Hopf twists and marginal deformations

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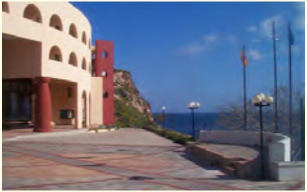

9th Crete Regional Meeting on String Theory
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Based on arXiv:1602.08061 (with H. Dlamini) and ongoing work


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Outline

- Discuss symmetries and integrability of the exactly marginal deformations of $\mathcal{N} = 4$ SYM
- Explain why it is useful to generalise from Lie algebras to Hopf algebras
- Define these algebras using a star product derived from a twist
- For a specific marginal deformation, use this star product and generalised geometry to construct the dual supergravity background

$\mathcal{N} = 4$ Super–Yang–Mills

- Fundamental role in the study of gauge and string theory
- Unique four–dimensional gauge theory with maximal global supersymmetry (16 supercharges)
- Many interesting properties:
  - Perturbative Finiteness
  - Nonperturbative Finiteness (4d SCFT)
  - AdS/CFT Correspondence
  - Planar Integrability
  - Planar Amplitudes
- Is it the unique theory with these features?
- Are there theories which share only *some* of these features?
- How does the knowledge accumulated for $\mathcal{N} = 4$ SYM help to understand more realistic theories?

Marginal Deformations of $\mathcal{N} = 4$ SYM

- Look for theories as close as possible to $\mathcal{N} = 4$ SYM
- Preserve conformal invariance $\Rightarrow$ Marginal Deformations
- Focus on superpotential deformations $\Rightarrow \mathcal{N} = 1$ SUSY
- In $\mathcal{N} = 1$ superspace:

$$\mathcal{L} = \int d^4\theta \mathrm{Tr} e^{gV}\overline{\Phi}_i e^{-gV}\Phi^i + \left(\int d^2\theta \mathcal{W} + \int d^2\bar{\theta}\overline{\mathcal{W}}\right) + \cdots$$

- Chiral Superfields $\Phi^i = \phi^i + \theta^\alpha \psi^i_\alpha + \theta^2 F^i, \; i = 1, 2, 3$
- $\mathcal{N} = 4$ superpotential:

$$\mathcal{W} = g\mathrm{Tr}\Phi^1[\Phi^2, \Phi^3] = \frac{g}{3}\epsilon_{ijk}\mathrm{Tr}\Phi^i\Phi^j\Phi^k$$

- Most general classically marginal deformation:

$$\delta\mathcal{W} = h_{ijk}\mathrm{Tr}\Phi^i\Phi^j\Phi^k,$$

where h_{ijk} is a symmetric tensor

Exactly Marginal Deformations

- Which of these deformations are *exactly marginal?*
- Perturbative approaches [Parkes, West '84],[Jones,Mezincescu '84]
- Non–perturbative proof by Leigh and Strassler ('95) using the NSVZ β function
- The *Leigh–Strassler* superpotential:

$$\mathcal{W}_{LS} = \kappa \mathrm{Tr} \left(\Phi^1 [\Phi^2, \Phi^3]_q + \frac{h}{3} \left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right) \right)$$

- q–commutator $[X, Y]_q = XY - qYX$
- Recover $\mathcal{N} = 4$ SYM for $q = 1$, $h = 0$, $\kappa = g$
- Finite if $f(g, \kappa, q, h) = 0$, where f unknown in general
- 1–loop finiteness condition

$$2g^2 = \kappa\bar{\kappa} \left[\frac{2}{N^2}(1+q)(1+\bar{q}) + \left(1 - \frac{4}{N^2}\right)(1 + q\bar{q} + h\bar{h}) \right]$$

- An interesting case: $q = e^{i\beta}, h = 0$ ("Real β deformation")

$\mathcal{N} = 4$ SYM vs. Leigh–Strassler?

- How do the LS theories compare with $\mathcal{N} = 4$ SYM?

	$\mathcal{N} = 4$ SYM	Leigh–Strassler
Conformally Invariant	$\surd$	$\surd$
AdS/CFT dual	$\surd$	complex β [Lunin,Maldacena'05]
Planar Integrability	$\surd$	basically real β

- What makes the real β deformation so special?
- Take a closer look at the symmetries
- Work at the level of the classical lagrangian

Symmetries: $\mathcal{N} = 4$ SYM

- 4d Superconformal group: $\mathrm{PSU}(2, 2|4)$
- Focus on the R–symmetry subgroup $\mathrm{SU}(4) \sim \mathrm{SO}(6)$
- In $\mathcal{N} = 1$ superspace notation, the $\mathcal{N} = 4$ theory has manifest $\mathrm{SU}(3) \times \mathrm{U}(1)_R$ symmetry

$$\mathcal{W} = g\mathrm{Tr}\Phi^1[\Phi^2, \Phi^3] = \frac{g}{3}\epsilon_{ijk}\mathrm{Tr}\Phi^i\Phi^j\Phi^k$$

- ϵ_{ijk} is the invariant tensor of $\mathrm{SU}(3)$

$$\epsilon_{ijk}U^i{}_l U^j{}_m U^k{}_n = (\det U)\epsilon_{lmn} = \epsilon_{lmn}$$

- Transforming $\Phi^i \to U^i{}_j\Phi^j$ leaves the superpotential invariant

Symmetries: Leigh–Strassler

- The generic LS deformation breaks $\mathrm{SU}(3)$ to a discrete subgroup

$$\mathcal{W}_{LS} = \kappa \mathrm{Tr}\left(\Phi^1[\Phi^2,\Phi^3]_q + \frac{h}{3}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right)\right)$$

- This superpotential has the following $\mathbb{Z}_3$ symmetries:

$$\mathbb{Z}_3^A: \ \Phi^1 \to \Phi^2 \quad , \quad \Phi^2 \to \Phi^3 \quad , \quad \Phi^3 \to \Phi^1$$

$$\mathbb{Z}_3^B: \ \Phi^1 \to \omega\Phi^1 \quad , \quad \Phi^2 \to \omega^2\Phi^2 \quad , \quad \Phi^3 \to \Phi^3 \quad (\omega^3 = 1)$$

- Together with a third $\mathbb{Z}_3$ within $\mathrm{U}(1)_R$ ($\Phi^i \to \omega\Phi^i$), they form a trihedral group known as Δ_{27} [Aharony et al. '02]
- For real β the symmetry group is enhanced to $\mathrm{U}(1)^3$
- Is this all?

q-deforming $SU(3)$

- In 0811.3755 (with T. Månsson) it was claimed that the global $SU(3)$ is not actually broken
- Rather, it is *deformed* to a Hopf algebra.

$$\Phi^i \to T^i{}_j \Phi^j \qquad \text{where} \quad T = \begin{pmatrix} t^1{}_1 & t^1{}_2 & t^1{}_3 \\ t^2{}_1 & t^2{}_2 & t^2{}_3 \\ t^3{}_1 & t^3{}_2 & t^3{}_3 \end{pmatrix}$$

is a symmetry if the components of T satisfy:

$$\text{(a)} \qquad t^a{}_c t^{a+1}_c - q t^{a+1}_c t^a{}_c + h t^{a-1}_c t^{a-1}_c = h\left(t^a{}_{c+1} t^{a+1}_{c-1} - \bar{q} t^a{}_{c-1} t^{a+1}_{c+1} + \bar{h} t^a{}_c t^{a+1}_c\right)$$

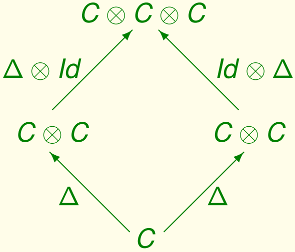
$$\text{(b)} \qquad q[t^{a+1}{}_{c+1}, t^a{}_c] = -q^2 t^{a+1}{}_c t^a{}_{c+1} + hq t^{a-1}{}_c t^{a-1}{}_{c+1} + h t^{a-1}{}_{c+1} t^{a-1}{}_c + t^a{}_{c+1} t^{a+1}{}_c$$

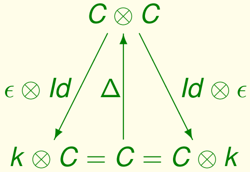
$$\text{(c)} \qquad -q t^{a+1}_c t^a{}_{c+1} + \bar{q} t^a{}_{c+1} t^{a+1}_c = \bar{h} t^a{}_{c-1} t^{a+1}_{c-1} - h t^{a-1}_c t^{a-1}_{c+1}$$

$$\text{(d)} \qquad h(t^a{}_{c+1} t^a{}_{c-1} - \bar{q} t^a{}_{c-1} t^a{}_{c+1}) = \bar{h}(t^{a+1}_c t^{a-1}_c - q t^{a-1}_c t^{a+1}_c)$$

Quantum Algebras

- Recall an *algebra* $\mathcal{C}$ is a vector space together with a product $\cdot : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ and a unit map $\eta : k \to \mathcal{C}$
- A *coalgebra* $\mathcal{C}$ is instead equipped with a coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and a counit $\epsilon : \mathcal{C} \to k$





- A *bialgebra* is both an algebra and a coalgebra in a compatible way
- A Hopf Algebra is a bialgebra equipped with an antipode $S : \mathcal{C} \to \mathcal{C}$

$$\cdot\,(S \otimes \mathrm{id}) \circ \Delta = \cdot\,(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon\,.$$

The coproduct

- Lie algebras have a trivial coproduct

$$\Delta(X) = X \otimes 1 + 1 \otimes X$$

- (Recall from QM:

$$\Delta(\vec{S})(|\psi_1\rangle \otimes |\psi_2\rangle) = (\vec{S}\,|\psi_1\rangle) \otimes |\psi_2\rangle + |\psi_1\rangle \otimes (\vec{S}\,|\psi_2\rangle) \quad)$$

- Commutative: $\tau \circ \Delta(X) = \Delta(X)$

- Non-commutative co-product (and product): Hopf Algebra

The R-matrix

- One can construct Hopf algebras using the RTT relations [Faddeev, Reshetikhin, Takhtajan]

$$R^{i}{}_{a}{}^{k}{}_{b} t^{a}_{j} t^{b}_{l} = t^{k}_{b} t^{i}_{a} R^{a}{}_{j}{}^{b}{}_{l} \,,$$

where $R : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is called an *R-matrix.*

- Quantum Yang–Baxter Equation (QYBE): *Quasitriangular* Hopf Algebra

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \qquad \left(R^{i}{}_{s}{}^{j}{}_{r} R^{s}{}_{l}{}^{k}{}_{p} R^{r}{}_{m}{}^{p}{}_{n} = R^{j}{}_{s}{}^{k}{}_{p} R^{i}{}_{r}{}^{p}{}_{n} R^{r}{}_{l}{}^{s}{}_{m} \right)$$

- YBE guarantees the resulting algebra is not too trivial

- R controls Δ non-commutativity: $\tau \circ \Delta(h) = R(\Delta(h))R^{-1}$

R-matrix for the general LS deformation

- Read off 1–loop spin chain Hamiltonian [Roiban '03]

$$H_{l,l+1} = \frac{1}{2d^2}\begin{pmatrix} h\bar{h} & 0 & 0 & 0 & 0 & \bar{h} & 0 & -\bar{h}q & 0 \\ 0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & h \\ 0 & 0 & q\bar{q} & 0 & -h\bar{q} & 0 & -\bar{q} & 0 & 0 \\ 0 & -\bar{q} & 0 & q\bar{q} & 0 & 0 & 0 & 0 & -h\bar{q} \\ 0 & 0 & -\bar{h}q & 0 & h\bar{h} & 0 & \bar{h} & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\ 0 & 0 & -q & 0 & h & 0 & 1 & 0 & 0 \\ -h\bar{q} & 0 & 0 & 0 & 0 & -\bar{q} & 0 & q\bar{q} & 0 \\ 0 & \bar{h} & 0 & -\bar{h}q & 0 & 0 & 0 & 0 & h\bar{h} \end{pmatrix}$$

- Define $\hat{R}^{ij}_{kl} = \delta^i_k \delta^j_l - H^{ij}_{kl} \quad \Longrightarrow \quad R^{i\ \,j}_{\ k\ \,l} = \hat{R}^{j\ \,i}_{\ k\ \,l}$

$$R = \frac{1}{2d^2}\begin{pmatrix} 1+q\bar{q}-h\bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 \\ 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 & 0 & 0 & 0 & 2h\bar{q} \\ 0 & 0 & 2q & 0 & -2h & 0 & q\bar{q}+h\bar{h}-1 & 0 & 0 \\ 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 2\bar{h}q & 0 & 1+q\bar{q}-h\bar{h} & 0 & -2\bar{h} & 0 & 0 \\ 2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 \\ 0 & 0 & 1-q\bar{q}+h\bar{h} & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\ -2h & 0 & 0 & 0 & 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 \\ 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 & 0 & 0 & 0 & 1+q\bar{q}-h\bar{h} \end{pmatrix}$$

$(2d^2 = 1 + \bar{q}q + \bar{h}h)$

General LS deformation

- *R* does not satisfy YBE in general
- YBE $\Longleftrightarrow$ known integrable deformations
- Still get a Hopf algebra for general case — checked up to cubic level [Månsson, KZ '08]
- Associativity will imply higher relations $\Longrightarrow$ danger of trivialising the algebra
- Has not yet been analysed in detail
- In this talk I'll take a complementary approach — derive the Hopf algebra using a *twist*

Our goals

- We are claiming that the LS theories have more global symmetry than is naively visible
- Lie algebra $\to$ Hopf algebra
- How do we work with such symmetries?
- Is this quantum symmetry visible on the gravity side?
- If so, can it be turned into a solution-generating technique for AdS/CFT duals?
- First step: Focus on the integrable cases

Our goals

- We are claiming that the LS theories have more global symmetry than is naively visible
- Lie algebra $\to$ Hopf algebra
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- First step: Focus on the integrable cases
- Derive them as *twists* of the $\mathcal{N}=4$ structure
- Drinfeld (Hopf) twist $F : \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$

$$\Delta_F(h) = F(\Delta(h))F^{-1} , \qquad R_F = F_{21} \cdot R \cdot F^{-1}$$

- Cocycle condition: $(1 \otimes F)(\mathrm{id} \otimes \Delta)F = (F \otimes 1)(\Delta \otimes \mathrm{id})F$
- For $\mathcal{N}=4$ SYM, $R = I \otimes I$

The β-deformation

- Gauge theory has $\mathrm{U}(1)^3$ symmetry
- $q = e^{i\beta}, h = 0$. For β real, the R-matrix is

$$R_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Satisfies YBE — quasitriangular case
- Dual background constructed by Lunin and Maldacena using TsT transformations
- Obtained as a twist of the $\mathcal{N} = 4$ case [Beisert, Roiban '05]

β-deformation from a twist

- The appropriate twist is simply

$$F_q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

- Can be expressed as

$$F_q = e^{i\frac{\beta}{2} H_1 \wedge H_2} , \qquad \text{where} \quad H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \; H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

- Depends on Cartan elements $\Longrightarrow$ Abelian
- Too simple. Would like a theory without the extra $\mathrm{U}(1)$'s

The w-deformation

- Set $q = 1 + w, h = w$
- Unitarily equivalent to real-β case [Bundzik,Mansson '05]
- Will not take advantage of this fact — "TST without TST"
- w-deformed R-matrix satisfies YBE

$$R_w = \frac{1+w}{1+w+w^2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{w}{1+w} & 0 & w & 0 \\ 0 & 1 & 0 & -\frac{w}{1+w} & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 1 & 0 & -\frac{w}{1+w} & 0 & w & 0 & 0 \\ 0 & w & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{w}{1+w} \\ 0 & 0 & w & 0 & 1 & 0 & -\frac{w}{1+w} & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{w}{1+w} & 0 \\ 0 & 0 & -\frac{w}{1+w} & 0 & w & 0 & 1 & 0 & 0 \\ -\frac{w}{1+w} & 0 & 0 & 0 & 0 & w & 0 & 1 & 0 \\ 0 & -\frac{w}{1+w} & 0 & w & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- More compactly:

$$R_w = \frac{1+w}{1+w+w^2}\left[I \otimes I + w\, U \otimes V - \frac{w}{1+w} V \otimes U\right] .$$

with

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V := U^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U^3 = I .$$

w-deformation from a twist

- The twist is a bit more complicated

$$F_w = \tilde{C}\begin{pmatrix} 1+w & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 \\ 0 & 1+w & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+w & 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+w & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 1+w & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+w & 0 & w & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 1+w & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 0 & 0 & 1+w & 0 \\ 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 1+w \end{pmatrix}$$

- More compact form:

$$F_w = \tilde{C}\big[(1+w)I \otimes I + wV \otimes U\,\big].$$

- $[V, U] = 0 \Longrightarrow$ Also abelian
- Inverse twist

$$F_w^{-1} = \frac{(1+w)^2}{\tilde{C}(1+2w)(1+w+w^2)}\left[I \otimes I - \frac{w}{1+w} V \otimes U + \frac{w^2}{(1+w)^2} U \otimes V\,\right].$$

Star product

- The twist can be used to define a star product
- Hopf algebra acts as derivation

$$g \triangleright (x \cdot y) = (g \triangleright x) \cdot y + x \cdot (g \triangleright y) \, .$$

- Compatibility of algebra and module product

$$\begin{aligned} g \triangleright (x \cdot y) &= m(\Delta(g) \triangleright [x \otimes y]) = m([g \otimes 1 + 1 \otimes g] \triangleright [x \otimes y]) \\ &= m((g \triangleright x) \otimes y + x \otimes (g \triangleright y)) = (g \triangleright x) \cdot y + x \cdot (g \triangleright y) \end{aligned}$$

- Twisting Δ also twists the module product

$$m_F(x \otimes y) = m(F^{-1} \triangleright x \otimes y) = (F^{-1}_{(1)} \triangleright x) \cdot (F^{-1}_{(2)} \triangleright y)$$

- Star product

$$x \star y = m_F(x \otimes y) \, .$$

Inverse star product

- Generate the LS superpotential from the $\mathcal{N} = 4$ SYM one

$$x * y = m(F \triangleright x \otimes y) \qquad \left(\text{in indices:} \quad z^i * z^j = F^{ji}_{lk} z^k z^l\right)$$

- Obtain:

$$\Phi^1 * \Phi^2 * \Phi^3 + \Phi^2 * \Phi^3 * \Phi^1 + \Phi^3 * \Phi^1 * \Phi^2 - \Phi^1 * \Phi^3 * \Phi^2 - \Phi^3 * \Phi^2 * \Phi^1 - \Phi^2 * \Phi^1 * \Phi^3$$
$$= \frac{1}{1+2w}\left[\Phi^1\Phi^2\Phi^3 + \Phi^2\Phi^3\Phi^1 + \Phi^3\Phi^1\Phi^2 - (1+w)\left[\Phi^1\Phi^3\Phi^2 - \Phi^3\Phi^2\Phi^1 - \Phi^2\Phi^1\Phi^3\right]\right.$$
$$\left. + w\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right)\right]$$

- Take the gauge theory trace

$$\mathrm{Tr}\left(\Phi^1 * \Phi^2 * \Phi^3 - \Phi^1 * \Phi^3 * \Phi^2\right)$$
$$= \frac{3}{1+2w}\mathrm{Tr}\left(\Phi^1\Phi^2\Phi^3 - (1+w)\Phi^1\Phi^3\Phi^2 + \frac{w}{3}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right)\right)$$

First-order star product

- Expand the full star product to first order

$$F^{i}{}_{k}{}^{j}{}_{l} = \delta^{i}{}_{k}\delta^{j}{}_{l} - \frac{i\beta}{2}\, r^{i}{}_{k}{}^{j}{}_{l}$$

- r is the classical R-matrix
- Deformed commutator

$$z^i * z^j - z^j * z^i = \Theta^{ij}$$

- Noncommutativity matrices

$$\Theta^{ij}_{\beta} = i\beta \begin{pmatrix} 0 & -z^1 z^2 & z^1 z^3 \\ z^1 z^2 & 0 & -z^2 z^3 \\ -z^1 z^3 & z^2 z^3 & 0 \end{pmatrix}$$

[Kulaxizi '06]

$$\Theta^{ij}_{w} = w \begin{pmatrix} 0 & (z^3)^2 - z^1 z^2 & z^1 z^3 - (z^2)^2 \\ z^1 z^2 - (z^3)^2 & 0 & (z^1)^2 - z^2 z^3 \\ (z^2)^2 - z^1 z^3 & z^2 z^3 - (z^1)^2 & 0 \end{pmatrix}$$

So far...

- Understood how to obtain the Hopf algebra structure via a twist
- The twist defines an all-orders star product
- Associative for the integrable cases
- Expanding to first order reproduces known results

- How about the gravity side?
- Can we see the quantum symmetry?
- Use the framework of generalised complex geometry

Generalised geometry

[Hitchin '02][Gualtieri '04][Grana,Minasian,Petrini,Tomasiello,Zabzine,Zaffaroni...]

- Generalises and unifies complex and symplectic geometry
- Metric and B-field treated on equal footing
- Very natural from string theory perspective
- Consider structures on $T \oplus T^*$
- 6d space transverse to D3 branes $\longrightarrow$ 12d $T \oplus T^*$

$$X = \{\iota_I, dz^I\} = \{\iota_1, \bar{\iota}_1, \iota_2, \ldots, d\bar{z}^2, dz^3, d\bar{z}^3\}$$

- Gamma matrices: $\Gamma_I = \iota_I$, $\Gamma_{I+6} = dz^I$

Generalised complex structures

- Holomorphic and Kähler forms

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3 \quad \text{and} \quad J = \frac{i}{2} \sum_{i=1}^{3} dz^i \wedge d\bar{z}^{\bar{i}}$$

- Pure spinors (here for flat space)

$\Phi^0_- = \Omega = dz^1 \wedge dz^2 \wedge dz^3$ and

$\Phi^0_+ = e^{-iJ} = 1 + \frac{1}{2} \sum_i dz^i \wedge d\bar{z}^i + \frac{1}{4} \sum_i dz^i \wedge d\bar{z}^i \wedge dz^{i+1} \wedge d\bar{z}^{i+1} + \frac{1}{8} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \wedge dz^3 \wedge d\bar{z}^3$

- For IIB $\mathcal{N} = 2$ backgrounds

$$d\Phi_\pm = 0$$

$$\langle \Phi_-, X\Phi_+ \rangle = 0 \,, \quad \langle \bar{\Phi}_-, X\Phi_+ \rangle = 0 \,, \quad \langle \bar{\Phi}_+, \Phi_+ \rangle = \langle \bar{\Phi}_-, \Phi_- \rangle$$

(Mukai pairing $\langle A, B \rangle = \sum (-1)^{[n/2]} A_n \wedge B_{6-n}$)

- Generalised complex structures

$$\mathcal{J}_{\pm\, MN} = \langle \overline{\Phi}_\pm, \Gamma_{MN} \Phi_\pm \rangle \qquad \mathcal{J}^2 = -1$$

Generalised metric

- Constructed from the two generalised complex structures

$$\mathcal{G}_{MN} = -\mathcal{J}_{+ML}\mathcal{J}_{-}{}^{L}{}_{N} \ .$$

- Can just read off the NS-NS fields

$$\mathcal{G}^{M}_{N} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \ , \qquad e^{2\Phi} = \sqrt{|\det g|}$$

- Example: Flat space

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \ , \quad \text{where} \quad g = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Twisting the pure spinors

- Use gauge-theory star product to twist flat-space pure spinors

$$dz^I \wedge_* dz^J = \left(1 - \frac{i}{2}\Theta^{KL}\iota_K \wedge \iota_L\right) dz^I \wedge dz^J = dz^I \wedge dz^J - i\Theta^{IJ}$$

- Equivalent to a bivector deformation (for our case!)

$\Phi_\pm = e^{\Theta^{IJ}\iota_I \wedge \iota_J}\Phi^0_\pm$ For real-β: [Halmagyi-Tomasiello '07]

- Leads to the deformed background!

$\Phi^*_- = dz^1 \wedge_* dz^2 \wedge_* dz^3$ and

$$\Phi^*_+ = 1 + \frac{1}{2}\sum_{i=1}^{3} dz^i \wedge_* d\bar{z}^i + \frac{1}{4}\sum_{i=1}^{3} dz^i \wedge_* d\bar{z}^i \wedge_* dz^{i+1} \wedge_* d\bar{z}^{i+1}$$
$$+ \frac{1}{8} dz^1 \wedge_* d\bar{z}^1 \wedge_* dz^2 \wedge_* d\bar{z}^2 \wedge_* dz^3 \wedge_* d\bar{z}^3$$

- Geometric manifestation of the CFT quantum symmetry

Results

- Using the real-β $*$-product we obtain the pure spinors of the $\mathcal{N}=2$ LM precursor background [Lunin-Maldacena '05]
- The w-deformed $*$-product leads to

$$\Phi_-^w = dz^1 \wedge dz^2 \wedge dz^3 - iw[(z^2z^3-(z^1)^2)dz^1 + (z^3z^1-(z^2)^2)dz^2 + (z^1z^2-(z^3)^2)dz^3] \,,$$

$$\begin{aligned}\Phi_+^w = \Phi_+^0 + \frac{iw}{4} \Big[&(z^3\bar{z}^3 - z^2\bar{z}^2)dz^1 \wedge d\bar{z}^1 + ((\bar{z}^3)^2 - \bar{z}^1\bar{z}^2)dz^1 \wedge dz^2 + (z^1\bar{z}^3 - \bar{z}^2z^3)dz^1 \wedge d\bar{z}^2 \\ &+ (z^2\bar{z}^3 - z^1\bar{z}^2)dz^1 \wedge d\bar{z}^3 + (z^1z^2 - (z^3)^2)d\bar{z}^1 \wedge d\bar{z}^2 + \text{cyclic}\Big] \\ + \frac{iw}{8} \Big[&(z^1\bar{z}^1 - z^2\bar{z}^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 - (\bar{z}^2\bar{z}^3 - (\bar{z}^1)^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge dz^3 \\ &+ (\bar{z}^1z^3 - z^1\bar{z}^2)dz^1 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 - (z^1\bar{z}^3 - \bar{z}^1z^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 \\ &+ (z^2z^3 - (z^1)^2)dz^1 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + \text{cyclic}\Big]\end{aligned}$$

- Compatible pair of pure spinors $\Longrightarrow$ $\mathcal{N}=2$ background!
- Put D3-brane at origin, near-horizon limit $\Longrightarrow$ $\mathrm{AdS}_5 \times \tilde{S}^5$
- Purely NS-NS. IIB equations of motion $\Longrightarrow$ R-R fields.

The w-deformed background

$$e^{2\Phi} = G \, . \qquad\qquad g_{\mu\nu} = G\, \tilde{g}_{\mu\nu} \, ,$$

$$G^{-1} = 1 + w^2 R^4 (1 - s_\alpha^2 c_\alpha^2 - s_\alpha^4 s_\theta^2 c_\theta^2 - 2 c_\alpha s_\alpha^3 c_\theta s_\theta^2 C_2 - 2 s_\alpha^2 c_\alpha^2 s_\theta c_\theta C_1 - 2 c_\alpha s_\alpha^3 c_\theta^2 s_\theta C_3)$$

$$\begin{aligned}
\tilde{g}_{\alpha\alpha} &= R^2 + \frac{w^2 R^6}{2} \Big[1 - 2c_\alpha^2 + 2 c_\alpha^2 s_\alpha^2 c_\theta^2 s_\theta^2 + 2 c_\alpha^4 + (4c_\alpha^2 - 3) c_\alpha s_\alpha c_\theta s_\theta^2 C_2 - c_\alpha s_\alpha c_\theta^3 C_{3,1} \\
&\qquad - c_\alpha s_\alpha s_\theta^3 C_{2,1} + (4c_\alpha^4 - 3c_\alpha^2 + 1) c_\theta s_\theta C_1 + (4c_\alpha^3 - 3c_\alpha) s_\alpha c_\theta^2 s_\theta C_3 \Big] \, , \\
\tilde{g}_{\alpha\theta} &= \frac{w^2 R^6}{2} \Big[c_\alpha s_\alpha^3 c_\theta s_\theta (c_\theta^2 - s_\theta^2) + s_\alpha^2 c_\alpha^2 s_\theta c_\theta^2 C_{3,1} + (2 s_\alpha^2 s_\theta^2 - s_\theta^2 - s_\alpha^2) s_\alpha^2 c_\theta C_3 \\
&\qquad - s_\alpha^2 c_\alpha^2 c_\theta s_\theta^2 C_{2,1} + c_\alpha^3 s_\alpha (1 - 2 s_\theta^2) C_1 + (2 s_\alpha^4 s_\theta^3 - s_\alpha^2 s_\theta^3 + s_\theta s_\alpha^2 c_\alpha^2) C_2 \Big] \, , \\
\tilde{g}_{\theta\theta} &= R^2 \Big[s_\alpha^2 + \frac{w^2 R^4}{2} \Big[s_\alpha^2 - 2 s_\alpha^4 s_\theta^2 + 2 s_\alpha^4 s_\theta^4 + c_\alpha s_\alpha^3 s_\theta (s_\alpha^2 s_\theta^2 + s_\theta^2 - s_\alpha^2) C_3 - c_\alpha^3 s_\alpha^3 s_\theta c_\theta^3 C_{2,1} \\
&\qquad - c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 C_{3,1} - c_\alpha s_\alpha^3 c_\theta (s_\theta^2 s_\alpha^2 - c_\theta^2) C_2 + (s_\alpha^6 + s_\alpha^4 - 2 s_\alpha^2) c_\theta s_\theta C_1 - s_\alpha^6 c_\theta s_\theta C_{3,2} \Big] \Big] \, , \\
\tilde{g}_{\alpha\phi_1} &= \frac{w^2 R^6}{2} \Big[c_\alpha^2 s_\alpha^2 s_\theta c_\theta^2 S_3 + c_\alpha^2 s_\alpha^2 s_\theta^3 S_{2,1} + c_\alpha (1 + c_\alpha^2) s_\alpha c_\theta s_\theta S_1 + c_\alpha^2 s_\alpha^2 c_\theta^3 S_{3,1} + c_\alpha^2 s_\alpha^2 c_\theta s_\theta^2 S_2 \Big] \, , \\
\tilde{g}_{\alpha\phi_2} &= \frac{w^2 R^6}{2} \Big[- s_\alpha^4 c_\theta s_\theta^4 S_2 - c_\alpha s_\alpha^3 c_\theta s_\theta^3 S_1 + s_\alpha^2 c_\alpha^2 s_\theta^3 S_{2,1} - s_\alpha^2 c_\alpha^2 c_\theta s_\theta^2 S_2 + c_\alpha^3 s_\alpha c_\theta s_\theta S_1 - s_\alpha^4 c_\theta^4 s_\theta S_3 \Big] \, , \\
\tilde{g}_{\alpha\phi_3} &= \frac{w^2 R^6}{2} \Big[- s_\alpha^4 c_\theta s_\theta^4 S_2 - s_\alpha^4 c_\theta^4 s_\theta S_3 - c_\alpha s_\alpha^3 c_\theta^3 s_\theta S_1 - s_\alpha^2 c_\alpha^2 c_\theta^2 s_\theta S_3 + c_\alpha^3 s_\alpha c_\theta s_\theta S_1 + s_\alpha^2 c_\alpha^2 c_\theta^3 S_{3,1} \Big] \, ,
\end{aligned}$$

The w-deformed background

$$\tilde{g}_{\theta\phi_1} = \frac{w^2 R^6}{2}\Big[c_\alpha^3 s_\alpha^3 s_\theta^3 S_{3,1} + c_\alpha s_\alpha^3 s_\theta (c_\alpha^2 c_\theta^2 - s_\alpha^2 s_\theta^2) S_2 - c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_3 + c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 S_{2,1}$$
$$-2c_\alpha^4 s_\alpha^2 s_\theta^2 S_1 - c_\alpha^3 s_\alpha^3 s_\theta S_{3,1} + c_\alpha s_\alpha^5 c_\theta S_3 + c_\alpha^4 s_\alpha^2 S_1\Big] \ ,$$

$$\tilde{g}_{\theta\phi_2} = \frac{w^2 R^6}{2}\Big[-c_\alpha s_\alpha^3 s_\theta (2s_\alpha^2 s_\theta^2 + c_\theta^2 - s_\alpha^2 s_\theta^4) S_2 - s_\alpha^2 s_\theta^2 (c_\alpha^2 - s_\alpha^2 s_\theta^2 c_\alpha^2) S_1$$
$$-c_\alpha s_\alpha^5 c_\theta s_\theta^4 S_3 + c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 S_{2,1} - s_\alpha^6 s_\theta^2 c_\theta^2 S_{3,2}\Big] \ ,$$

$$\tilde{g}_{\theta\phi_3} = \frac{w^2 R^6}{2}\Big[c_\alpha s_\alpha^5 c_\theta^4 s_\theta S_2 - c_\alpha^3 s_\alpha^3 c_\theta^2 s_\theta S_{3,1} - c_\alpha s_\alpha^5 c_\theta^5 S_3 - s_\alpha^4 c_\alpha^2 c_\theta^4 S_1 - s_\alpha^6 c_\theta^2 s_\theta^2 S_{3,2}$$
$$-c_\alpha s_\alpha^3 (1 - 2s_\alpha^2) c_\theta^3 S_3 + s_\alpha^2 c_\alpha^2 c_\theta^2 S_1 + c_\alpha s_\alpha^3 c_\theta S_3\Big] \ ,$$

$$\tilde{g}_{\phi_1\phi_1} = R^2\Big[c_\alpha^2 + \frac{w^2 R^4}{2}\Big[-c_\alpha^3 s_\alpha^3 c_\theta^2 s_\theta C_3 + c_\alpha^3 s_\alpha^3 s_\theta^3 C_{2,1} - (c_\alpha^2 s_\alpha^4 + 6c_\alpha^4 s_\alpha^2) c_\theta s_\theta C_1$$
$$+c_\alpha^3 s_\alpha^3 c_\theta^3 C_{3,1} - c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 C_2 + c_\alpha^2 s_\alpha^4 + c_\alpha^4 s_\alpha^2 + 2c_\alpha^6\Big]\Big] \ ,$$

$$\tilde{g}_{\phi_1\phi_2} = -\frac{w^2 R^6}{2}\Big[c_\alpha^4 s_\alpha^2 s_\theta^2 + c_\alpha^2 s_\alpha^4 s_\theta^4 - 2c_\alpha^2 s_\alpha^4 c_\theta^2 s_\theta^2 + (c_\alpha s_\alpha^5 c_\theta s_\theta^4 - c_\alpha^3 s_\alpha^3 s_\theta^2 c_\theta) C_2$$
$$+(c_\alpha^4 s_\alpha^2 c_\theta s_\theta - c_\alpha^2 s_\alpha^4 c_\theta s_\theta^3) C_1 - c_\alpha^3 s_\alpha^3 s_\theta^3 C_{2,1} + c_\alpha s_\alpha^5 c_\theta^4 s_\theta C_3\Big] \ ,$$

$$\tilde{g}_{\phi_1\phi_3} = -\frac{w^2 R^6}{2}\Big[c_\alpha s_\alpha^5 c_\theta s_\theta^4 C_2 - 2c_\alpha^2 s_\alpha^4 c_\theta^2 s_\theta^2 + c_\alpha s_\alpha^5 c_\theta^4 s_\theta C_3 - c_\alpha^2 s_\alpha^4 c_\theta^3 s_\theta C_1 - c_\alpha^3 s_\alpha^3 c_\theta^2 s_\theta C_3$$
$$+c_\alpha^4 s_\alpha^2 c_\theta s_\theta C_1 + c_\alpha^2 s_\alpha^4 c_\theta^4 - c_\alpha^3 s_\alpha^3 c_\theta^3 C_{3,1} + c_\alpha^4 s_\alpha^2 c_\theta^2\Big] \ ,$$

$$\tilde{g}_{\phi_2\phi_2} = R^2\Big[s_\alpha^2 s_\theta^2 + \frac{w^2 R^4}{2}\Big[s_\alpha^2 s_\theta^2 (1 - s_\alpha^2 s_\theta^2 + 2s_\alpha^4 s_\theta^4) + s_\alpha^6 c_\theta^3 s_\theta^3 C_{3,2} - c_\alpha s_\alpha^5 s_\theta^3 c_\theta^2 C_3$$
$$-c_\alpha^2 s_\alpha^4 c_\theta s_\theta^3 C_1 + c_\alpha^3 s_\alpha^3 s_\theta^3 C_{2,1} - c_\alpha s_\alpha^3 c_\theta s_\theta^2 (6s_\alpha^2 s_\theta^2 + s_\alpha^2 c_\theta^2 + c_\alpha^2) C_2\Big]\Big] \ ,$$

The w-deformed background

$$\tilde{g}_{\phi_2\phi_3} = \frac{w^2R^6}{2}\Big[2c_\alpha^2 s_\alpha^4 c_\theta^2 s_\theta^2 - s_\alpha^6 c_\theta^2 s_\theta^2 + c_\alpha s_\alpha^5 c_\theta s_\theta^2(c_\theta^2 - s_\theta^2)C_2 + s_\alpha^6 c_\theta^3 s_\theta^3 C_{3,2}$$
$$+c_\alpha s_\alpha^5 c_\theta^2 s_\theta(s_\theta^2 - c_\theta^2)C_3 - c_\alpha^4 s_\alpha^2 c_\theta s_\theta C_1\Big] ,$$

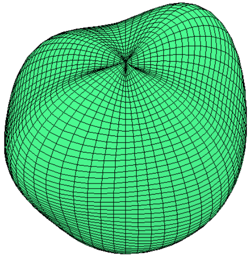
$$\tilde{g}_{\phi_3\phi_3} = R^2\Big[s_\alpha^2 c_\theta^2 + \frac{w^2R^4}{2}\Big[s_\alpha^2 c_\theta^2(1 - s_\alpha^2 c_\theta^2 + 2s_\alpha^4 c_\theta^4) + s_\alpha^6 c_\theta^3 s_\theta^3 C_{3,2} - s_\alpha^4 c_\alpha^2 c_\theta^3 s_\theta C_1$$
$$-5c_\alpha s_\alpha^5 c_\theta^4 s_\theta C_3 - c_\alpha s_\alpha^3 c_\theta^2 s_\theta C_3 - c_\alpha s_\alpha^5 c_\theta^3 s_\theta^2 C_2 + c_\alpha^3 s_\alpha^3 c_\theta^3 C_{3,1}\Big]\Big] .$$

$$B_{\alpha\theta} = wR^4G\left(s_\alpha^2 s_\theta S_2 + s_\alpha^2 c_\theta S_3 + c_\alpha s_\alpha S_1\right)/2 ,$$
$$B_{\alpha\phi_1} = wR^4G\left(2c_\alpha s_\alpha s_\theta^2 - c_\alpha s_\alpha\right)/2 ,$$
$$B_{\alpha\phi_2} = wR^4G\left(s_\alpha^2 c_\theta s_\theta^2 C_2 + c_\alpha s_\alpha s_\theta^2 - s_\alpha^2 c_\theta^2 s_\theta C_3 - c_\alpha s_\alpha c_\theta s_\theta C_1\right)/2 ,$$
$$B_{\alpha\phi_3} = wR^4G\left(s_\alpha^2 c_\theta s_\theta^2 C_2 + c_\alpha s_\alpha s_\theta c_\theta C_1 - s_\alpha^2 c_\theta^2 s_\theta C_3 - c_\alpha s_\alpha c_\theta^2\right)/2 ,$$
$$B_{\theta\phi_1} = wR^4G\left(2c_\alpha^2 s_\alpha^2 c_\theta s_\theta - c_\alpha s_\alpha^3 s_\theta C_2 - c_\alpha s_\alpha^3 c_\theta C_3 + c_\alpha^2 s_\alpha^2 c_1\right)/2 ,$$
$$B_{\theta\phi_2} = wR^4G\left(-c_\alpha s_\alpha^3 s_\theta^3 C_2 + c_\alpha s_\alpha^3 c_\theta s_\theta^2 C_3 + s_\alpha^2 c_\alpha^2 s_\theta^2 C_1 + s_\alpha^2(1 - 2s_\alpha^2)c_\theta s_\theta\right)/2 ,$$
$$B_{\theta\phi_3} = wR^4G\left(c_\alpha s_\alpha^3 c_\theta^2 s_\theta C_2 + s_\alpha^2(1 - 2s_\alpha^2)c_\theta s_\theta - c_\alpha s_\alpha^3 c_\theta^3 C_3 + s_\alpha^2 c_\alpha^2 c_\theta^2 C_1\right)/2 ,$$
$$B_{\phi_1\phi_2} = wR^4G\left(-c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_2 + c_\alpha s_\alpha^3 c_\theta^2 s_\theta S_3 - c_\alpha^2 s_\alpha^2 c_\theta s_\theta S_1\right)/2 ,$$
$$B_{\phi_1\phi_3} = wR^4G\left(-c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_2 + c_\alpha s_\alpha^3 c_\theta^2 s_\theta S_3 + c_\alpha^2 s_\alpha^2 c_\theta s_\theta S_1\right)/2 ,$$
$$B_{\phi_2\phi_3} = wR^4G\left(-c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_2 - c_\alpha s_\alpha^3 c_\theta^2 s_\theta S_3 + c_\alpha^2 s_\alpha^2 c_\theta s_\theta S_1\right)/2 .$$

+ RR fields

The w-deformed background

- Diffeomorphic to the LM background — only appears to be more complicated
- Deformed sphere, depending on angles $\alpha, \theta, \phi_1, \phi_2, \phi_3$
- Visualise by taking $\phi_i = 0$ and plotting the scalar curvature as a function of α, θ. Here for $wR = 0.3$



Summary

- Hopf algebra structure underlying the general Leigh–Strassler deformation
- The $\mathrm{SU}(3) \times \mathrm{U}(1)$ R–symmetry of $\mathcal{N} = 4$ SYM is not broken, just q–deformed
- This quantum symmetry appears at the level of the classical Lagrangian
- Also a symmetry of the 1–loop spin chain Hamiltonian
- Focused on an integrable deformation: $q = 1 + w$, $h = w$
- Superficially more complicated than $q = e^{i\beta}, h = 0$, though unitarily equivalent
- Exhibited the star product leading to this deformation
- Showed how it can be applied on the supergravity side to obtain the dual $\mathcal{N} = 1$ background

Outlook

- Extend to non-integrable cases $\longrightarrow$ non-associativity?
- Could the appropriate structure be a quasi-Hopf algebra?
  [Drinfel'd '89], [Mack, Schomerus '92]
  - Non-associative, quasi-Hopf QYBE

$$R_{12}\Phi_{312}R_{13}\Phi^{-1}_{132}R_{23}\Phi_{123} = \Phi_{321}R_{23}\Phi^{-1}_{231}R_{13}\Phi_{213}R_{12}$$

- Construction of more general dual backgrounds
- Relation to other solution generating techniques (CYBE, λ-deformations)
- Higher loops? Non-planar?
- Relation between quantum symmetry and perturbative finiteness ?

Details of the star product

- Choose basis z^i, $i = 1, 2, 3$, use index notation

$$z^i \star z^j = (F^{-1})^{j\ i}_{l\ k} z^l z^k$$

- Quadratic products for w-deformation

$$z^1 \star z^2 = z^1 z^2 - \frac{w}{1+w}(z^3)^2 + \frac{w^2}{(1+w)^2} z^1 z^2 = \left(1 + \frac{w^2}{(1+w)^2}\right) z^1 z^2 - \frac{w}{1+w}(z^3)^2$$

$$z^2 \star z^1 = z^1 z^2 - \frac{w}{1+w} z^1 z^2 + \frac{w^2}{(1+w)^2}(z^3)^2 = \left(1 - \frac{w}{1+w}\right) z^1 z^2 + \frac{w^2}{(1+w)^2}(z^3)^2$$

$$z^3 \star z^3 = (z^3)^2 - \frac{w}{1+w} z^1 z^2 = \frac{w^2}{(1+w)^2} z^1 z^2 = -\frac{w}{1+w}\left(1 - \frac{w}{1+w}\right) z^1 z^2 + (z^3)^2$$

- Cubic products — careful with associativity!

$$(z^1 \star z^2) \star z^3 = (F^{-1})^{32}_{k'j'} (F^{-1})^{k'1}_{ni'} (F^{-1})^{j'i'}_{ml} z^l z^m z^n$$

$$z^1 \star (z^2 \star z^3) = (F^{-1})^{21}_{j'i'} (F^{-1})^{3i'}_{k'l} (F^{-1})^{k'j'}_{nm} z^l z^m z^n \ .$$

- Integrable twist $\Longrightarrow$ associative product

$$z^1 \star z^2 \star z^3 = (F^{-1})^{21}_{j'i'} (F^{-1})^{3i'}_{k'l} (F^{-1})^{k'j'}_{nm} z^l z^m z^n \ .$$

More details of the star product

- By explicit computation

$$F_{12}^{-1}F_{13}^{-1}F_{23}^{-1} = \frac{1+w}{(1+2w)^2(1+w+w^2)}\big[(1+4w+5w^2+w^3)I\otimes I\otimes I$$
$$+ w^2(1+w)(I\otimes U\otimes V + U\otimes V\otimes I + U\otimes I\otimes V + U\otimes U\otimes U + V\otimes V\otimes V)$$
$$- w(1+3w+3w^2)(I\otimes V\otimes U + V\otimes U\otimes I) - \frac{w(1+3w+2w^2-w^3)}{1+w}V\otimes I\otimes U\big]$$

- Apply to $\mathcal{N}=4$ superfields

$$\Phi^1\star\Phi^2\star\Phi^3+\Phi^2\star\Phi^3\star\Phi^1+\Phi^3\star\Phi^1\star\Phi^2-(1+w)\left(\Phi^1\star\Phi^3\star\Phi^2+\Phi^3\star\Phi^2\star\Phi^1+\Phi^2\star\Phi^1\star\Phi^3\right)$$
$$+w\left(\Phi^1\star\Phi^1\star\Phi^1+\Phi^2\star\Phi^2\star\Phi^2+\Phi^3\star\Phi^3\star\Phi^3\right)$$
$$=(1+2w)\left[\Phi^1\Phi^2\Phi^3+\Phi^2\Phi^3\Phi^1+\Phi^3\Phi^1\Phi^2-\Phi^1\Phi^3\Phi^2-\Phi^3\Phi^2\Phi^1-\Phi^2\Phi^1\Phi^3\right]$$

- Take gauge theory trace

$$\kappa\mathrm{Tr}\left(\Phi^1\star\Phi^2\star\Phi^3-(1+w)\Phi^1\star\Phi^3\star\Phi^2+\frac{w}{3}\left((\Phi^1)^3_\star+(\Phi^2)^3_\star+(\Phi^3)^3_\star\right)\right)=\mathrm{Tr}\left(\Phi^1[\Phi^2,\Phi^3]\right)$$