Matrix Quantum Mechanics

and the $S^1/\mathbb{Z}_2$ orbifold

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Motivation: Cosmology in string theory

- Is it possible to understand (resolve) spacelike cosmological singularities in string theory?
- Is there a set of possible consistent initial conditions of the Universe?
- What happens to spacetime as one approaches the singularity? Are there any more fundamental physical notions behind space and time?
- Many works in the past related to these fundamental problems but no conclusive answer...

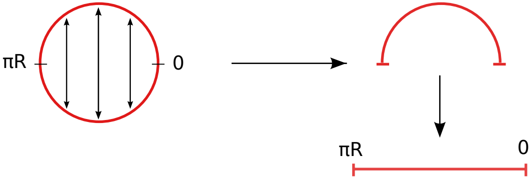
We will be more modest and try to ask some of these deep questions in the toy model of 2D string theory/$c = 1$ Matrix Quantum Mechanics where concrete calculations are possible

Reasonable motivation: Toy model of a Bang-Crunch Universe

- The main idea is to model a 2D Big Bang - Big Crunch universe using Matrix Quantum Mechanics.

- One can relate Matrix Quantum Mechanics (MQM) to 2D string theory. We will soon describe the emergence of an extra spacelike dimension ϕ through the eigenvalues of the Matrix.

- We will try to model the Bang-Crunch Universe using an $S^1/\mathbb{Z}_2$ orbifold of Euclidean time with a subsequent analytic continuation. The identification is: $X^0 \sim -X^0$ and $X^0 \sim X^0 + 2\pi R$.



- In string theory we get extra twisted states at the orbifold fixed points. These might be related to the states at the Big-Bang and Big-Crunch singularities of the 2D toy universe upon analytic continuation of Euclidean time.

What is Matrix Quantum Mechanics

Klebanov[9108019v2]

- MQM (gauged) is a $0+1$ dimensional quantum mechanical theory of $N \times N$ Hermitian matrices $M(t)$ and a non dynamical gauge field $A(t)$.
- The Path Integral is:

$$e^{-iW} = \int \mathcal{D}M\mathcal{D}A \exp\left[-iN\int_{t_{in}}^{t_f} dt \mathrm{Tr}\left(\frac{1}{2}\left(D_t M\right)^2 + \frac{1}{2}M^2 - \frac{\kappa}{3!}M^3 + \ldots\right)\right]$$

- One can diagonalise M by a unitary transformation $M(t) = U(t)\Lambda(t)U^\dagger(t)$ where $\Lambda(t)$ is diagonal and $U(t)$ unitary.
- One then picks up a Jacobian from the path integral measure $(\forall t)$

$$\mathcal{D}M = \mathcal{D}U \prod_{i=1}^{N} d\lambda_i \Delta^2(\Lambda), \quad \Delta(\Lambda) = \prod_{i<j}^{N}(\lambda_i - \lambda_j)$$

- This Vandermonde determinant will make the wavefunctions fermionic.
- We will return to the Path Integral in a while, let us now discuss the Hamiltonian picture.

Fermionic Well

Set the gauge field to zero - impose the gauss-law constraint $\delta S/\delta A = i[M, \dot{M}] = 0$ (singlet sector projection).

- The Hamiltonian is:

$$H = -\frac{1}{2\Delta^2(\lambda)}\frac{d}{d\lambda_i}\Delta^2(\lambda)\frac{d}{d\lambda_i} + \sum_{i<j}\frac{\Pi_{ij}\Pi_{ji}}{(\lambda_i - \lambda_j)^2} + V(\lambda_i) \ ,$$

  Constraint $\Rightarrow \Pi_{ij} = 0$ on physical states (momenta of $SU(N)$ rotations)

- Upon rescaling $\lambda \to \frac{\sqrt{N}}{\kappa}\lambda$ and redefining the wavefunction as $\tilde{\Psi}(\lambda) \equiv \Delta(\lambda)\Psi(\lambda)$, the Schrödinger equation now reads:

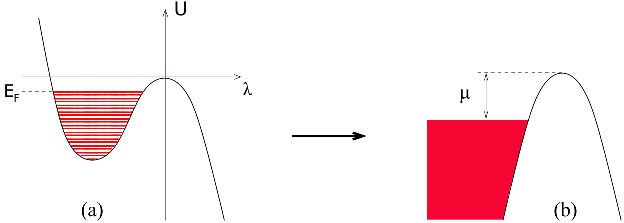
$$\left(-\frac{1}{2}\frac{d^2}{d\lambda_i^2} - \frac{1}{2}\lambda_i^2 + \frac{\sqrt{\hbar}}{3!}\lambda_i^3 + \ldots\right)\tilde{\Psi}(\lambda) = \hbar^{-1}E\tilde{\Psi}(\lambda), \qquad \hbar^{-1} = \frac{N}{\kappa^2}$$

- This describes N non interacting fermions in the potential $V(\lambda)$.

Connection with string theory- Double scaling limit

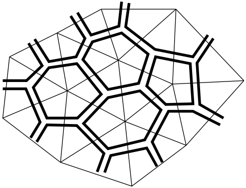
[Kazakov, Migdal...]

- Consider an initial state where the energy states are populated up to some Fermi energy E_F below the top of the barrier, and send $\hbar \to 0$, $N \to \infty$, such that $E_F \to 0$.
- The eigenvalues are about to spill over the top of the potential barrier.
- Enough to focus on the quadratic maximum of the potential. We hold $\mu/\hbar = -E_F/\hbar$ fixed in the limit.
- The result is quantum mechanics of free fermions in an inverted harmonic oscillator potential, with Fermi level $-\mu < 0$.
- At this limit the model is perturbatively stable in $1/N \to 0$ expansion, since by WKB we can see that the tunneling probability is exponentially suppressed.



Non-critical String theory

- The continuum limit of the matrix path integral is in the same universality class as the $2D$ string theory defined via the worldsheet path integral for Liouville theory coupled to $c = 1$.
  - The double scaling limit produces smooth surfaces while at the same time keeping all higher genera. It is defined by $\hbar, \mu \to 0$ as we discussed, while keeping $\frac{\mu}{\hbar} = g_{st}^{-1}$ fixed.



- The string theory at play is called NCST or c=1 Liouville theory. It can also be interpreted as a 2D critical string theory in a linear dilaton background.
- It contains D_0 branes whose excitations are a "Tachyon" and a 1-d gauge field. MQM describes their dynamics. [McGreevy, Verlinde]
- The double scaling limit is the "analogue" of the Maldacena decoupling limit. The matrix eigenvalues λ are related with the coordinate ϕ. [Seiberg]

$c = 1$ Liouville - (NCST) and 2d Critical String Theory

[Nakayama:0402009v7]

- Gauge fix only the worldsheet diffeos and keep the conformal mode of the metric dynamical. $h_{ab} = e^{\phi(\sigma)} \hat{h_{ab}}$.
- $\mathcal{D}h$ is not invariant under $h_{ab} \to e^{\rho(\sigma)} h_{ab}$.
- Exponentiating the conformal anomaly from the measure, the total action becomes

$$S_{CFT} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{h}} \left[\hat{h}^{ab} (\partial_a X^\mu \partial_b X_\mu + \partial_a \phi \partial_b \phi) + Q\hat{R}\phi + \mu e^{\gamma\phi} \right] + ghosts$$

- $Q = \sqrt{\frac{25-d}{6}}$. This is a conformal theory under the simultaneous transformation $h_{ab} \to e^{\rho(\sigma)} h_{ab}$, $\phi(\sigma) \to \phi(\sigma) - \rho(\sigma)$ iff $\gamma = -\frac{1}{6} \left(\sqrt{25-d} - \sqrt{1-d} \right)$ which is real up to $d = c = 1$.
- Consider the arbitrary background sigma model

$$S = \frac{1}{4\pi} \int d^2\sigma \sqrt{h} \left[h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \Lambda T(X) + R\Phi(X) \right]$$

- The two actions match upon identifying $G_{\mu\nu} = \eta_{\mu\nu}$, $\Phi = 2\phi$, $\Lambda T(\phi) = \mu e^{\gamma\phi}$ and $X^0, X^1 = \phi$ are the two coordinates.
- This linear dilaton background is an exact CFT background.

Let us now move on to the Orbifold.

$S^1/\mathbb{Z}_2$ - Liouville computation

We will now discuss the torus contribution to the partition function computed from Liouville.

- The modular partition function has the following form:

$$Z_{orb}\left(R,\tau\right)=\frac{1}{2}Z_{cir}\left(R,\tau\right)+\left\{\left|\frac{\eta\left(\tau\right)}{\theta_{00}\left(0,\tau\right)}\right|+\left|\frac{\eta\left(\tau\right)}{\theta_{01}\left(0,\tau\right)}\right|+\left|\frac{\eta\left(\tau\right)}{\theta_{10}\left(0,\tau\right)}\right|\right\}$$

- The full torus partition function comes from coupling the above with the ghost and the Liouville modes and integrating over the torus moduli τ.

$$\mathcal{Z}_{orb}\left(R\right)=\frac{V_\phi}{2}\int d^2\,\tau\left(\frac{\left|\eta\left(\tau\right)\right|^4}{2\tau_2}\right)\left(2\pi\sqrt{2\tau_2}\left|\eta\left(\tau\right)\right|^2\right)^{-1}Z_{orb}\left(R,\tau\right)$$

- Performing the integration one gets:

$$\mathcal{Z}_{orb}\left(R\right)=\frac{1}{2}\mathcal{Z}_{circ}\left(R\right)+C,\quad \mathcal{Z}_{circ}\left(R\right)=-\frac{1}{24}\left(R+\frac{1}{R}\right)\log\mu$$

- Using the fact that $Z_{orb}\left(R=1,\tau\right)=Z_{cir}\left(R=2,\tau\right)$ `[Ginsparg]`, we finally get

$$Z_{orb}=-\frac{1}{48}\left(R+\frac{1}{R}\right)\log\mu-\frac{1}{16}\log\mu$$

$S^1/\mathbb{Z}_2$ - MQM

- To implement orbifolding on the matrix model we gauge the Z_2 reflection symmetry $t \leftrightarrow -t$ by combining it with a Z_2 subgroup of the gauge group [Ramgoolam, Waldram, Gürsoy, Liu]:

$$\Omega = \begin{pmatrix} -1_{n\times n} & 0 \\ 0 & 1_{(N-n)\times(N-n)} \end{pmatrix} * \quad with, \; *f(t) = f(-t)*, \;\; 0 \leq n \leq \frac{N}{2}$$

and then requiring:

$$\Omega A(t) \Omega^{-1} = -A(t), \quad \Omega M(t) \Omega^{-1} = M(t)$$

- This naturally splits the matrices into (even/odd) blocks under $t \to -t$ that satisfy different boundary conditions.

$$M(t) = \begin{pmatrix} even & odd \\ odd & even \end{pmatrix} \quad A(t) = \begin{pmatrix} odd & even \\ even & odd \end{pmatrix}$$

- The breaking $U(N) \to U(n) \times U(N-n)$ at the endpoints, will lead to two separate sets of n and $N-n$ fermions.

MQM Path Integrals

- We study MQM in an inverted harmonic oscillator (H.O.) potential that captures the universal physics in the double scaling limit.
- We define $K^N(\lambda_i;\lambda'_j,\beta) = \langle \lambda'_j, \beta | \lambda_i, 0\rangle$ the oscillator propagator between the N eigenvalues.
- S^1 partition function ($M'(\beta) = M(0)$)

$$Z = \int \mathcal{D}M(0)\mathcal{D}U \langle U M(0) U^\dagger | M(0)\rangle = \frac{1}{N!}\int \prod_{i=1}^{N} d\lambda_i \det K^N(\lambda_i;\lambda_j)$$

- Orbifold for generic n (where $\prod_{i=1}^n dx_i \equiv d^n x$, $\overline{x} = (x,y)$)

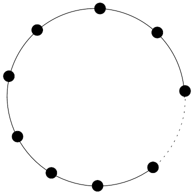
$$Z \sim \int d^n x d^{N-n} y d^n x' d^{N-n} y' \frac{\Delta(x)\Delta(y)}{\prod_{i,j}(x_i - y_j)} \det K(\overline{x}_i;\overline{x}'_j) \frac{\Delta(x')\Delta(y')}{\prod_{i,j}(x'_i - y'_j)}$$

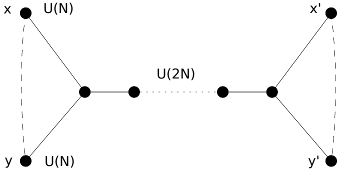
  We get two kinds of fermions (with $q_i = \pm 1$ charge - or spin).
- This structure admits a natural analytic continuation to a transition amplitude

$$\langle \Psi_{out}; T | \Psi_{in}\rangle = \int d\overline{x} d\overline{x}' \Psi_{out}(\overline{x}') \det\left[\langle \overline{x}'_j; T | \overline{x}_i\rangle\right] \Psi_{in}(\overline{x})$$

Orbifold Partition function ($n = N/2$ is special!)

- Wavefunctions of the form $\prod_{i,j}(\lambda_i - \lambda_j)^{q_i q_j}$ have arisen in studies of Quiver/Super Matrix Models. [Dijkgraaf, Vafa, Kostov..]
- The most special case is $n = N/2$ (regular representation). The wavefunctions are square integrable as $\lambda_i \to \infty$ in this case. (Zero -charge condition of the 1-d Coulomb gas).
- The breaking of $U(N) \to U(N/2) \times U(N/2)$ at the end points is also encountered in studies of $\hat{D}_m$ quiver matrix models. [Kostov, Moriyama..]
- Our description is implementing the same gauge symmetry breaking at the end-points and holds for arbitrary radius.





Matrix models defined on the $\hat{A}_m$ quiver: Integrand is a determinant. [Marino..]

Matrix models defined on the $\hat{D}_m$ quiver: Integrand is a Pfaffian. [Moriyama..]

Orbifold Partition function

Taking inspiration of this we use the Cauchy identity

$$\frac{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}{\prod_{i,j}(x_i - y_j)} = \det \frac{1}{x_i - y_j},$$

then

$$Z \sim \int d^n \overline{x} d^n \overline{x}' \det_{n\times n}\left(\frac{1}{x_i - y_j}\right) \det_{2n\times 2n}\begin{pmatrix} K^n(x_i, x_j') & K^n(x_i, y_j') \\ K^n(y_i, x_j') & K^n(y_i, y_j') \end{pmatrix} \det_{n\times n}\left(\frac{1}{x_i' - y_j'}\right)$$

We integrate over (y, y') [Andreief, Moriyama] (Determinant Id).
To perform the x' integrations one uses a formula by de Bruijn(Pfaffian Id) to get

$$Z \sim (-1)^{n+\frac{1}{2}(n-1)n} \int d^n x \,\mathrm{pf}\, P, \quad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

$$P_{11} = -(K \circ N \bullet K + K \bullet N \circ K), \qquad P_{12} = (K \circ N \bullet K + K \bullet N \circ K) \bullet M,$$
$$P_{21} = -M \bullet (K \circ N \bullet K + K \bullet N \circ K), \quad P_{22} = M \bullet (K \circ N \bullet K + K \bullet N \circ K) \bullet M.$$

where K the H.O. propagator, $\bullet$ stands for either the y integration or the y' integration distinguished by $M(x_i, y) = \frac{1}{x_i - y}$, $N(y', x_j') = \frac{1}{y' - x_j'}$.

Introduce the chemical potential μ (to perform the double scaling).

The Grand Free Energy is (Borodin, Ishikawa)

$$e^{J_o(\beta,\mu)} = \sum_{n=0}^{\infty} z^n \int \frac{d^n x}{n!} (-1)^{\frac{1}{2}(n-1)n}\, \mathrm{pf}\, P = \mathrm{pf}(\overline{\Omega} + z\hat{P}) = \sqrt{\det\big(\overline{I} - z\,\hat{\rho}\big)},$$

with $\overline{\Omega} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \overline{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \hat{\rho} = \overline{\Omega}\,\hat{P}, \quad z = -e^{\beta\mu}$.

On the right-hand we have simultaneously the 2×2 and Fredholm determinant of the kernel $\hat{\rho}$.

It is instructive to compare this result with the one obtained for the circle which is just a Fredholm determinant of the H.O. propagator as kernel

$$e^{J_c(\beta,\mu)} = \det\left(I + z\hat{K}\right) = \prod_{n=0}^{\infty}\left(1 + zq^{n+\frac{1}{2}}\right)$$

with $q = e^{-\beta\omega}$ and $\hat{K} = e^{-\beta\hat{H}}$.

Kernel

Let us analyse the kernel. If we define the bi-local operator $\langle x|\hat{\mathcal{O}}|y\rangle = \frac{1}{x-y}$ the kernel is written as

$$\hat{\rho} = \begin{pmatrix} -\hat{\mathcal{O}}e^{-\frac{\beta}{2}\hat{H}}\hat{\mathcal{O}}e^{-\frac{\beta}{2}\hat{H}} & \hat{\mathcal{O}}e^{-\frac{\beta}{2}\hat{H}}\hat{\mathcal{O}}e^{-\frac{\beta}{2}\hat{H}}\hat{\mathcal{O}} \\ e^{-\frac{\beta}{2}\hat{H}}\hat{\mathcal{O}}e^{-\frac{\beta}{2}\hat{H}} & -e^{-\frac{\beta}{2}\hat{H}}\hat{\mathcal{O}}e^{-\frac{\beta}{2}\hat{H}}\hat{\mathcal{O}} \end{pmatrix},$$

Using the Mehler resolution $\langle x|\hat{K}|x'\rangle = \sum_{n=0}^{\infty} q^{n+\frac{1}{2}}\psi_n(\sqrt{\omega}x)\psi_n^{\star}(\sqrt{\omega}x')$ with $\psi_n(\sqrt{\omega}x) = \mathcal{N}e^{-\frac{\omega}{2}x^2}H_n(\sqrt{\omega}x)$ and H_n the Hermite polynomials, we find that (principal value prescription)

$$\langle x|\hat{\mathcal{O}}|\psi_n\rangle = \mathcal{P}\int_{-\infty}^{\infty} dy \frac{\psi_n(y)}{x-y}$$

- The effect of the bi-local operator acting at the endpoints is a Hilbert-transform of the H.O. wavefunctions. $\mathcal{O}$ relates odd with even modes in the energy basis.
- Remember that we further need to send $\omega \to i\omega$ in order to discuss the inverse H.O. This has the effect of turning Hermite functions into Parabolic cylinder functions.

Alternative description

Diagonalise U and integrate out M's in the previous expression

$$Z = \int \mathcal{D}M \mathcal{D}M' \mathcal{D}U \langle UM'U^\dagger, \beta | M, 0 \rangle = \int \mathcal{D}U\, I(U)$$

We define $A = 1/\tanh(\omega\beta)$, $B = 1/\sinh(\omega\beta)$. The measure for $\mathcal{D}U$ can be found by defining the metric on the tangent space of the group $ds^2 = \mathrm{tr}(UdU^\dagger UdU^\dagger)$ and then computing its determinant to get

$$J = \prod_{i<j}^{n} \sin^2(\theta_i - \theta_j) \sin^2(\theta_i + \theta_j) \prod_{i=1}^{n} \sin 2\theta_i \sin^{2(N-2n)} \theta_i$$

Again we see that $n = \frac{N}{2}$ is special. By defining $q = e^{-\omega\beta}$ and $x_i = e^{i\theta_i}$ and using Schur's Pfaffian identity

$$Z_n = \frac{1}{n!} \int_0^\pi \prod_{k=1}^{n} d\theta_k \prod_{k=1}^{n} \frac{1}{\sqrt{(\cosh(\beta) - \cos(2\theta_k))}}\, \mathrm{pf} \begin{pmatrix} \frac{q^{1/2}(x_i - x_j)}{1 - q x_i x_j} & \frac{q^{1/2}(x_i - x_j^*)}{1 - q x_i x_j^*} \\ \frac{q^{1/2}(x_i^* - x_j)}{1 - q x_i^* x_j} & \frac{q^{1/2}(x_i^* - x_j^*)}{1 - q x_i^* x_j^*} \end{pmatrix}$$

Kernel in the new variables

- We can again pass to the grand canonical ensemble $e^{-J_o} = \sqrt{\det(1 + z\hat{\rho})}$.
- It is useful to write the action of $\hat{\rho}$ to functions $X(\theta)$.

$$\hat{\rho}\left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right](\theta) = \int_0^\pi d\mu(\theta') \begin{pmatrix} \rho_{11}(\theta,\theta') & \rho_{11}(\theta,-\theta') \\ -\rho_{11}(-\theta,\theta') & -\rho_{11}(-\theta,-\theta') \end{pmatrix} \begin{pmatrix} X_1(\theta') \\ X_2(\theta') \end{pmatrix}$$

$$\rho_{11}(\theta,\theta') = \frac{1}{q^{-1/2}e^{i\theta} - q^{1/2}e^{i\theta'}} + \frac{1}{q^{1/2}e^{-i\theta} - q^{-1/2}e^{-i\theta'}},$$

and $d\mu(\theta') = \frac{q d\theta'}{\sqrt{(1-qe^{i2\theta'})(1-qe^{-i2\theta'})}}$.

- The square root in the measure introduces branch cuts in the complex $x' = e^{i\theta'}$ plane.
- The integration is from 0 to π so that the matrix entries have support in the lower/upper half plane.

Properties of the kernel

- For the orbifold we can also write the action of the kernel

$$\hat{\rho}[\vec{X}](x) = \oint \frac{dx'}{\pi i x'} e^{-\sum_n \frac{q^n}{2n} x^n - \sum_n \frac{q^n}{2n} x^{-n}} \rho(\sqrt{x}, \sqrt{x'}) \vec{X}(x')$$

- The measure thus can be thought of as containing Vortex perturbations with $t_n = t_{-n} = q^n/2n,\ 1 \leq n$.
- The matrix piece of the kernel now contains terms of the form $\frac{1}{q^{-\frac{1}{2}}\sqrt{x} - q^{\frac{1}{2}}\sqrt{x'}}$.
- To be able to identify a Hierarchy one should thus make contact with fermionic correlators containing square-roots.
- One can use Jacobi-elliptic functions to rewrite the kernel. This simplifies the measure completely. [Baxter, Zamolodchikov...]. The kernel is naturally defined on a torus.

Summary of the Results

- We implemented orbifolding in the Matrix model and passed on to grand-canonical ensemble with two independent methods. The physics is encoded in an integral kernel.

- The trace of the kernel is found to be the same by both methods:

$$\mathrm{tr}\,\hat{\rho} = \frac{1}{\sinh\frac{\beta}{2}}\arctan\left(\frac{1}{\sinh\frac{\beta}{2}}\right) = \int d\epsilon\,[\rho_{HO}(\epsilon) + \rho_{twist}(\epsilon)]\,e^{-\beta\epsilon}$$

  From this we have managed to read-off the density of states that contains an untwisted $\rho_{HO}(\epsilon)$ and a twisted piece $\rho_{twist}(\epsilon)$.

- Non-trivial physics is encoded either in the Hilbert transform operators at the end of time (which commute with the $SL(2,\mathbb{R})$ generators of fractional linear transformations of the matrix eigenvalues), or in the branch-cuts/ torus monodromy (which has an $SL(2,\mathbb{Z})$ symmetry).

- The kernel has a very interesting 2×2 matrix structure. It can also be thought of as containing Vortex perturbations of arbitrary order (Higher spins? [Sen, Mukhi, Pakman...]).

Large radius results

- We have also matched the result for the twisted states computed by Liouville theory
- This contribution can be isolated by considering the limit $\beta \to \infty$. This limit reduces the free energy to $\beta\mathcal{F} = \beta E_{ground}/2 + \Theta$, where the β-independent constant piece Θ in this expression is the twisted state contribution that arises from the wavefunctions at the end-points.
- For $n = N/2$ and the normal oscillator the radius independent term looks like "entropy"

$$\Theta = \frac{1}{2}\,\mathrm{tr}\log\left(\mathcal{O}^2\right) = N\log 2 \approx \frac{1}{2}\,\mathrm{tr}\log\rho_{twisted}$$

where $\mathcal{O}^2$ near the diagonal tends to

$$\mathcal{O}^2(n_1, n_2) \to \frac{2\sin(\pi(n_1 - n_2))}{\pi(n_1 - n_2)}$$

(and is non zero only when $n_1 = n_2$ for $n \in \mathbb{Z}$).

- This is the famous sine-kernel which is the analogue of the Dirac-δ for time/frequency banded functions.

Large radius results - Sine kernel

(Parabolic cylinder)

- We studied also the delta-function normalised even and odd parabolic cylinder functions.
- Using identities and integrals involving ${}_1F_1$, ${}_2F_1$ we computed $\langle \epsilon_1 | \mathcal{O} | \epsilon_2 \rangle$ to be

$$\mathcal{O}(\epsilon_1, \epsilon_2) = N(\epsilon_1) N(\epsilon_2) \frac{\sinh\left(\pi(\epsilon_2 - \epsilon_1)\right)}{\pi(\epsilon_1 - \epsilon_2)} .$$

- The spectrum is continuous, we obtain a discrete one by putting a cutoff/wall at Λ which is then send to infinity.
- The different normalisation contributes non-perturbatively $\sim e^{-A\mu}$ in $1/\mu$. It also reflects the branch structure of the ${}_2F_1$. Here we will be interested only for the asymptotic expansion in $g_{st} \sim 1/\mu$.
- By reinstating the oscillator frequency ω, one can now rotate between the normal and the inverted oscillator up to such ambiguities.
- The full partition function depends only on the combination $\omega\beta \Rightarrow$ Link between Euclidean and Lorentzian description

Sine kernel

- The Fredholm determinant of the sine kernel gives the universal level spacing distribution of random matrices in the bulk of the spectrum.
- The determinant we are after corresponds to the probability that all the energy eigenvalues lie outside $(-\mu, 0)$ and thus form the Fermi sea.
- The determinant of this kernel can be calculated in a $1/\mu$ expansion. [Dyson '76]

$$\Theta = -\frac{1}{32}\mu^2 - \frac{1}{16}\log\mu + \frac{1}{48}\log 2 + \frac{3}{4}\zeta'(-1) + O\left(\frac{1}{\mu^{2m}}\right).$$

- The twisted state contribution to the torus level partition function $-\frac{1}{16}\log\mu$ matches precisely the world-sheet result.
- This is a non-trivial check of the duality we propose between the $n = N/2$ representation of the orbifold matrix quantum mechanics and the 2D non-critical string theory on $S_1/\mathbb{Z}_2$.

Future Directions

- Solve the integral equation for the spectrum/find exact partition function to all genera for the orbifold.
- What do different n's correspond to ?
- Supersymmetric case (0B - 0A)?
- Is there any underlying integrable structure? (BKP-DKP [Orlov])
- Connect with target space physics. What is the initial state in collective field theory language? (Fermi sea with folds?)
- What is the form of the semi-classical metric near the end-points in time? Role of $SL(2,R)$ symmetry?
- Analytic continuation of the result in time?
- T-duality? It relates the universe with the black hole ($\frac{SL(2,R)}{U(1)}$ coset) [Kiritsis], [Giveon, Porrati, Rabinovici], [Vafa, Tseytlin]
- Lots of non-perturbative physics to be understood!

Thank you!

Double scaling

- 't Hooft limit- $g_s \to 0$, $N \to \infty$, $t = g_s N$ fixed in

$$F \sim \sum_h F_h(t) g_s^{2h-2} \tag{1}$$

- Double scaling for $c < 1$ (Matrix models)

$$F \sim \sum_h F_h \left[N(g - g_c)^{(2-\gamma)/2} \right]^{2-2h} = \sum_h F_h g_s^{2h-2} \tag{2}$$

  Thus send $N \to \infty$, $g \to g_c$, g_s fixed.

- Double scaling for $c = 1$. We define the cosmological constant $\Delta \sim \kappa_c^2 - \kappa^2$ and $\hbar^{-1} = N/\kappa^2$.
- We are after the non-analytic piece of $\mathcal{F}(\Delta)$ which gives surfaces with a diverging number of triangles.
- One introduces the density of states $\rho(\epsilon) = \hbar \sum_k \delta(\epsilon - \epsilon_k)$ and sets the chemical potential μ. One finds

$$\kappa^2 = \hbar N = \int_{-\infty}^{\infty} \rho(\epsilon) \frac{1}{1 + e^{2\pi R \hbar^{-1}(\epsilon - \mu)}} d\epsilon \tag{3}$$

Double scaling

- We also have the following equations

$$\frac{\partial \mathcal{F}}{\partial \Delta} = \hbar^{-1}\mu, \quad \mathcal{F}_G(\mu) = \hbar^{-1}\Delta\mu - \mathcal{F}(\Delta) \tag{4}$$

- One finds

$$\frac{\partial^2 \mathcal{F}_G}{\partial \mu^2} = \frac{\partial \Delta}{\partial \mu} = \int_{-\infty}^{\infty} d\epsilon \rho(\epsilon) \frac{\pi R \hbar^{-1}}{2\cosh^2\left(\pi R \hbar^{-1}(\mu - \epsilon)\right)} \tag{5}$$

- For the H.O. we have

$$\rho(\epsilon) = \frac{1}{\pi}\Re\Psi(i\epsilon + \frac{1}{2}) = \frac{1}{\pi}\Re\int_0^{\infty} dt \frac{e^{-i\epsilon t}}{2\sinh\frac{t}{2}} \tag{6}$$

- The idea is to determine $\mathcal{F}_G$ as a series in $\hbar^{-1}\mu$ and then go back to find $\mathcal{F}(\Delta)$.
- It is convenient to use $\Delta = \mu_0|\log\mu_0|$. One can also find $\mu(\mu_0)$ and then compute $\mathcal{F}(\mu_0)$.
- Note that all these are asymptotic expansion manipulations.

Genus expansions- finite radius

- For the circle of finite radius we get

$$\frac{\partial^2 \mathcal{F}_G}{\partial \mu^2} = \int_0^\infty dt \int_{-\infty}^{\infty} d\epsilon \frac{e^{-i\epsilon t}}{2\sinh\frac{t}{2}} \frac{\pi R \hbar^{-1}}{2\cosh^2\left(\pi R\hbar^{-1}(\mu-\epsilon)\right)} \tag{7}$$

$$= \frac{1}{\pi}\Im \int_0^\infty dt e^{-i\hbar^{-1}\mu t} \frac{1}{2\sinh t/2} \frac{t/2R}{\sinh t/2R} \tag{8}$$

The most convenient expression to work with is

$$\frac{\partial^3 \mathcal{F}_G}{\partial \mu^3} = \frac{1}{\pi}\Im \int_0^\infty dt e^{-it} \frac{t/2\hbar^{-1}\mu}{\sinh(t/2\hbar^{-1}\mu)} \frac{t/2R\hbar^{-1}\mu}{\sinh(t/2R\hbar^{-1}\mu)} \tag{9}$$

which one expands in $1/\hbar^{-1}\mu$ and then performs the integrals term by term. In the end we get

$$\mathcal{F}(\mu_0, R) = -\frac{R}{2}\mu_0^2 \log\mu_0 - \frac{1}{24}\left(R + \frac{1}{R}\right)\log\mu_0 + R\sum_{k=2}^{\infty} \mu_0^{-2k+2} f_k(R) \tag{10}$$

which is T-dual under $R \to 1/R$, $\mu_0 \to R\mu_0$.

- The effective string theory coupling is now $g_{eff}(R) = 1/(\mu_0\sqrt{R})$.

- One can take the $R \to \infty$ limit of the previous expression or directly consider

$$E_0 = \int^{\mu} \rho(\epsilon)\epsilon d\epsilon, \quad (11)$$

expand in powers of $1/\mu$, integrate term by term and change variables to get

$$-E_0 = \frac{1}{8\pi}\left[(2\mu_0)^2 \log \mu_0 - \frac{1}{3}\log \mu_0 + \sum_{n=0}^{\infty} \frac{2^{(2n+1}-1)|B_{2n+2}|}{n(n+1)(2n+1)}(2\mu_0)^{-2n}\right] \quad (12$$

the terms diverge as $(2n+2)!$ - closed string factorial growth. This is correct up to non-perturbative terms.

DOS

- We found the following partition function from the 1-particle kernel:

$$Z_o(\omega,\beta) = \frac{1}{2\sinh(\omega\beta/2)} \arctan\left(\frac{1}{\sinh(\omega\beta/2)}\right) = \frac{1}{2}\int_0^\pi d\theta \frac{\cos(\theta/2)}{\cosh(\omega\beta) - \cos(\theta} \tag{13}$$

The analogous formula for the circle is:

$$Z_c(\omega,\beta) = \frac{1}{2\sinh(\omega\beta/2)} = \frac{1}{2\pi}\int_0^\pi d\theta \frac{1}{\cosh(\omega\beta/2) - \cos(\theta)} \tag{14}$$

To discuss the inverted H.O. one needs to set $\omega = i$ in these formulas.

- We can write this in terms of twisted partition function and dos [Boulatov, Kazakov]

$$Z_o(\omega = i,\beta) = \int_0^\pi d\theta \cos(\theta/2) Z(\theta,\beta) = \int_0^\pi d\theta \cos(\theta/2) \int_{-\infty}^{+\infty} d\epsilon\, \mathrm{e}^{-\beta\epsilon} \rho(\theta,\epsilon) \tag{15}$$

with

$$\rho(\theta,\epsilon) = \sum_{m=-\infty}^{\infty} e^{im\theta}\rho^{(m)}(\epsilon) = \frac{1}{\pi}\sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty} \frac{e^{im\theta}(\frac{|m|+1}{2}+k)}{(\epsilon)^2 + (k+\frac{|m|+1}{2})^2} + \delta(\theta)\log\tilde{\Lambda}^2 \tag{16}$$

DOS

- We see that the circle and the orbifold differ by different averaging.
- For the circle one finds that the $m = 0$ contribution only survives

$$\rho_c(\epsilon) = \frac{1}{\pi} \Re \Psi(i\epsilon + \frac{1}{2}) \tag{17}$$

- For the orbifold we get

$$\rho_o(\epsilon) = \rho_c(\epsilon) + \frac{1}{\sinh(\pi\epsilon)} \left(\Im\, \Psi \left(\frac{1}{2} i\epsilon + \frac{1}{4} \right) - \Im\, \Psi \left(\frac{1}{2} i\epsilon + \frac{3}{4} \right) \right) \tag{18}$$

with Ψ the digamma function.

- This also looks similar to the contribution coming from the diagonal piece of the full kernel in the energy basis.
- With this the Free energy can be written as

$$\mathcal{F} = \frac{1}{2} \int_{-\infty}^{\infty} d\epsilon \rho_o(\epsilon) \log \left(1 + e^{\beta(\mu - \epsilon)} \right) \tag{19}$$

which is the starting point for the genus expansion computation.

Alternative description

There is an alternative description of the PF ⇒ sanity check.

- Diagonalise U and integrate out M's in the previous expression

$$Z = \int \mathcal{D}M\mathcal{D}M'\mathcal{D}U \langle UM'U^\dagger, \beta | M, 0\rangle = \int \mathcal{D}U I(U)$$

- We define $A = 1/\tanh(\omega\beta),\ B = 1/\sinh(\omega\beta)$, and remember to use blocks for the matrices after orbifolding we get

$$I(U) = \omega^{-\frac{1}{2}(N-2n)^2} \left(\frac{B}{2\pi}\right)^{N^2/2} \int dM_1 dM_2 dM_1' dM_2' e^T, \quad U = \begin{pmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{pmatrix}$$

$$T = -\frac{A}{2}\mathrm{tr}(M_1^2 + M_1^{2\prime}) + B\mathrm{tr}(M_1 U_1 M_1' U_1^\dagger + M_1 U_{12} M_2' U_{12}^\dagger) + (1 \leftrightarrow 2).$$

- The $U's$ are complex but can be diagonalised by bi-unitary transformations that leave the measure invariant.

$$U = \begin{pmatrix} \delta_{ij}\cos\theta_i & -\delta_{ij}\sin\theta_i \\ \delta_{ij}\sin\theta_i & \delta_{ij}\cos\theta_i \end{pmatrix},\ 0 \leq \theta_i \leq \frac{\pi}{2},$$

(or just exponentiate the zero mode of the gauge field!).

Orbifold Partition function

In terms of angles

Integrate out M's, to find

$$I = \left(\frac{B}{\omega}\right)^{\frac{(N-2n)^2}{2}} \prod_i^n \left(\frac{B^2}{1+B^2\sin^2\theta_i}\right)^{N-2n} \prod_{i,j}^n \left(\frac{B^4}{(1+B^2\sin^2(\theta_i+\theta_j)(1+B^2\sin^2(\theta_i-\theta_j)}\right)^{\frac{1}{2}}$$

The measure for $\mathcal{D}U$ can be found by defining the metric on the tangent space of the group $ds^2 = \mathrm{tr}(UdU^\dagger UdU^\dagger)$ and then computing its determinant to get

$$J = \prod_{i<j}^n \sin^2(\theta_i-\theta_j)\sin^2(\theta_i+\theta_j)\prod_{i=1}^n \sin 2\theta_i \sin^{2(N-2n)}\theta_i$$

Again we see that $n = \frac{N}{2}$ is special. In this case we find

$$Z_n = \frac{1}{n!}\int_0^\pi \prod_i^n d\theta_i J \prod_{i,j}\left(\frac{1}{(\cosh\beta-\cos(\theta_i+\theta_j)(\cosh\beta+\cos(\theta_i-\theta_j)}\right)^{\frac{1}{2}}.$$

Still this looks quite daunting!

Schur's Pfaffian

There is still a way forward. One unfolds the denominator using for example

$$\frac{1}{\cosh\beta - \cos(\theta_i + \theta_j)} \sim \frac{q}{(1 - q x_i x_j)(1 - q x_i^* x_j^*)}, \quad q = e^{-\omega\beta},\ x_i = e^{i\theta_i}$$

and similarly the measure

$$J = \prod_{i<j}^{n} (x_i - x_j)(x_i - x_j^*)(x_i^* - x_j)(x_i^* - x_j^*) \prod_{k}^{n} (x_k - x_k^*)$$

[

noframenumbering]Then one uses Schur's Pfaffian identity

$$\mathrm{pf}\left(\frac{x_i - x_j}{1 - x_i x_j}\right)_{1\le i,j\le 2n} = \prod_{i<j}^{2n} \frac{x_i - x_j}{1 - x_i x_j}$$

to compactly write

$$Z_n = \frac{1}{n!}\int_0^{\pi} \prod_{k=1}^{n} d\theta_k \prod_{k=1}^{n} \frac{1}{\sqrt{(\cosh(\beta) - \cos(2\theta_k))}}\, \mathrm{pf}\begin{pmatrix} \frac{q^{1/2}(x_i - x_j)}{1 - q x_i x_j} & \frac{q^{1/2}(x_i - x_j^*)}{1 - q x_i x_j^*} \\ \frac{q^{1/2}(x_i^* - x_j)}{1 - q x_i^* x_j} & \frac{q^{1/2}(x_i^* - x_j^*)}{1 - q x_i^* x_j^*} \end{pmatrix}$$

Comparison with the circle and 2d Black Hole

- For the S^1 the kernel is $(x = e^{i\theta})$

$$\hat{K}_c[f](x) = -\oint \frac{dx'}{2\pi i} \frac{f(x')}{q^{1/2}x - q^{-1/2}x'}$$

The eigenfunctions are polynomials x^n with eigenvalues $q^{n+\frac{1}{2}}$.

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- The 2d BH was described as a correlator of two Polyakov lines/winding modes that induce vortex perturbations $(\mathrm{tr}(U^n) = e^{in\theta} = x^n)$ [Kazakov, Kostov, Kutasov]

$$\hat{K}_{BH}[f](x) = -\oint \frac{dx'}{2\pi i} \frac{e^{u(x)+u(x')}}{q^{1/2}x - q^{-1/2}x'} f(x'), \quad u(x) = \sum_{n\neq 0} t_n x^n$$

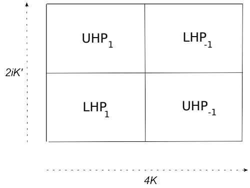
  and restricting to t_1, t_{-1}. In this case the partition function was identified as a τ function of the Toda Hierarchy. This arose from realising that the denominator is a free fermion correlator

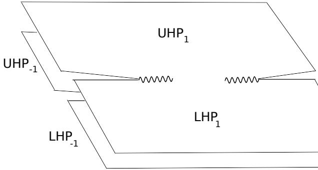
$$\langle \psi(q^{\frac{1}{2}}x)\psi(q^{-\frac{1}{2}}x')\rangle = \frac{1}{(q^{\frac{1}{2}}x - q^{-\frac{1}{2}}x')}$$

  and $t_{\pm 1}$ can be identified as Toda "times". This realization gives differential equations for the partition function with respect to t's.

Elliptic parametrization

The measure is very reminiscent of Elliptic integrals. In fact in the $x' = e^{i\theta'}$ plane we find that the measure $d\mu(\theta')$ has branch points at $\pm q^{-\frac{1}{2}}, \pm q^{\frac{1}{2}}$.





The domain is a two-sheeted Riemann surface.

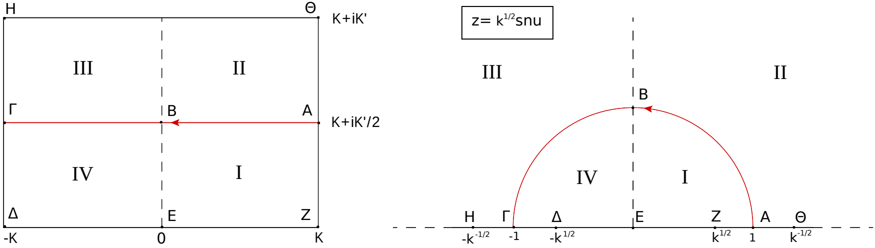
- We make the elliptic substitution $e^{i\theta'} = q^{\frac{1}{2}} \operatorname{sn}(v, q)$ with sn, Jacobi's elliptic sine and $q = k$ the elliptic modulus. The measure becomes $d\mu(\theta') \to -q^{\frac{1}{2}} dv$!
- Jacobi's sine has a double periodicity $\operatorname{sn}(4mK + 2inK' + u) = \operatorname{sn}(u)$. The functions are now naturally defined on a torus.
- We are only left with the matrix piece of the kernel which is

$$\rho(u, v) = \begin{pmatrix} \rho_{11}(u, v) & \rho_{11}(u, v + iK') \\ -\rho_{11}(u + iK', v) & -\rho_{11}(u + iK', v + iK') \end{pmatrix},$$

$$\rho_{11}(u, v) = \frac{1 - k \operatorname{sn} u \operatorname{sn} v}{\operatorname{sn} u - k \operatorname{sn} v}$$

The Torus

This is the mapping of the contour of integration.



The integral equation thus becomes

$$\lambda \begin{pmatrix} X_1(u) \\ X_2(u) \end{pmatrix} = -k^{\frac{1}{2}} \int_{K+iK'/2}^{-K+iK'/2} dv \rho(u,v) \begin{pmatrix} X_1(v) \\ X_2(v) \end{pmatrix}$$

One finds the following consistency condition $X_1(u) + X_2(u - iK') = 0$.

This gets rid of the matrix structure for a single equation.

$$\lambda X(u) = -k^{\frac{1}{2}} \left[\int_{K+iK'/2}^{-K+iK'/2} + \int_{-K-iK'/2}^{K-iK'/2} \right] dv \rho_{11}(u,v) X(v)$$

MQM Path Integral

We will now study MQM in an inverted harmonic oscillator (H.O.) potential that captures the universal physics in the double scaling limit.

- The (Euclidean) propagator for a matrix harmonic oscillator is

$$\langle M', \beta | M, 0 \rangle = \left(\frac{\omega}{2\pi \sinh \omega\beta} \right)^{N^2/2} e^{-\frac{\omega}{2\sinh\omega\beta}\left[\left(\mathrm{Tr}M^2 + \mathrm{Tr}M'^2\right)\cosh\omega\beta - 2\mathrm{Tr}MM'\right]}$$

- One can also show that

$$\int_{M(0)}^{M'(\beta)} \mathcal{D}A\mathcal{D}M e^{-\int_0^\beta d\tau \mathrm{Tr}\frac{1}{2}(D_\tau M)^2 - V(M)} = \int_{U(N)} \mathcal{D}U \langle UM'U^\dagger, \beta | M, 0 \rangle$$

- This is useful because one can use the Harish-Chandra-Itzykson-Zuber integral

$$\int_{U(N)} DU \exp\left(g \mathrm{Tr} MUM'U^\dagger\right) = \prod_{n=1}^{N-1} (p!)\ g^{-\frac{1}{2}N(N-1)} \frac{\det e^{g\lambda_i\lambda_j'}}{\Delta(\lambda)\Delta(\lambda')}$$

- This will help us to reduce the path integral to eigenvalues ($N^2 \to N$ dofs)

Orbifold Partition function

Matrix models defined on the $\hat{A}_m$ quiver: Integrand is a determinant.

[Marino..]

Matrix models defined on the $\hat{D}_m$ quiver: Integrand is a Pfaffian.

[Moriyama..]

Taking inspiration of this we use the Cauchy identity

$$\frac{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}{\prod_{i,j}(x_i - y_j)} = \det \frac{1}{x_i - y_j},$$

then

$$Z \sim \int d^n\overline{x} d^n\overline{x}' \det_{n\times n}\left(\frac{1}{x_i - y_j}\right) \det_{2n\times 2n}\begin{pmatrix} K^n(x_i, x_j') & K^n(x_i, y_j') \\ K^n(y_i, x_j') & K^n(y_i, y_j') \end{pmatrix} \det_{n\times n}\left(\frac{1}{x_i' - y_j'}\right)$$

Now integrate over (y, y') using the following lemma [Andreief, Moriyama]:

$$\frac{1}{n!}\int \prod_{k=1}^{n} dx_k \cdot \det\left[(\phi_i(x_k))_{\substack{1\le i\le n+r \\ 1\le k\le n}} \ (\zeta_{iq})_{\substack{1\le i\le n+r \\ 1\le q\le r}}\right] \cdot \det(\psi_j(x_k))_{1\le j,k\le n}$$

$$= \det\left[(m_{ij})_{\substack{1\le i\le n+r \\ 1\le j\le n}} \ (\zeta_{iq})_{\substack{1\le i\le n+r \\ 1\le q\le r}}\right], \qquad m_{ij} = \int dx \phi_i(x)\psi_j(x).$$

Orbifold Partition function

We obtain

$$Z \sim (-1)^n \int d^n x d^n x' \det \begin{pmatrix} K(x_i, x'_j) & (K \bullet N)(x_i, x'_j) \\ (M \bullet K)(x_i, x'_j) & (M \bullet K \bullet N)(x_i, x'_j) \end{pmatrix},$$

where $\bullet$ stands for either the y integration or the y' integration distinguished by $M(x_i, y) = \frac{1}{x_i - y}$, $N(y', x'_j) = \frac{1}{y' - x'_j}$.

To perform the x' integrations one uses a formula by de Bruijn

$$\int \frac{d^N x}{N!} \det \begin{pmatrix} (\phi_a(x_i))_{\substack{1 \le a \le 2N \\ 1 \le i \le N}} & (\psi_a(x_i))_{\substack{1 \le a \le 2N \\ 1 \le i \le N}} \end{pmatrix} = (-1)^{\frac{1}{2}(N-1)N} \operatorname{pf} P_{ab},$$

with $P_{ab} = \int dx(\phi_a(x)\psi_b(x) - \phi_b(x)\psi_a(x))$ skew symmetric, to get

$$Z \sim (-1)^{n + \frac{1}{2}(n-1)n} \int d^n x \operatorname{pf} P, \quad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

$$P_{11} = -(K \circ N \bullet K + K \bullet N \circ K), \qquad P_{12} = (K \circ N \bullet K + K \bullet N \circ K) \bullet M,$$
$$P_{21} = -M \bullet (K \circ N \bullet K + K \bullet N \circ K), \quad P_{22} = M \bullet (K \circ N \bullet K + K \bullet N \circ K) \bullet M.$$