Off shell $(4, q)$ Supersymmetric Sigma models

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$(4,1)$ and $(4,0)$

$$\phi^i : \Sigma \to \mathcal{T} \quad : \quad x^\mu \mapsto \phi^i(x)$$

d= 2, The number of supersymmetries $(left, right) = (p, q)$ restrict the target space geometry $(g, H = dB, J^{(A)}, ...)$

Typically (hyper)Kähler with torsion.

$(4,4)$ in $(2,2)$ in *Gates, Hull and Roček* 1984

(p,q) Susy in *Hull and Witten* 1985

$(1,0)$ superspace discussion of the geometry in
Howe and Papadopoulos 1988.

The action we consider in $(2,1)$ superspace is

$$S = \int d^2xd^3\theta \left(k_\alpha D_-\varphi^\alpha + \bar{k}_{\bar{\alpha}} D_-\bar{\varphi}^{\bar{\alpha}}\right) \ .$$

The fields φ^α are $(2,1)$ chiral

$$\bar{\mathbb{D}}_+\varphi^\alpha = 0 \ ,$$

and $\bar{\varphi}^{\bar{\alpha}} = (\varphi^\alpha)^*$.

The 1-form potential $k_\alpha(\varphi,\bar\varphi)$ with $\bar k_{\bar\alpha}=(k_\alpha)^*$, is defined up to

$$k_\alpha(\varphi,\bar\varphi)\to k_\alpha(\varphi,\bar\varphi)+\partial_\alpha h(\varphi,\bar\varphi)+f_\alpha(\varphi)\ .$$

The metric g and B field are

$$\begin{aligned} g_{\alpha\bar\beta} &= i(\partial_\alpha \bar k_{\bar\beta}-\partial_{\bar\beta}k_\alpha)\\ B^{(2,0)}_{\alpha\beta} &= i(\partial_\alpha k_\beta-\partial_\beta k_\alpha)\\ B &= B^{(2,0)}+B^{(0,2)} \end{aligned}$$

(4, 1) Susy:

$$\delta\varphi^{\alpha} = \bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}f^{\alpha}(\varphi,\bar{\varphi})$$
$$\delta\bar{\varphi}^{\bar{\alpha}} = \epsilon^{+}\mathbb{D}_{+}\bar{f}^{\bar{\alpha}}(\varphi,\bar{\varphi})$$

Off-shell closure of the algebra,

$$[\delta_1,\delta_2]\varphi^{\alpha} = 2i\epsilon^{+}_{[2}\bar{\epsilon}^{+}_{1]}\partial_{+\!\!+}\varphi^{\alpha},$$

implies

$$f^{\alpha}_{\ \bar{\beta}}\bar{f}^{\bar{\gamma}}_{\ \alpha} = -\delta^{\bar{\gamma}}_{\ \bar{\beta}}\ , \quad \bar{f}^{\bar{\alpha}}_{\ \beta}f^{\gamma}_{\ \bar{\alpha}} = -\delta^{\gamma}_{\ \beta}\ ,$$
$$f^{\alpha}_{\ [\bar{\alpha}}f^{\beta}_{\ \bar{\beta}]\alpha} = 0\ , \quad \bar{f}^{\bar{\alpha}}_{\ [\alpha}\bar{f}^{\bar{\beta}}_{\ \beta]\bar{\alpha}} = 0\ .$$

Invariance of the action gives

$$f^{\beta}_{\ (\bar\alpha}g_{\bar\gamma)\beta} = 0\ , \quad \Rightarrow f^{\beta}_{\ \bar\alpha}g_{\bar\gamma\beta} = \omega_{\bar\gamma\bar\alpha}\ ,$$

and

$$\nabla^{(+)}_{\bar\beta}\omega_{\bar\alpha\bar\gamma} = 0\ , \quad \nabla^{(+)}_{\beta}\omega_{\bar\alpha\bar\gamma} = 0\ ,$$

$$\Gamma^{(+)} := \Gamma^{(0)} + \tfrac{1}{2}g^{-1}H\ , \quad H = dB.$$

In terms of the complex structures

$$\mathbb{J}^{(1)} = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}, \quad \mathbb{J}^{(2)} = \begin{pmatrix} 0 & f^{\alpha}{}_{\bar{\beta}} \\ \bar{f}^{\bar{\alpha}}{}_{\beta} & 0 \end{pmatrix}, \quad \mathbb{J}^{(3)} = \begin{pmatrix} 0 & if^{\alpha}{}_{\bar{\beta}} \\ -i\bar{f}^{\bar{\alpha}}{}_{\beta} & 0 \end{pmatrix}$$

we have

$$\mathbb{J}^{t(A)} g \mathbb{J}^{(A)} = g \,, \quad \nabla^{(+)} \mathbb{J}^{(A)} = 0 \,, \quad \mathcal{N}(\mathbb{J}^{(A)}) = 0 \,,$$

and

$$\mathbb{J}^{(A)} \mathbb{J}^{(B)} = -\delta^{AB} + \epsilon^{ABC} \mathbb{J}^{(C)} \,.$$

The (4, 1) algebra is

$$\{\mathbb{D}_{+a}, \bar{\mathbb{D}}_{+}^{b}\} = 2i\delta_a^b \partial_{+\!\!+} , \quad a, b, = 1, 2.$$
$$(D_-)^2 = i\partial_= .$$

The (4, 1) multiplet obtained by truncating the (4, 4) multiplet of GHR is

$$\bar{\mathbb{D}}_{+}^{1}\phi = 0 = \mathbb{D}_{+2}\phi , \quad \bar{\mathbb{D}}_{+}^{1}\chi = 0 = \mathbb{D}_{+2}\chi ,$$
$$\bar{\mathbb{D}}_{+}^{2}\chi = -i\bar{\mathbb{D}}_{+}^{1}\bar{\phi} , \quad \bar{\mathbb{D}}_{+}^{2}\phi = i\bar{\mathbb{D}}_{+}^{1}\bar{\chi} .$$

This is an off shell multiplet and gives a geometry with

$$\mathbb{J}^{(A)} = \mathbb{I}^{(A)} \otimes \mathbf{1}_{d\times d}$$

where

$$\mathbb{I}^{(1)} = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}, \ \mathbb{I}^{(2)} = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \ \mathbb{I}^{(3)} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}.$$

The metric and torsion arise from and action. A convenient formulation is via projective superspace.
$\zeta \in \mathbb{CP}^1$

$$\nabla_+ := \mathbb{D}_{+1} + \zeta \mathbb{D}_{+2} \, ,$$
$$\check{\nabla}_+ := \bar{\mathbb{D}}^1_+ - \zeta^{-1} \bar{\mathbb{D}}^2_+ \, ,$$
$$\eta = \sum \eta_m \zeta^m \, , \qquad \nabla_+ \eta = \check{\nabla}_+ \eta = 0 \, ,$$

where

$$S = i \int d^2x \oint_C \frac{d\zeta}{2\pi i \zeta} \overbrace{\Delta_+ \check{\Delta}_+}^{\mathbb{D}_+ \bar{\mathbb{D}}_+} D_- \left(\lambda_i(\eta, \check{\eta}; \zeta) D_- \eta^i - \check{\lambda}_i(\eta, \check{\eta}; \zeta) D_- \check{\eta}^i \right) .$$

For the multiplet introduced,

$$\eta^i = \bar{\phi}^i + \zeta\chi^i$$
$$\breve{\eta}^i = \phi^i - \zeta^{-1}\bar{\chi}^i \ .$$

From the action we read off the one form potential

$$k_{\phi^i} = -\oint_C \frac{d\zeta}{2\pi i\zeta}\breve{\lambda}_i \ , \quad \bar{k}_{\bar{\phi}^i} = \oint_C \frac{d\zeta}{2\pi i\zeta}\lambda_i$$
$$k_{\chi^i} = \oint_C \frac{d\zeta}{2\pi i\zeta}\zeta\lambda_i \ , \quad \bar{k}_{\bar{\chi}^i} = \oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^{-1}\breve{\lambda}_i \ .$$

The geometry is hyperkähler with torsion.

When the model has $(4,2)$ susy, there is a scalar potential K such that the one forms are

$$k_i = K_{,i}$$

Ex:

$$i\lambda = -i\breve{\lambda} = \frac{1}{3!}(\eta + \breve{\eta})^3$$

gives

$$\begin{aligned} g_{\phi\bar{\phi}} &= g_{\chi\bar{\chi}} = (\phi + \bar{\phi})^2 - 2\chi\bar{\chi} \\ B_{\phi\chi} &= -2(\phi + \bar{\phi})\bar{\chi} \ , \\ B_{\bar{\phi}\bar{\chi}} &= -2(\phi + \bar{\phi})\chi \ . \end{aligned}$$

The $(4,0)$ action in $(2,0)$ superspace is

$$S = \int d^2xd^2\theta \left(k_\alpha \partial_= \varphi^\alpha + \bar{k}_{\bar{\alpha}} \partial_= \bar{\varphi}^{\bar{\alpha}} + e_{\mu\nu} \Lambda^\mu_- \Lambda^\nu_- + G_{\mu\bar{\nu}} \Lambda^\mu_- \bar{\Lambda}^{\bar{\nu}}_- + e_{\bar{\mu}\bar{\nu}} \bar{\Lambda}^{\bar{\mu}}_- \bar{\Lambda}^{\bar{\nu}}_- \right) ,$$

$(2,0)$ chiral scalar and fermion superfields

$$\bar{\mathbb{D}}_+ \varphi^\alpha = 0 \ , \quad \bar{\mathbb{D}}_+ \Lambda^\mu_- = 0 \ .$$

$G_{\mu\bar{\nu}}$ fibre metric, $e_{\mu\nu}$ antisymmetric field, both related to a vector bundle with connection $A_i{}^M{}_N$.

$(4, 0)$ susy:

$$\bar{\delta}\varphi^{\alpha} = \bar{\epsilon}^{+}\bar{\mathbb{D}}_{+} f^{\alpha}$$

$$\bar{\delta}\Lambda^{\mu} = \bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}\left(I^{\mu}{}_{N}\Lambda^{N}_{-}\right) \ .$$

Bosonic part: Same as for $(4, 1)$.

Fermionic part:
Closure:

$$I^{\mu}{}_{\bar{\mu}} I^{\bar{\mu}}{}_{\rho} = -\delta^{\mu}_{\rho}$$

$$I^{\mu}{}_{N\bar{\alpha}} \bar{f}^{\bar{\alpha}}_{,\beta} + I^{\mu}{}_{\bar{\mu}} I^{\bar{\mu}}{}_{N\beta} = 0$$

$$I^{\mu}{}_{N\alpha[\bar{\alpha}} f^{\alpha}_{,\bar{\beta}]} + I^{\mu}{}_{\nu[\bar{\alpha}} I^{\nu}{}_{N\bar{\beta}]} = 0 \ ,$$

Invariance:

$$G_{\rho(\bar{\mu}}I^{\rho}{}_{\bar{\nu})} = 0 \ ,$$

$$\begin{aligned}
&e_{\mu\nu,\alpha[\bar{\beta}}f^{\alpha}_{,\bar{\alpha}]} + 2e_{\rho[\nu,[\bar{b}}I^{\rho}{}_{\mu],\bar{\alpha}]} = 0 \\
&f^{\alpha}_{,\bar{\beta}}G_{\nu\bar{\rho},\alpha} + 2e_{\rho\nu,\bar{\beta}}I^{\rho}{}_{\bar{\rho}} + G_{\rho\bar{\rho}}I^{\rho}{}_{\nu\bar{\beta}} = 0 \\
&2f^{\alpha}_{,\bar{\beta}}e_{\bar{\nu}\bar{\rho},\alpha} + G_{\rho\bar{\nu}\bar{\beta}}I^{\rho}{}_{\bar{\rho}} + G_{\rho\bar{\rho}}I^{\rho}{}_{\bar{\nu}\bar{\beta}} = 0 \ .
\end{aligned}$$

Complex structures:

Bosonic sector: $\mathbb{J}^{(A)}$ as for $(4,1)$.

Fermionic:

$$\mathbb{I}^{(2)} = \left(\begin{array}{cc} 0 & I^{\mu}_{\bar{\nu}} \\ \bar{I}^{\bar{\mu}}_{\nu} & 0 \end{array} \right) , \quad \mathbb{I}^{(3)} = \left(\begin{array}{cc} 0 & iI^{\mu}_{\bar{\nu}} \\ -i\bar{I}^{\bar{\mu}}_{\nu} & 0 \end{array} \right) .$$

Again they form an $SU(2)$ algebra together with the canonical complex structure.
Together they also form complex structures on the bundle space.
Integrability and hermiticity of the metrics from closure and invariance.

The $(4,0)$ algebra is

$$\{\mathbb{D}_{+a}, \bar{\mathbb{D}}_+^b\} = 2i\delta_a^b\partial_{\#}, \quad a, b, = 1, 2.$$

The $(4,0)$ multiplet, obtained by truncating the $(4,1)$ is

$$\bar{\mathbb{D}}_+^1\phi = 0 = \mathbb{D}_{+2}\phi, \quad \bar{\mathbb{D}}_+^1\chi = 0 = \mathbb{D}_{+2}\chi,$$
$$\bar{\mathbb{D}}_+^2\chi = -i\bar{\mathbb{D}}_+^1\bar{\phi}, \quad \bar{\mathbb{D}}_+^2\phi = i\bar{\mathbb{D}}_+^1\bar{\chi}.$$

In addition a fermi multiplet $\Lambda_- = (\psi_-, \lambda_-)$ is

$$\bar{\mathbb{D}}^1_+ \psi_- = 0 = \mathbb{D}_{+2}\psi_- \ , \quad \bar{\mathbb{D}}^1_+ \lambda_- = 0 = \mathbb{D}_{+2}\lambda_- \ ,$$
$$\bar{\mathbb{D}}^2_+ \lambda_- = -i\bar{\mathbb{D}}^1_+ \bar{\psi}_- \ , \quad \bar{\mathbb{D}}^2_+ \psi_- = i\bar{\mathbb{D}}^1_+ \bar{\lambda}_- \ .$$

Off shell system with

$$\mathbb{I}^{(1)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes \mathbf{1} \ , \quad \mathbb{I}^{(2)} = \begin{pmatrix} 0 & i\hat{\sigma}_2 \\ i\hat{\sigma}_2 & 0 \end{pmatrix} \ , \quad \mathbb{I}^{(3)} = \begin{pmatrix} 0 & -\hat{\sigma}_2 \\ \hat{\sigma}_2 & 0 \end{pmatrix}$$

where $\mathbf{1} = \mathbf{1}_{d\times d}$ when acting on the bosonic superfields and $\mathbf{1} = \mathbf{1}_{n\times n}$ when acting on the fermionic superfields. Each satisfy the quaternion algebra

$$\mathbb{I}^{(I)}\mathbb{I}^{(J)} = -\delta^{IJ} + \epsilon^{IJK}\mathbb{I}^{(K)} \ .$$

Together they define a quaternionic structure on the total space.

The metrics and torsion and connection arise from and action. A convenient formulation is again via projective superspace adapted to $(2,0)$:
$\zeta \in \mathbb{CP}^1$

$$\nabla_+ := \mathbb{D}_{+1} + \zeta \mathbb{D}_{+2} ,$$
$$\check{\nabla}_+ := \bar{\mathbb{D}}_+^1 - \zeta^{-1} \bar{\mathbb{D}}_+^2 ,$$

We consider $(4,0)$ projective superfields $\eta^i(x,\theta,\zeta), \rho_-^{\underline{a}}(x,\theta,\zeta)$ satisfying

$$\nabla_+ \eta^i = 0 , \quad \check{\nabla}_+ \eta^i = 0 , \quad \nabla_+ \rho_-^{\underline{a}} = 0 , \quad \check{\nabla}_+ \rho_-^{\underline{a}} = 0 ,$$

The action is

$$S = \int d^2x (\mathcal{L}_b + \mathcal{L}_f)$$

with

$$\mathcal{L}_b = i\oint_C \frac{d\zeta}{2\pi i\zeta}\overbrace{\Delta_+\check{\Delta}_+}^{\mathbb{D}_+\bar{\mathbb{D}}_+}\Big(\lambda_i(\eta,\breve{\eta};\zeta)\partial_= \eta^i - \check{\lambda}_i(\eta,\breve{\eta};\zeta)\partial_=\breve{\eta}^i\Big)$$

and

$$\mathcal{L}_f = i\oint_C \frac{d\zeta}{2\pi i\zeta}\overbrace{\Delta_+\check{\Delta}_+}^{\mathbb{D}_+\bar{\mathbb{D}}_+}\Big(\rho_-^a h_{ab}\rho_-^b + \rho_-^a h_{a\bar{b}}\breve{\rho}_-^{\bar{b}} + \breve{\rho}_-^{\bar{a}}\check{h}_{\bar{a}b}\rho_-^b + \breve{\rho}_-^{\bar{a}}\check{h}_{\bar{a}\bar{b}}\breve{\rho}_-^{\bar{b}}\Big)$$

The new multiplet is described by

$$\eta^i = \bar{\phi}^i + \zeta\chi^i\ , \quad \bar{\eta}^i := \breve{\eta}^i = \phi^i - \zeta^{-1}\bar{\chi}^i$$
$$\rho_-^a = \bar{\psi}_-^a + \zeta\lambda_-^a\ , \quad \bar{\rho}_-^a := \breve{\rho}_-^a = \psi_-^a - \zeta^{-1}\bar{\lambda}^a$$

The formulae for the one forms are identical to the $(4,1)$ case.

The fibre tensors are

$$
\begin{aligned}
G_{\psi^a\bar\psi^{\bar b}} &= -\oint_C \frac{d\zeta}{2\pi i\zeta} H_{\bar a b}\ , &\quad G_{\psi^a\bar\lambda^{\bar b}} &= -2\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^{-1} h_{\bar a\bar b}\ ,\\
G_{\lambda^a\bar\psi^{\bar b}} &= 2\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta h_{ab}\ , &\quad G_{\lambda^a\bar\lambda^{\bar b}} &= -\oint_C \frac{d\zeta}{2\pi i\zeta} H_{a\bar b}\ ,\\
e_{\psi^a\psi^b} &= \oint_C \frac{d\zeta}{2\pi i\zeta} h_{\bar a\bar b}\ , &\quad e_{\lambda^a\psi^b} &= \tfrac{1}{2}\oint_C \tfrac{d\zeta}{2\pi i\zeta}\zeta H_{a\bar b}\ ,\\
e_{\lambda^a\lambda^b} &= \oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^2 h_{ab}\ , &\quad e_{\bar\psi^{\bar a}\bar\psi^{\bar b}} &= \oint_C \frac{d\zeta}{2\pi i\zeta} h_{ab}\\
e_{\bar\psi^{\bar a}\bar\lambda^{\bar b}} &= -\tfrac{1}{2}\oint_C \tfrac{d\zeta}{2\pi i\zeta}\zeta^{-1} H_{a\bar b}\ , &\quad e_{\bar\lambda^{\bar a}\bar\lambda^{\bar b}} &= \oint_C \tfrac{d\zeta}{2\pi i\zeta}\zeta^{-2} h_{\bar a\bar b}\ .
\end{aligned}
$$

They satisfy all the conditions for closure and invariance.