}$$

- 4+1 dim, s-th Landau level

$\mathbb{CP}^2 = SU(3)/U(2);$ Abelian gauge field

$$\begin{aligned} S_{5d}^{(s)} \quad = \quad & \frac{i^3 (s+1)}{(2\pi)^2} \int \left\{ \frac{1}{3!} \Big(A + (s+1)\omega^0 \Big) \left[d \Big(A + (s+1)\omega^0 \Big) \right]^2 \right. \\ & \left. - \frac{1}{12} \Big(A + (s+1)\omega^0 \Big) \left[(d\omega^0)^2 - \left[((s+1)^2 - \frac{3}{2} \right] \mathrm{Tr}(\tilde{R} \wedge \tilde{R}) \right] \right\} \end{aligned}$$

- 6+1 dim, lowest Landau level

  $\mathbb{CP}^3 = SU(4)/U(3)$; Abelian gauge field

$$
\begin{aligned}
S_{7d}^{\mathrm{LLL}} \quad = \quad & \frac{1}{(2\pi)^3} \int \left\{ \frac{1}{4!} \left(A + \frac{3}{2}\omega^0 \right) \left[d \left(A + \frac{3}{2}\omega^0 \right) \right]^3 \right. \\
& - \frac{1}{16} \left(A + \frac{3}{2}\omega^0 \right) d \left(A + \frac{3}{2}\omega^0 \right) \left[(d\omega^0)^2 + \frac{1}{3}\mathrm{Tr}(\tilde{R} \wedge \tilde{R}) \right] \\
& \left. + \frac{1}{1920}\omega^0 d\omega^0 \left[17(d\omega^0)^2 + 14\mathrm{Tr}(\tilde{R} \wedge \tilde{R}) \right] + \frac{1}{720}\omega^0 \mathrm{Tr}(\tilde{R} \wedge \tilde{R} \wedge \tilde{R}) \right\} \\
& + \frac{1}{120} \int (CS)_7(\tilde{\omega})
\end{aligned}
$$

- Extend QHE to higher dimensions; physical realizations of fuzzy spaces
- Universal matrix action $\rightarrow$ noncommutative field theory description of LL dynamics
- At large N limit action describes dynamics of abelian/nonabelian quantum Hall droplets with gauge fluctuations
- anomaly free bulk/edge dynamics
- Use index theorems to introduce metric perturbations
- Applications to fluids and higher dim transport coefficients ?