Non-Associativity in M-theory and the magnetic monopole/R-flux paradigm

DIETER LÜST (LMU, MPI)

Ninth Crete Regional Meeting on String Theory, July 13th, 2017

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I. Bakas, D. L., arXiv:1309.3172
M. Günaydin, D.L., E. Malek, arXiv:1607.06474
D.L., E. Malek, R. Szabo, arXiv:1705.09639

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This talk is dedicated to my friend Ioannis Bakas

6 common papers:

- Horava-Lifshitz gravity:

I. Bakas, F. Bourliot, D.L., M. Petropoulos, arXiv:0911.2665

I. Bakas, F. Bourliot, D.L., M. Petropoulos, arXiv:1002.0062

I. Bakas, D.L., arXiv:1103.5693

- T-duality, non-associativity:

I. Bakas, D.L., arXiv:1309.3172

I. Bakas, D.L., arXiv:1505.04004

I. Bakas, D.L., E. Plauschinn, arXiv:1602.07705

3-Cocycles, Non-Associative Star-Products and the Magnetic Paradigm of *R*-Flux String Vacua

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December 6, 2013

ABSTRACT

We consider the geometric and non-geometric faces of closed string vacua arising by T-duality from principal torus bundles with constant H-flux and pay attention to their double phase space description encompassing all toroidal coordinates, momenta and their dual on equal footing. We construct a star-product algebra on functions in phase space that is manifestly duality invariant and substitutes for canonical quantization. The 3-cocycles of the Abelian group of translations in double phase space are seen to account for non-associativity of the star-product. We also provide alternative cohomological descriptions of non-associativity and draw analogies with the quantization of point-particles in the field of a Dirac monopole or other distributions of magnetic charge. The magnetic field analogue of the R-flux string model is provided by a constant uniform distribution of magnetic charge in space and non-associativity manifests as breaking of angular symmetry. The Poincaré vector comes to rescue angular symmetry as well as associativity and also allow for quantization in terms of operators and Hilbert space only in the case of charged particles moving in the field of a single

Ioannis also played an instrumental role for the regional meetings.

Nauplia 2015:

Outline:

II) Non-associative, octonionic algebra in M-theory from non-geometric R-flux - M2 brane phase space

II) Non-associative, octonionic algebra in M-theory from non-geometric Kaluza-Klein monopoles - M-wave phase space

III) Free M-theory phase space

II) Non-associative, octonionic algebra in M-theory from non-geometric R-flux - M2 brane phase space

Geometry in general depends on what kind of objects you test it.

Point particles in classical Einstein gravity „see" continuous Riemannian manifolds:

- $[x^i, x^j] = 0$

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Point particles in classical Einstein gravity „see“ continuous Riemannian manifolds:

- $$[x^i, x^j] = 0$$

Strings, membranes may see space-time in a different way $\Rightarrow$ new string geometry.

Open strings in 2-dim. B-field background:

$$[x^i, x^j] \sim B^{ij}$$

F. Ardalan, H. Arfaei, M. Sheikh-Jabbari (1999); N. Seiberg, E. Witten (19999

Consider closed strings on three-dimensional string flux backgrounds:

Chain of three T-duality transformations:

(Hellerman, McGreevy, Williams (2002); C. Hull (2004); Shelton, Taylor, Wecht (2005); Dabholkar, Hull, 2005)

$$H_{ijk} \xrightarrow{T_i} f^i_{jk} \xrightarrow{T_j} Q^{ij}_k \xrightarrow{T_k} R^{ijk}\,, \qquad (i,j,k = 1,\ldots,3)$$

R-flux: locally non-geometric background (similar to asymmetric orbifolds).

SO(3,3) Double field theory: $\quad R^{ijk} = 3\hat{\partial}^{[k}\beta^{ij]}$

bi-vector

Non-associative 6-dim. phase space of a probe string with 3 momenta and 3 coordinates in R-flux background:

$$[x^i, x^j] = i\frac{l_s^3}{\hbar} R^{ijk} p_k$$

R. Blumenhagen, E. Plauschinn, arXiv:1010.1263.
D.L., arXiv:1010.1361;
D. Mylonas, P. Schupp, R.Szabo, arXiv:1207.0926
I. Bakas, D.L., arXiv:1309.3172

$$[x^i, p^j] = i\hbar\delta^{ij}\,, \qquad [p^i, p^j] = 0$$

$$\Longrightarrow \quad [x^i, x^j, x^k] \equiv \frac{1}{3}\left[[x^1, x^2], x^3\right] + \text{cycl. perm.} = l_s^3 R^{ijk}$$

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I. Bakas, D.L., arXiv:1309.3172

This algebra can be derived from closed string CFT and applying duality symmetries.

R. Blumenhagen, A. Deser, D.L. , E. Plauschinn, F. Rennecke, arXiv:1106.0316
C. Condeescu, I. Florakis, D. L., arXiv:1202.6366
D. Andriot, M. Larfors, D.L. , P. Patalong:arXiv:1211.6437
C. Blair, arXiv:1405.2283
I. Bakas, D.L., arXiv:1505.04004

Mathematical framework to describe this non-associative phase space structure:

- Group theory cohomology - Hochschild (1945), Cartan, Eilenberg (1956)

$\Rightarrow$ 3-cocycles, 2-cochains:

Quantization:

D. Mylonas, P. Schupp, R.Szabo, arXiv:1207.0926, arXiv:1312.1621

I. Bakas, D.L., arXiv:1309.3172

P. Aschieri, R.Szabo, arXiv:1504.03915

$$[x,x] \sim R\,p \quad \Rightarrow \quad \star_p - \text{product}$$

$$[x,x,x] \sim R \quad \Rightarrow \quad \triangle - \text{product}$$

$\Rightarrow$ Non-associative quantum mechanics

Volume quantization.

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How is the R-flux algebra related to other known non-associative algebras, in particular to the algebra of the octonions?

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Can one lift the R-flux algebra of closed strings to M-theory?

Two questions:

On the mathematical side:

How is the R-flux algebra related to other known non-associative algebras, in particular to the algebra of the octonions?

Our conjecture:

the answers to these two questions are closely related

On the physics side:

Can one lift the R-flux algebra of closed strings to M-theory?

R-flux algebra from octonions:

There exist four normed division algebras over $\mathbb{R}$:

$$\mathbb{R}\,,\mathbb{C}\,,\mathbb{Q}\,,\mathbb{O}$$

Division algebra of real octonions $\mathbb{O}$: non-commutative,
non-associative

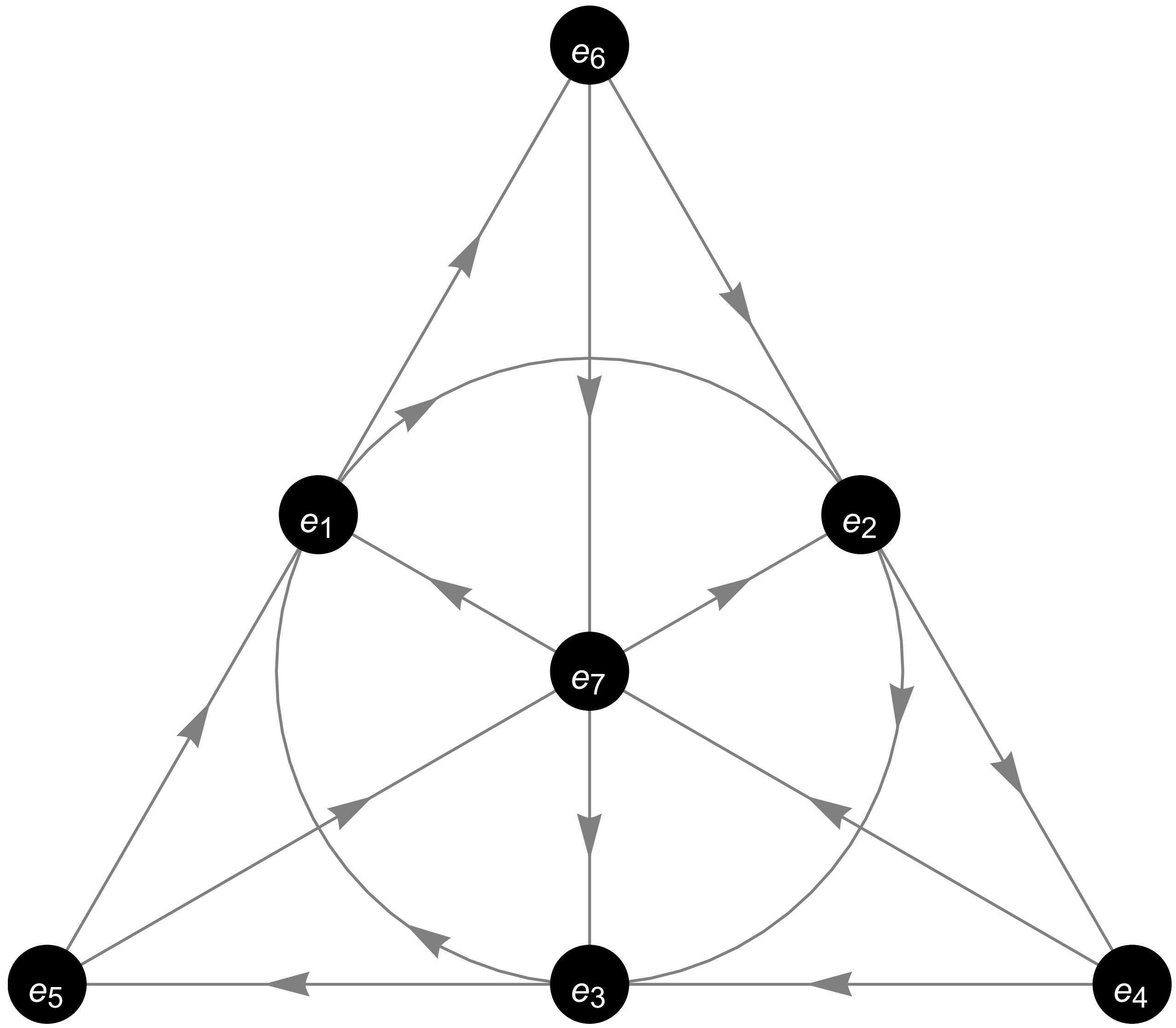
Besides the identity, there are seven imaginary units e_A

$$e_A e_B = \eta_{ABC}\, e_C \qquad (A = 1\,\ldots,7)$$

$$\eta_{ABC} = 1 \Longleftrightarrow (ABC) = (123),\,(516),\,(624),\,(435),\,(471),\,(572),\,(673)$$

This algebra is invariant under $G_2 \subset SO(7)$.

Triangle of the seven octonions:



Split indices: $e_i \,, \quad e_{(i+3)} = f_i \,, \qquad \text{for } i = 1, 2, 3$

and e_7

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Commutators:

$$
\begin{aligned}
[e_i, e_j] &= 2\epsilon_{ijk} e_k \; , & [e_7, e_i] &= 2 f_i \; , \\
[f_i, f_j] &= -2\epsilon_{ijk} e_k \; , & [e_7, f_i] &= -2 e_i \; , \\
[e_i, f_j] &= 2\delta_{ij} e_7 - 2\epsilon_{ijk} f_k
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3-brackets:

$$\begin{aligned}
[e_i, e_j, f_k] &= 4\epsilon_{ijk} e_7 - 8\delta_{k[i} f_{j]}\,,\\
[e_i, f_j, f_k] &= -8\delta_{i[j} e_{k]}\,,\\
[f_i, f_j, f_k] &= -4\epsilon_{ijk} e_7\,,\\
[e_i, e_j, e_7] &= -4\epsilon_{ijk} f_k\,,\\
[e_i, f_j, e_7] &= -4\epsilon_{ijk} e_k\,,\\
[f_i, f_j, e_7] &= 4\epsilon_{ijk} f_k
\end{aligned}$$

Associator $[X, Y, Z] \equiv (XY)Z - X(YZ)$

Now we rename the octonions in the following way:

$$p_i = -i\lambda\frac{1}{2}e_i\,, \qquad x^i = i\lambda^{1/2}\frac{\sqrt{R}}{2}f_i \quad , \quad I = i\lambda^{3/2}\frac{\sqrt{R}}{2}e_7$$

Contraction of octonionic algebra: $\lambda \to 0$

$$[f_i, f_j] = -2\epsilon_{ijk}e_k \quad \Longrightarrow \quad [x^i, x^j] = iR\epsilon^{ijk}p_k$$

$$[e_i, e_j] = 2\epsilon_{ijk}e_k \quad \Longrightarrow \quad [p_i, p_j] = 0$$

$$[f_i, e_j] = -\delta^i_j e_7 + \epsilon^i{}_{jk}f_k \quad \Longrightarrow \quad [x^i, p_j] = i\delta^i_j I$$

$$[x_i, I] = 0 = [p_i, I]$$

$$[f_i, f_j, f_k] = -4\epsilon_{ijk}e_7 \quad \Longrightarrow \quad [x^i, x^j, x^k] = R\epsilon^{ijk}I$$

Agrees with non-associative R-flux algebra !

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- e^7 additional M-theory coordinate x^4

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Lift of string R-flux algebra to non-geometric M-theory background:

- e^7 additional M-theory coordinate x^4

  $\Rightarrow$ Four coordinates: f_1, f_2, f_3, e_7

- but no additional momentum.

  $\Rightarrow$ Three momenta: e_1, e_2, e_3

M-theory uplift of R-flux background:

Consider IIA string: the duality chain splits into two possible T-dualities:

$$H_{ijk} \xrightarrow{T_{ij}} Q^{ij}_{k}\,, \qquad\qquad f^{i}_{jk} \xrightarrow{T_{jk}} R^{ijk}$$

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Uplift to M-theory: add additional circle $S^1_{x^4}$

- 3-dim IIA flux background $\Leftrightarrow$ 4-dim M-theory flux background
- two T-dualities $\Leftrightarrow$ 3 U-dualities

(Need third duality along the M-theory circle to ensure right dilaton shift.)

Using SL(5) exceptional field theory one can obtain the R-flux in M-theory.

C. Blair, E. Malek, arXiv:1412.0635.

Non-geometric R-flux background in M-theory, which is dual to twisted torus:

R-flux: $$R^{\alpha,\beta\gamma\delta\rho} = 4\hat{\partial}^{\alpha[\beta}\Omega^{\gamma\delta\rho]}$$

tri-vector

This is supposed to be still a consistent M-theory background, which can be probed by M2 branes.

Four coordinates: x^1, x^2, x^3, x^4

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- Alternatively consider Freed-Witten anomaly:

R-Flux with probe momentum p_4 along x^4

$$\Updownarrow$$

R-Flux with D0 branes.

Dualize to IIB: H-flux with D3-branes

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This is forbidden by the Freed-Witten anomaly.

$\Rightarrow$ No momentum modes along the x^4 direction !

So we see that the phase space space of a M2-brane in the R-flux background in M-theory is seven-dimensional:

$$x^1, x^2, x^3, x^4\,;\, p_1, p_2, p_3$$

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Missing momentum constraint in covariant terms:

$$F \equiv p_\alpha R^{\alpha,\beta\gamma\delta\rho} = 0$$

Non-associative phase space algebra of M2-brane probes in R-flux background:

$$[x^i, x^j] = \frac{\ell_s^3}{\hbar}\, R^{4,ijk4}\, p_k \ , \qquad [x^4, x^i] = \frac{\lambda\,\ell_s^3}{\hbar}\, R^{4,1234}\, p^i \ ,$$

$$[x^i, p_j] = \hbar\,\delta^i_j\, x^4 + \hbar\,\lambda\,\varepsilon^i{}_{jk}\, x^k \ , \qquad [p_i, x^4] = \hbar\,\lambda^2\, x_i \ ,$$

$$[p_i, p_j] = -\,\hbar\,\lambda\,\varepsilon_{ijk}\, p^k \ ,$$

$$[\![x^i, x^j, x^k]\!] = -\,\ell_s^3\, R^{4,ijk4}\, x^4 \ ,$$

$$[\![x^i, x^j, x^4]\!] = \lambda^2\,\ell_s^3\, R^{4,ijk4}\, x_k \ ,$$

$$[\![p_i, x^j, x^k]\!] = -\,\lambda\,\ell_s^3\, R^{4,1234}\,\big(\delta_i^j\, p^k - \delta_i^k\, p^j\big) \ ,$$

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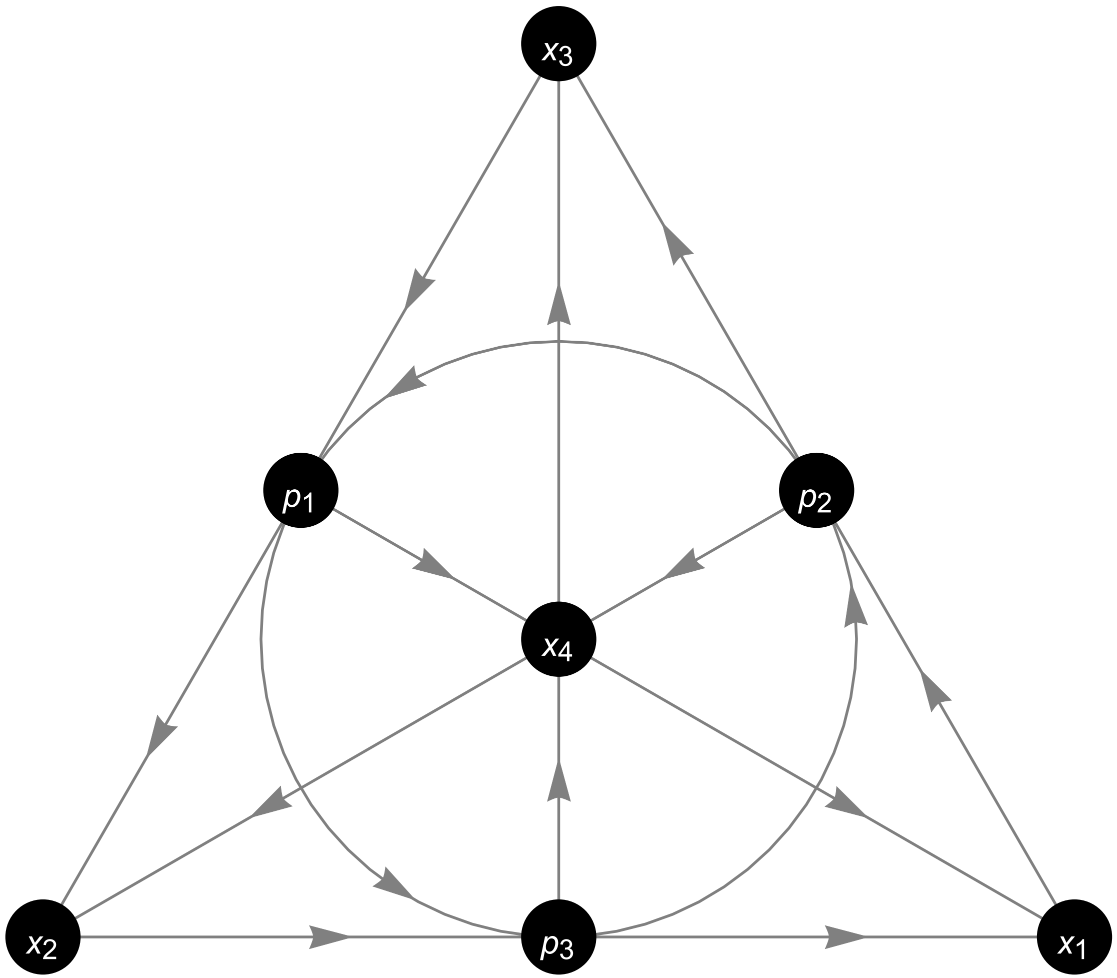
$$[\![p_i, p_j, p_k]\!] = 0 \ ,$$

Corresponding octonionic triangle:



Natural identification of contraction parameter λ :

$$\lambda = \frac{\ell_{\mathrm{P}}}{\hbar}$$

Standard identification of M-theory parameters:

$$\ell_s^2 = \frac{\ell_{\mathrm{P}}^3}{r_{11}} \, , \qquad g_s = \left(\frac{r_{11}}{\ell_{\mathrm{P}}} \right)^{3/2}$$

(E. Witten, 1995)

Reduction to string theory

$$\ell_{\mathrm{P}} \sim r_{11}^{1/3} \longrightarrow 0 \, , \qquad g_s \sim r_{11} \longrightarrow 0$$

Corresponds indeed to $\lambda \longrightarrow 0 \, ,$

II) Non-associative, octonionic algebra in M-theory from non-geometric Kaluza-Klein monopoles - M-wave phase space

Now we consider the phase space of an electron moving in the field of a magnetic monopole:

$$\boxed{\begin{gathered}[x^i, x^j] = 0 \,, \qquad [x^i, p_j] = \hbar\, \delta^i_j \,, \qquad [p_i, p_j] = \frac{e\,\hbar}{c}\, \varepsilon_{ijk}\, B^k \,, \\ [\![p_i, p_j, p_k]\!] = -\frac{e\,\hbar^2}{c}\, \varepsilon_{ijk}\, \nabla \cdot \vec{B} \end{gathered}}$$

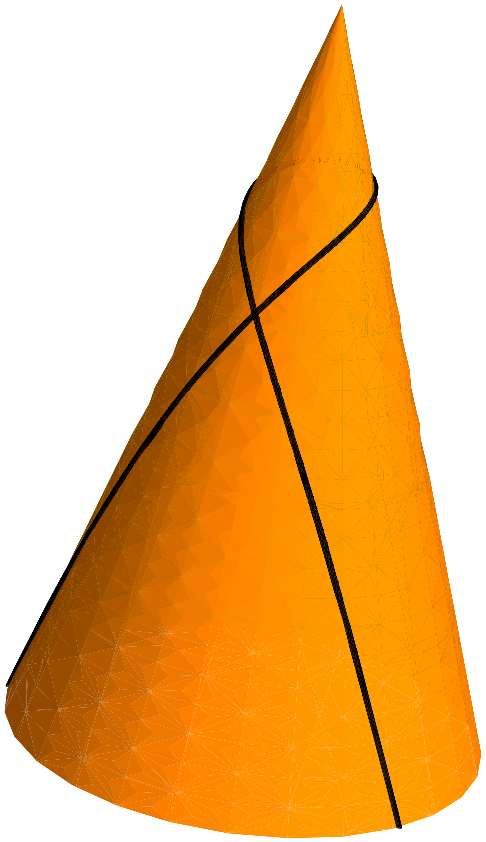
p_i : gauge invariant momenta.

R. Jackiw (1985); B. Grossman (1985); Y. Wu, A. Zee (1985); J. Mickelsson (1986), M. Günaydin, B. Zumino (1985)

B^k : magnetic field of magnetic monopole

Isolated magnetic charges: the electron avoids the non-associativity by excising these points on position space - electron never reaches the magnetic monopole:

I. Bakas, D.L. (2013)



However we like to consider a constant smeared magnetic charge density ρ : magnetic charge is uniformly distributed over space.

$\Rightarrow$ If you now wanted remove the magnetic sources from position space you would end up with empty space.

(Signal of local non-geometry just like R-flux.)

This is described by a constant magnetic charge gerbe that is realized as a family of Dirac monopole gerbes.

Phase space algebra of electron in constant magnetic charge density ρ :

$$[x^i, x^j] = 0 \ ,$$

$$[x^i, p_j] = \hbar\, \delta^i_j \ ,$$

$$[p_i, p_j] = \frac{\hbar\, e}{c}\, \rho\, \varepsilon_{ijk}\, x^k \ ,$$

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This algebra is isomorphic to R-flux algebra via the following (canonical) transformation:

$$x^i_R \longleftrightarrow p_{M\,i} \ , \qquad p_{R\,i} \longleftrightarrow -x^i_M \ ,$$

$$\ell_s^3 \longleftrightarrow \frac{e\,\hbar^2}{c} \ , \qquad R \longleftrightarrow \rho$$

M-theory up-lift of magnetic monopole algebra:

Conjecture: the magnetic charge algebra in M-theory is again provided by the non-associative algebra of the seven octonions.

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But now there will be four momenta and three positions!

(i) What is the M-theory background which realizes this picture?

(ii) What are the relevant M-theory probes that have this octonionic algebra as phase space?

(i) KK monopole as M-theory background:

$$ds^2 = ds^2_{M_7} + U\, d\vec{y}\cdot d\vec{y} + U^{-1}\left(dz + \vec{A}\cdot d\vec{y}\,\right)^2 \ ,$$

$$A = \vec{A}(\vec{y}\,)\cdot d\vec{y} \qquad\qquad (\vec{y}, z) \in R^3 \times S^1$$

$$F = *_3\, dU \qquad\qquad A : \text{U(1) gauge connection}$$

We need a smeared KK monopole with constant magnetic charge density.

$\Rightarrow$ No local, geometric expression for metric and A:

A in non-local, smeared form:

$$A_i(\vec{y}\,) = \frac{1}{4\pi}\int A_i^{\mathrm{D}}(\vec{y} - \vec{y}\,')\; d^3\vec{y}\,' = \frac{N}{4\pi}\,\varepsilon_{ijk}\int \frac{(y_j - y_j')}{|\vec{y} - \vec{y}\,'|\left((y_k - y_k') + |\vec{y} - \vec{y}\,'|\right)}\; d^3\vec{y}\,' \ ,$$

(ii) M-wave as M-theory probe:

We need a probe that is electrically charged under the U(1) gauge field.

The electric dual to the KK monopole NUT charge is a graviton momentum mode $\Rightarrow$ M-wave

This M-wave travels along the x^4 direction.

$\Rightarrow$ No well-defined, local position with respect to x^4 .

(IIA: D6 $\leftrightarrow$ D0 , $p_4 = \dfrac{\hbar e}{r_{11}}$)

Seven-dimensional phase space

$$(x^1, x^2, x^3; p_1, p_2, p_3, p_4)$$

So we obtain as phase space algebra of a M-wave in the non-geometric KK-monopole background:

$$[x^i, x^j] = -\hbar\,\lambda\,\varepsilon^{ijk}\,x_k \ ,$$

$$[p_i, x^j] = \hbar\,\delta_i^j\,p_4 + \hbar\,\lambda\,\varepsilon_i{}^{jk}\,p_k \ , \qquad [x^i, p_4] = \hbar\,\lambda^2\,p^i \ ,$$

$$[p_i, p_j] = \frac{\hbar\,e}{c}\,\rho\,\varepsilon_{ijk}\,x^k \ , \qquad [p_4, p_i] = \frac{\hbar\,\lambda\,e}{c}\,\rho\,x_i \ ,$$

$$[x^i, x^j, x^k] = -2\,\hbar^2\lambda\,\varepsilon^{ijk}\,x^4 \ , \qquad [x^i, x^j, x^4] = \frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,x_k \ ,$$

$$[p^i, x^j, x^k] = -\frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,p_4 - \frac{\hbar^2\,\lambda}{2}\,\big(\delta^{ij}\,p^k - \delta^{ik}\,p^j\big) \ ,$$

$$[p_i, x^j, x^4] = -\frac{\hbar^2\,\lambda}{2}\,\delta_i^j\,p_4 - \frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,p_k \ ,$$

$$[p_i, p_j, x^k] = \frac{\hbar^2\,\lambda^2\,e}{2c}\,\rho\,\varepsilon_{ij}{}^k\,x^4 + \frac{\hbar^2\,\lambda\,e}{2c}\,\rho\,\big(\delta_j^k\,x_i - \delta_i^k\,x_j\big) \ ,$$

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$$[p_4, x^i, x^j] = -\frac{\hbar^2\,\lambda^3}{2}\,\varepsilon^{ijk}\,p_k \ , \qquad [p_4, x^i, x^4] = -\frac{\hbar^2\,\lambda^3}{2}\,p^i \ ,$$

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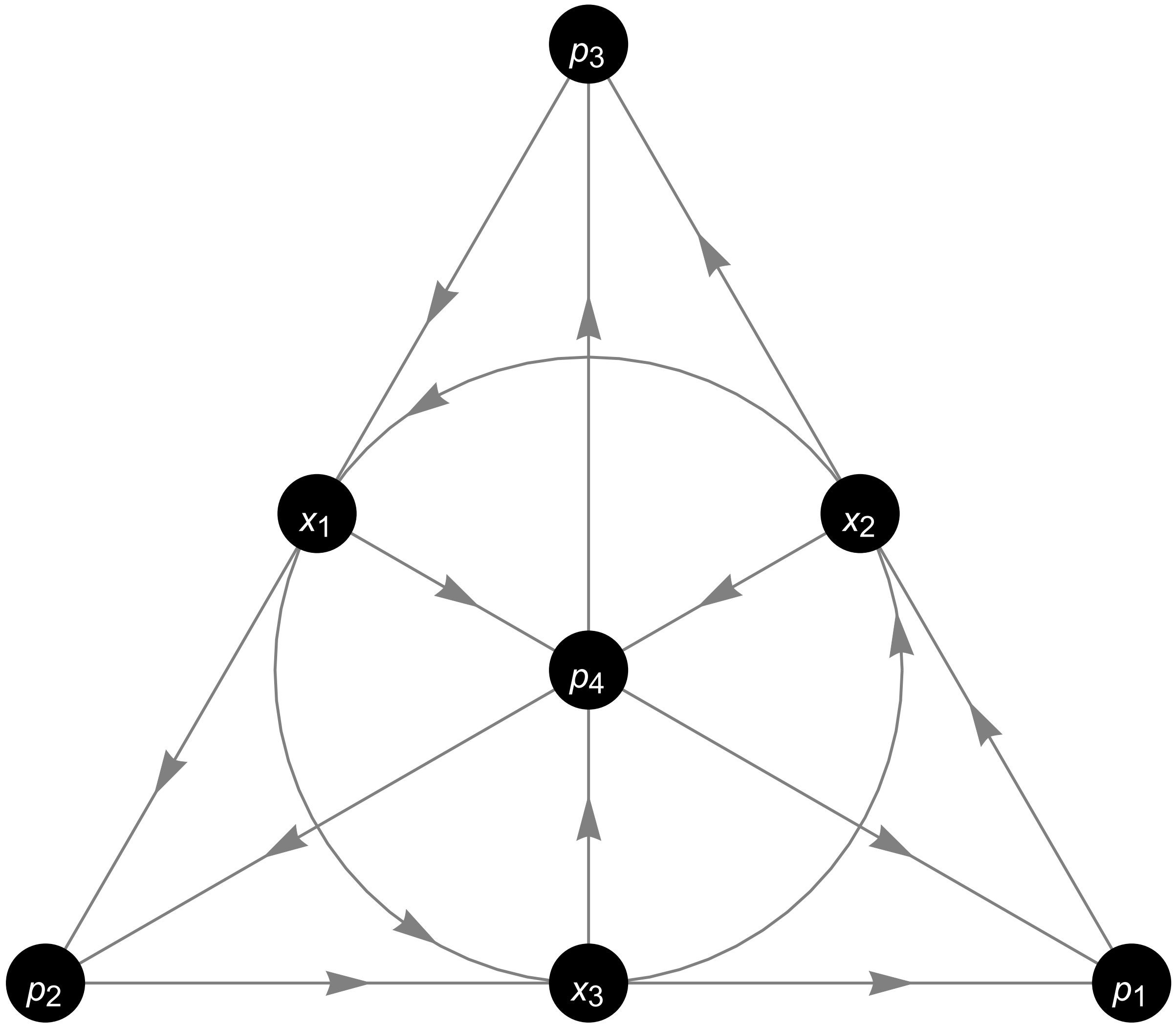
$\lambda \to 0 \ (l_P, r_{11} \to 0) :$ magnetic charge algebra

Corresponding octonionic triangle:



Magnetic monopoles and quantum gravity:

Restrict momenta to be on the unit sphere:

Then we obtain in the limit of $\rho \to 0$:

$$[x^i, x^j] = -\hbar\,\lambda\,\varepsilon^{ijk}\,x_k \;,$$

$$[x^i, q_j] = \hbar\,\sqrt{1-\lambda^2\,|\vec{q}\,|^2}\;\delta^i_j + \hbar\,\lambda\,\varepsilon^i{}_{jk}\,q^k \;,$$

$$[q_i, q_j] = 0 \;.$$

L. Freidel, E. Livine (2005)

These are the commutation relations of Ponzano-Regge spin foam model of three-dimensional quantum gravity.

Octonionic magnetic algebra can be viewed as algebra of monopoles in quantum gravity.

III) Free M-theory phase space

See also: V. Kupriyanov, R. Szabo, arXiv:1701.02574

Assume no missing momentum/coordinate condition.

$p_4\,(x^4)$ corresponds to the 8th. (identity) octonion e_8 .

3-algebra: $$\left[\Xi_{\hat A},\Xi_{\hat B},\Xi_{\hat C}\right] = -2\,\hbar^2\,\phi_{\hat A\hat B\hat C\hat D}\,\Xi_{\hat D}$$

$\hat A,\hat B,\hat C = 1,\dots,8$

This algebra is invariant under $Spin(7)\subset SO(8)$.

Failure of defining Nambu-Poisson bracket:

$$\begin{aligned}
[\![\Xi_{\hat A},\Xi_{\hat B},\Xi_{\hat C},\Xi_{\hat D},\Xi_{\hat E}]\!] := \tfrac{1}{12}\,\big(&\big[[\Xi_{\hat A},\Xi_{\hat B},\Xi_{\hat C}],\Xi_{\hat E},\Xi_{\hat D}\big] + \big[[\Xi_{\hat A},\Xi_{\hat B},\Xi_{\hat D}],\Xi_{\hat C},\Xi_{\hat E}\big] \\
&+ \big[[\Xi_{\hat A},\Xi_{\hat B},\Xi_{\hat E}],\Xi_{\hat C},\Xi_{\hat D}\big] + \big[[\Xi_{\hat C},\Xi_{\hat D},\Xi_{\hat E}],\Xi_{\hat A},\Xi_{\hat B}\big]\,\big) \\
= \hbar^4\,\big(&\delta_{\hat A\hat C}\,\phi_{\hat D\hat E\hat B\hat F} + \delta_{\hat A\hat D}\,\phi_{\hat E\hat C\hat B\hat F} + \delta_{\hat A\hat E}\,\phi_{\hat C\hat D\hat B\hat F} \\
&- \delta_{\hat B\hat C}\,\phi_{\hat D\hat E\hat A\hat F} - \delta_{\hat B\hat D}\,\phi_{\hat E\hat C\hat A\hat F} - \delta_{\hat B\hat E}\,\phi_{\hat C\hat D\hat A\hat F}\,\big)\,\Xi^{\hat F} \\
&- \hbar^4\,\big(\phi_{\hat B\hat C\hat D\hat E}\,\Xi_{\hat A} - \phi_{\hat A\hat C\hat D\hat E}\,\Xi_{\hat B}\big)\ .
\end{aligned}$$

Any constraint $f(\mathit{\Xi}) = 0$ breaks

$$Spin(7) \longrightarrow G_2 \qquad\qquad \mathbf{8} = 7 \oplus 1$$

$p_4 = 0 \implies$ M-theory R-flux algebra.

$x^4 = 0 \implies$ M-theory monopole algebra.

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The exchange of coordinates and momenta corresponds to a Spin(7) transformation - exchange of two SU(2)'s :

$$Spin(7) \supseteq SU(2)^3 \longrightarrow SU(2)^2 \subseteq G_2$$

$$\mathbf{8}\big|_{SU(2)^3} = (2,1,2) \oplus (1,2,2) \longrightarrow \mathbf{8}\big|_{SU(2)^2} = (2,2) \oplus (1,1) \oplus (1,3)$$

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R-flux: $(2,1,2) \downarrow x^\alpha$, $(1,2,2) \downarrow p_\alpha$, $(2,2) \downarrow x^\alpha$, $(1,3) \downarrow p_i$

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monopole: p_α $\quad x^\alpha$ $\quad p_\alpha$ $\quad x^i$

Consider also the limit $R \to 0 \; (\rho \to 0)$:

$$[p_i, p_j, x^k] = \frac{\ell_{\rm P}^2}{2}\, \varepsilon_{ij}{}^k\, x^4 + \frac{\hbar\, \ell_{\rm P}}{2}\, \big(\delta_j^k\, x_i - \delta_i^k\, x_j\big) \ , \qquad [p_i, p_j, x^4] = -\frac{\ell_{\rm P}^3}{2\hbar}\, \varepsilon_{ijk}\, x^k \ ,$$

$$[p_i, p_j, p_k] = \ -2\,\hbar\,\ell_{\rm P}\,\varepsilon_{ijk}\,p_4 \ , \qquad [p_4, p_i, x^j] = -\frac{\hbar\,\ell_{\rm P}}{2}\,\delta_i^j\,x^4 - \frac{\ell_{\rm P}^2}{2}\,\varepsilon_i{}^{jk}\,x_k \ ,$$

$$[p_4, p_i, x^4] = -\frac{\ell_{\rm P}^3}{2\hbar}\, x_i \ , \qquad [p_4, p_i, p_j] \ = \ -\frac{\ell_{\rm P}^2}{2}\,\varepsilon_{ijk}\,p^k \ ,$$

This algebra can be considered as the free, non-associative, eight-dimensional phase space algebra of M-theory.

It becomes trivial in the limit $l_P \to 0$.

Can one also get the full Spin(7) algebra with $R, \rho \neq 0$?

This seems to be possible in decompactification limit:

$$\frac{1}{V} R^{\mu,\nu\rho\alpha\beta} p_\mu = 0 \qquad \text{e.g.} \qquad \frac{R\, p_4}{V} = 0$$

$$V \to \infty \quad \Rightarrow \quad R \neq 0\,, \quad p_4 \neq 0$$

Full Spin(7) algebra:

$$[x^i, x^j, x^k] = -\frac{\ell_s^3}{2}\, R\, \varepsilon^{ijk}\, x^4 \ , \qquad [x^i, x^j, x^4] = \frac{\lambda^2\, \ell_s^3}{2}\, R\, \varepsilon^{ijk}\, x_k \ ,$$

$$[p^i, x^j, x^k] = -\frac{\lambda^2\, \ell_s^3}{2}\, R\, \varepsilon^{ijk}\, p_4 - \frac{\lambda\, \ell_s^3}{2}\, R\, \big(\delta^{ij}\, p^k - \delta^{ik}\, p^j\big) \ ,$$

$$[p_i, x^j, x^4] = -\frac{\lambda^2\, \ell_s^3}{2}\, R\, \delta_i^j\, p_4 - \frac{\lambda^2\, \ell_s^3}{2}\, R\, \varepsilon^{ijk}\, p_k \ ,$$

$$[p_i, p_j, x^k] = \frac{\hbar^2\, \lambda^2}{2}\, \varepsilon_{ij}{}^k\, x^4 + \frac{\hbar^2\, \lambda}{2}\, \big(\delta_j^k\, x_i - \delta_i^k\, x_j\big) \ ,$$

$$[p_i, p_j, x^4] = -\frac{\hbar^2\, \lambda^3}{2}\, \varepsilon_{ijk}\, x^k \ , \qquad [p_i, p_j, p_k] = -2\, \hbar^2\, \lambda\, \varepsilon_{ijk}\, p_4 \ ,$$

$$[p_4, x^i, x^j] = \frac{\lambda\, \ell_s^3}{2}\, R\, \varepsilon^{ijk}\, p_k \ , \qquad [p_4, x^i, x^4] = -\frac{\lambda^2\, \ell_s^3}{2}\, R\, p^i \ ,$$

$$[p_4, p_i, x^j] = -\frac{\hbar^2\, \lambda}{2}\, \delta_i^j\, x^4 - \frac{\hbar^2\, \lambda^2}{2}\, \varepsilon_i{}^{jk}\, x_k \ ,$$

$$[p_4, p_i, x^4] = -\frac{\hbar^2\, \lambda^3}{2}\, x_i \ , \qquad [p_4, p_i, p_j] \ = \ -\frac{\hbar^2\, \lambda^2}{2}\, \varepsilon_{ijk}\, p^k \ ,$$

V) Outlook & open questions

Non-associative algebras occur in M-theory at many places:

- Multiple M2-brane theories and 3-algebras

J. Bagger, N. Lambert (2007)

- $Spin(7), G_2$ backgrounds

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Generalization to higher dimensional exceptional field theory? Work in progress by D.L., E. Malek, M. Syväri

Interesting proposal for the quantization of non-geometric M-theory background by deriving a phase space star product for the non-associative algebra of octonions.

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Many thanks !