To the Memory of Ioannis Bakas

O(d,d) and T-duality

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9th Regional Meeting in String Theory
9-15 July 2017
Crete

13 July 2017

Material based on
GP: arXiv:1402.2586; arXiv:1412.1146
PS Howe and GP arXiv:1612.07968

Main Question

- Can T-duality rules arise from patching conditions?
- If they can, what are the spaces that arise?

Questions prompted from investigations on the global patching of DFT

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Expectations

What are the requirements?

- The final theory should produce both local and global properties of T-duality as described by the Buscher rules
- The theory should exhibit a $O(d,d)$ symmetry
- The patching of theory and its associated space requires for consistency the Dirac quantisation property of the 3-form flux
- The final theory and its associated space satisfies the topological geometrisation condition
- Generalised geometry emerges naturally

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Buscher Rules

Amongst the criteria the first one on Buscher rules is perhaps the most conservative.
Given a common sector background with an isometry $X = \partial_\theta$, the geometry

$$
\begin{aligned}
ds^2 &= V^2(d\theta + q_i dx^i)^2 + g_{ij} dx^i dx^j \ , \\
B &= (d\theta + q_i dx^i) \wedge p_j dx^j + \frac{1}{2} b_{ij} dx^i \wedge dx^j \ .
\end{aligned}
$$

transforms under T-duality to

$$
\begin{aligned}
d\tilde{s}^2 &= V^{-2}(d\tilde{\theta} + p_i dx^i)^2 + g_{ij} dx^i dx^j \ , \\
\tilde{B} &= (d\tilde{\theta} + p_i dx^i) \wedge q_j dx^j + \frac{1}{2} b_{ij} dx^i \wedge dx^j \ , \\
e^{2\tilde{\Phi}} &= e^{2\Phi} V^{-2} \ ,
\end{aligned}
$$

$\tilde{\theta}$ is a new angular coordinate.

Buscher Rules

Spacetime M has coordinates (θ, x^i) while the T-dual $\tilde{M}$ has coordinates $(\tilde{\theta}, x^i)$. $\tilde{\theta}$ is the T-dual coordinates of θ.

- M and $\tilde{M}$ may have different topology and/or geometry

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KK paradigm

Let us review the construction of Kaluza-Klein (KK) space. Consider a closed 2-form F^2, $dF^2 = 0$, and a good cover $\{U_\alpha\}_{\alpha \in I}$ on spacetime M.

Then consider the Čech-de Rham decomposition of F^2.

$$
\begin{aligned}
F^2 &= dA^1_\alpha\ , \quad U_\alpha \\
-A^1_\alpha + A^1_\beta &= da^0_{\alpha\beta}\ , \quad U_{\alpha\beta} = U_\alpha \cap U_\beta \\
a^0_{\beta\gamma} - a^0_{\alpha\gamma} + a^0_{\alpha\beta} &= 2\pi n_{\alpha\beta\gamma}\ , \quad U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma
\end{aligned}
$$

If all $n_{\alpha\beta\gamma} \in \mathbb{Z}$, then $\frac{1}{2\pi}[F^2] \in H^2(M, \mathbb{Z})$

KK space

The construction of KK space $\hat{M}$ can be done by introducing an angular coordinate θ_α at every open set U_α and imposing the patching conditions

$$-\theta_\alpha + \theta_\beta = a^0_{\alpha\beta} \mod 2\pi\mathbb{Z}$$

Consistency at $U_{\alpha\beta\gamma}$ requires that

$$a^0_{\beta\gamma} - a^0_{\alpha\gamma} + a^0_{\alpha\beta} = 0 \mod 2\pi\mathbb{Z}$$

which is satisfied iff $n_{\alpha\beta\gamma} \in \mathbb{Z}$ and so $\frac{1}{2\pi}[F^2] \in H^2(M, \mathbb{Z})$.
$\hat{M}$ is a circle bundle over M with curvature F^2 and $c_1(\hat{M}) = \frac{1}{2\pi}[F^2]$.

KK space

Some properties are

- The construction of KK space requires the Dirac quantisation condition as $\frac{1}{2\pi}[F^2] \in H^2(M, \mathbb{Z})$
- KK space satisfies the topological geometrisation condition

$$-\theta_\alpha + \theta_\beta = a^0_{\alpha\beta} \mod 2\pi\mathbb{Z} \Longrightarrow d\theta_\alpha - A^1_\alpha = d\theta_\beta - A^1_\beta$$

and so $d\theta - A^1$ is globally defined on $\hat{M}$. Moreover

$$F^2 = -d(d\theta - A^1)$$

and so F^2 is exact on $\hat{M}$.

Čech-de Rham

The Čech-de Rham decomposition of a NS-NS closed 3-form flux H^3, $dH^3 = 0$, is

$$
\begin{aligned}
H^3_\alpha &= dB^2_\alpha\,, \quad U_\alpha \\
-B^2_\alpha + B^2_\beta \equiv (\delta B^2)_{\alpha\beta} &= da^1_{\alpha\beta}\,, \quad U_{\alpha\beta} \\
a^1_{\beta\gamma} - a^1_{\alpha\gamma} + a^1_{\alpha\beta} \equiv (\delta a^1)_{\alpha\beta\gamma} &= da^0_{\alpha\beta\gamma}\,, \quad U_{\alpha\beta\gamma} \\
a^0_{\beta\gamma\delta} - a^0_{\alpha\gamma\delta} + a^0_{\alpha\beta\delta} - a^0_{\alpha\beta\gamma} \equiv (\delta a^0)_{\alpha\beta\gamma\delta} &= 2\pi n_{\alpha\beta\gamma\delta}\,, \quad U_{\alpha\beta\gamma\delta}
\end{aligned}
$$

δ is the Čech cohomology differential, $\delta^2 = 0$.
Again if $n_{\alpha\beta\gamma\delta} \in \mathbb{Z}$, then $\frac{1}{2\pi}[H^3] \in H^3(M, \mathbb{Z})$.

C-spaces

Let H^3 be a closed 3-form with transition function $a^1_{\alpha\beta}$, $a^0_{\alpha\beta\gamma}$ and $n_{\alpha\beta\gamma\delta}$. Introduce coordinates y^1_α and angular coordinates $\theta_{\alpha\beta}$ and impose the patching conditions

$$\begin{aligned}(\delta y^1)_{\alpha\beta} + d\theta_{\alpha\beta} &= a^1_{\alpha\beta}\ , \quad U_{\alpha\beta} \\ (\delta\theta)_{\alpha\beta\gamma} &= a^0_{\alpha\beta\gamma}\ \mathrm{mod} 2\pi\mathbb{Z}\ , \quad U_{\alpha\beta\gamma}\end{aligned}$$

where $(\delta y^1)_{\alpha\beta} = -y^1_\alpha + y^1_\beta$.

- The compatibility of the first condition of triple overlaps is implied by the second
- The compatibility of the second on 4-fold overlaps requires that $n_{\alpha\beta\gamma\delta} \in \mathbb{Z}$ and so $\frac{1}{2\pi}[H^3] \in H^3(M,\mathbb{Z})$
- C-spaces are independent from the choice of representative for $\frac{1}{2\pi}[H^3]$

Alternative patching

Suppose that the additional coordinates $\tilde{x}$ are patched as

$$\delta\tilde{x}^1_{\alpha\beta} = a^1_{\alpha\beta}$$

Then

$$\delta a^1_{\alpha\beta\gamma} = 0$$

$a^1_{\alpha\beta}$ is a Čech cocycle.

- If $\delta a^1 = 0$, then H is exact

If $\delta a^1 = 0$, then

$$\tilde{B}^2_\alpha = B^2_\alpha + d(\sum_\gamma \rho_\gamma a^1_{\alpha\gamma})$$

is a globally defined 2-form

$$-\tilde{B}^2_\alpha + \tilde{B}^2_\beta = da^1_{\alpha\beta} - d(\sum_\gamma \rho_\gamma(a^1_{\alpha\gamma} - a^1_{\beta\gamma})) = da^1_{\beta\gamma} - d(\sum_\gamma \rho_\gamma a^1_{\alpha\beta}) = 0$$

But $H^3 = dB^2 = d\tilde{B}^2$ and so it is exact. $\{\rho_\alpha\}_{\alpha\in I}$ partition of unity.

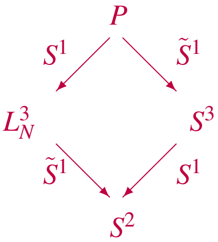
Other patching

Another suggestion is

$$\delta\tilde{x}^1_{\alpha\beta} = 0$$

Consider the T-dual pair S^3 with N-units of H-charge and L^3_N with 1-unit of $\tilde{H}$ charge. Arises after using the Buscher rules with isometry along the Hopf fibre of S^3.

- The Hopf fibre of S^3 twists over the Lens space L^3_N



To investigate this note that the cohomology of S^3 and L^3_N are

$$H^0(L^3_N,\mathbb{Z}) = H^3(L^3_N,\mathbb{Z}) = \mathbb{Z}\ , \quad H^1(L^3_N,\mathbb{Z}) = 0\ , \quad H^2(L^3_N,\mathbb{Z}) = \mathbb{Z}_N\ .$$

As $H^2(S^3,\mathbb{Z}) = 0$, P is a topologically trivial bundle over S^3 and so $P = S^3 \times S^1$. In particular

$$H^2(P,\mathbb{Z}) = 0$$

Suppose now that $P = S^1 \times L^3_N$, the Künneth formula for computing the cohomology of a topological product would have implied that

$$H^2(P,\mathbb{Z}) = H^2(L^3_N, H^0(S^1,\mathbb{Z})) = H^2(L^3_N,\mathbb{Z}) = \mathbb{Z}_N$$

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C-spaces and topological geometrization condition

- The construction of C-spaces requires the Dirac quantisation condition!
- The C-spaces satisfy the topological geometrisation condition. From the first equation

$$dy^1_\alpha - B_\alpha = dy^1_\beta - B_\beta$$

and

$$H^3 = -d(dy^1 - B)$$

ie it is exact!

- Generalised geometry emerges from C-spaces

C-spaces and generalized geometry

The patching condition for y^1 which gives $\delta(d\theta - a^1) = 0$ can be solved to yield

$$y^1_\alpha = \tilde{y}^1_\alpha + \sum_\gamma \rho_\gamma(d\theta_{\alpha\gamma} - a^1_{\alpha\gamma}) \, ,$$

where $\tilde{y}^1_\alpha = \tilde{y}^1_\beta$ transforms like a 1-form.
This gives $dy^1_\alpha = (dy^1_\alpha)_i \wedge dx^i_\alpha$.
Consider the bundle $\mathcal{E}$ with sections $\mathbf{X}_\alpha = Y^i_\alpha \frac{\partial}{\partial x^i_\alpha} + (w_\alpha)_i \frac{\partial}{\partial (y^1_\alpha)_i}$ where $\langle dy^1_{\alpha i}, \frac{\partial}{\partial (y^1_\alpha)_j} \rangle = \delta^i{}_j$. The patching conditions yield

$$Y^i_\alpha = \frac{\partial x^i_{\alpha\beta}}{\partial x^j_\beta} Y^j_\beta \, , \quad (w_\alpha)_i = \frac{\partial x^j_\beta}{\partial x^i_{\alpha\beta}} \big((w_\beta)_j - (B_\beta)_{jk} Y^k_\beta \big)$$

Thus

$$0 \to T^*M \to \mathcal{E} \to TM \to 0$$

as in generalized geometry.

- $\tilde{y}^1$ can be identified with the doubled coordinates $\tilde{x}$ according to the

Gerbes

Consider an open cover $\{U_\alpha\}$ of M which is not necessarily a good cover. A gerbe [Hitchin-Chatterjee] is

- an assignment of a circle bundle $P_{\alpha\beta}$ at each $U_{\alpha\beta}$ with $P_{\beta\alpha} = P_{\alpha\beta}^{-1}$
- on triple overlaps $U_{\alpha\beta\gamma}$ the circle bundle $P_{\alpha\beta}P_{\beta\gamma}P_{\gamma\alpha}$ admits a section $g_{\alpha\beta\gamma}$
- On 4-fold overlaps $U_{\alpha\beta\gamma\delta}$, it satisfies the condition

$$g_{\beta\gamma\delta}g_{\alpha\gamma\delta}^{-1}g_{\alpha\beta\delta}g_{\alpha\beta\gamma}^{-1} = 1$$

To make connection with C-spaces $\theta_{\alpha\beta}$ are the fibre coordinates of the bundles and $g_{\alpha\beta\gamma} = \exp ia^0_{\alpha\beta\gamma}$

- Gerbes have a notion of equivalence under refinement. This allows to define a gerbe at any refinement of the original open cover
- Any open cover admits a good refinement

T-duality

Consider S^3 with N units of H^3 flux [Murray]. Cover S^3 with the stereographic cover $S^3 = U_0 \cup U_1$. Then $U_{01} = U_0 \cap U_1$ is $S^2 \times I$ where I is an open interval. The circle bundles over U_{01} are classified by

$$H^2(U_{01}, \mathbb{Z}) = H^2(S^2, \mathbb{Z}) = \mathbb{Z}$$

Consider the circle bundle P_{01} with first Chern class N represented by the 2-form F_{01}. A representative of the 3-form flux $\hat{H}$ can be given using the Mayer-Vietoris construction as

$$\hat{H}^3_0 = -d\rho_1 \wedge F_{01} \ , \quad \text{on} \quad U_0; \quad \hat{H}^3_1 = d\rho_0 \wedge F_{01} \ , \quad \text{on} \quad U_1$$

$\{\rho_0, \rho_1\}$ is a partition of unity subordinate to the cover $\{U_0, U_1\}$.
$\hat{H}$ is globally defined on S^3 as on U_{01}

$$-\hat{H}_0 + \hat{H}_1 = d(\rho_1 + \rho_0) \wedge F_{01} = d1 \wedge F_{01} = 0 \ .$$

T-dual coordinates

- Furthermore Stoke's theorem reveals that $[H^3] = [\hat{H}^3]$. The rest of the compatibility conditions for the gerbe are trivially satisfied.
- The bundle space of P_{01} restricted on S^2 is the lens space L^3_N which in turn is Buscher dual to S^3 with N units of H flux
- The Buscher dual angular coordinate $\tilde{\theta}$ is identified with the θ_{01} coordinate of the gerbe in the C-space!
- The "gebre space" is the union of $S^3 \cup L^3_N$.

General case

Take M to be a circle bundle over Q, $\pi : M \to Q$, with 3-form flux H such that $[H] = aw$, where $w \in H^2(Q, \mathbb{Z})$ and $H^1(S^1, \mathbb{Z}) = \mathbb{Z}\langle a \rangle$.

- Consider a trivialization of M

  $$\pi^{-1}(W_\alpha) = \varphi_\alpha^{-1}(W_\alpha \times S^1)$$

  and the cover of M with the two open sets

  $$U_0 = \cup_\alpha \varphi_\alpha^{-1}(W_\alpha \times V_0)\ , \quad U_1 = \cup_\alpha \varphi_\alpha^{-1}(W_\alpha \times V_1)$$

  where $S^1 = V_0 \cup V_1$
- Consider a representative F of the class $w \in H^2(Q)$ and its pull back to M with the projection map π. Take $F_{01} = F|_{U_{01}}$ and construct a representative $\hat{H}$ of the H flux as before
- The Buscher T-dual space $\tilde{M}$ is the circle bundle over Q with first Chern class w.
- The T-dual angular coordinate $\tilde{\theta}$ is identified with the θ_{01} angular coordinate of the gerbe in C-space.

General case

- The gerbe construction gives the same T-dual space under a variety of choice of open covers. This can be turned into a statement of covariance for the Buscher T-duality rules
- The gerbe construction does not need the isometries of the Buscher construction. So potentially T-dual spaces can be identified for manifolds without isometries
- Although the gerbe construction can always be done, it is not always obvious that it will lead to an identification of a “T-dual space”

There are other proposals [Blumenhagen, Hassler, Lüst; Hassler] and [Cederwall]

Conclusion

- C-spaces proposal has all the characteristics required for the definition of manifestly T-duality covariant theory: Dirac quantisation of the H^3-flux, topological geometrisation condition, generalized geometry and $O(d,d)$ covariance, and incorporates T-duality via the identification of T-dual angular coordinates with the gerbe circle fibres
- The gerbes open a window to understanding T-duality beyond the isometry set up of Buscher rules which may be useful in the context of mirror symmetry.
- It requires more coordinates than double of those of spacetime and the underlying space may not be a manifold.

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What do all these mean for DFT?

- One option is to view DFT in terms only of generalized geometry.
  - This will allow for a consistent $O(d,d)$ formulation on the spacetime. However smooth $O(d,d)$ covariance cannot be identified with T-duality.
  - In such a formulation the doubled space is not essential. But adding new coordinates may lead to interesting algebraic structures
  - The solutions of this theory will produce all the T-dual pairs, as the standard Einstein formulation, but smooth $O(d,d)$ transformations will not relate different T-dual pairs
- If the incorporation Buscher T-duality rules is not negotiable, then an alternative way of formulating the theory must be found in which the $\theta_{\alpha\beta}$ coordinates have an essential role

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