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Topological Amplitudes and the Ω background

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based on:

1702.04998 with
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Outlook

work in progress with
I. Antoniadis

Dedicated to Ioannis Bakas

Yassen Stanev

The dynamics of (supersymmetric) gauge theories is a fascinating subject.

$\mathcal{N}$=2 theories in D=4 offer a rich playground for studying exact dynamics

Since the solution of Seiberg and Witten a lot of activity on the subject

In their seminal paper, Seiberg and Witten solved $\mathcal{N}=2$ theories in $D=4$ using smart arguments (the famous *Seiberg-Witten curve*) to compute the full pre-potential, including perturbative and non-perturbative corrections

In the early 2000, Nekrasov offered a microscopic interpretation of the elliptic curve by performing the explicit instanton sum

To achieve his task and get a sensible answer, Nekrasov had to deform the theory:

$$ds^2 = A\,dz\,d\bar{z} + g_{IJ}\left(dx^I + \Omega^I{}_K x^K dz + \bar{\Omega}^I{}_K x^K d\bar{z}\right)\left(dx^J + \Omega^J{}_L x^L dz + \bar{\Omega}^J{}_L x^L d\bar{z}\right)$$

Curving the background provided an IR regularisation and allowed to localise the instanton calculus

Since then, increasing activity to understand/reproduce the Ω background from String Theory

Nekrasov has conjectured a connection of the topological amplitudes with the free energy of $\mathcal{N}$=2 gauge theories on special (Ω) background.

In particular, for a single-parameter deformation

$$\log \mathscr{Z}_{\mathrm{Nek}}(\epsilon_+ = 0, \epsilon_- = g_{\mathrm{s}}) = \sum_{g=0}^{\infty} F_g \, g_{\mathrm{s}}^{2g-2} \Big|_{\text{field theory}}$$

In the topological-amplitude language ε_- corresponds to the graviphoton field strength

The topological string partition function

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{i=1}^{3(g-1)} |G^-(\mu_i)|^2 \rangle$$

computes half-BPS F-terms in the effective action

$$\int d^4x\, d^4\theta\, F_g(X)\, (W_{\mu\nu}^{ij} W_{ij}^{\mu\nu})^g$$

[Antoniadis, Gava, Narain, Taylor 1993]

What about the relation with the two-parameter deformation?

$$\log \mathscr{Z}_{\rm Nek}(\epsilon_+,\epsilon_-) = \sum_{g,n=0}^{\infty} F_{g,n}\, \epsilon_-^{2g-2}\, \epsilon_+^{2n}\Big|_{\text{field theory}}$$

What is the refinement of the topological string?

Defined via suitable topological amplitudes

$$F_{g,n} = \langle R_-^2\, F_-^{2g-2}\, V_+^{2n}\rangle$$

[Antoniadis, Florakis, Hohenegger, Narain, Zein Assi 2013]

Until now, this conjecture has been tested for pure SU(2) SYM

$$\log \mathscr{Z}_{\mathrm{Nek}}(\epsilon_+,\epsilon_-) = \sum_{g,n=0}^{\infty} F_{g,n}\,\epsilon_-^{2g-2}\,\epsilon_+^{2n}\Big|_{\text{field theory}}$$

In this talk, I shall provide more evidence, by computing topological amplitudes for (non-)Abelian $\mathcal{N}$=2* theory in D=4,5 with one/two deformations.

Outline

A string realisation of $\mathcal{N}$=2* theories in D=5,4

The topological amplitude for a single deformation

The topological amplitude for the two deformations

Non-Abelian generalisations and non-perturbative contributions

A type I realisation of 4d Abelian $\mathcal{N}=2^*$ SYM

a one-parameter interpolating theory

$m = 0$ U(1) vector multiplet $m \to \infty$

$\longleftrightarrow$

$\mathcal{N}=4$ Maxwell + neutral hypermultiplet with mass m $\mathcal{N}=2$ Maxwell

Reminiscent of Scherk-Schwarz compactifications

Realise it as a freely acting K3 orbifold

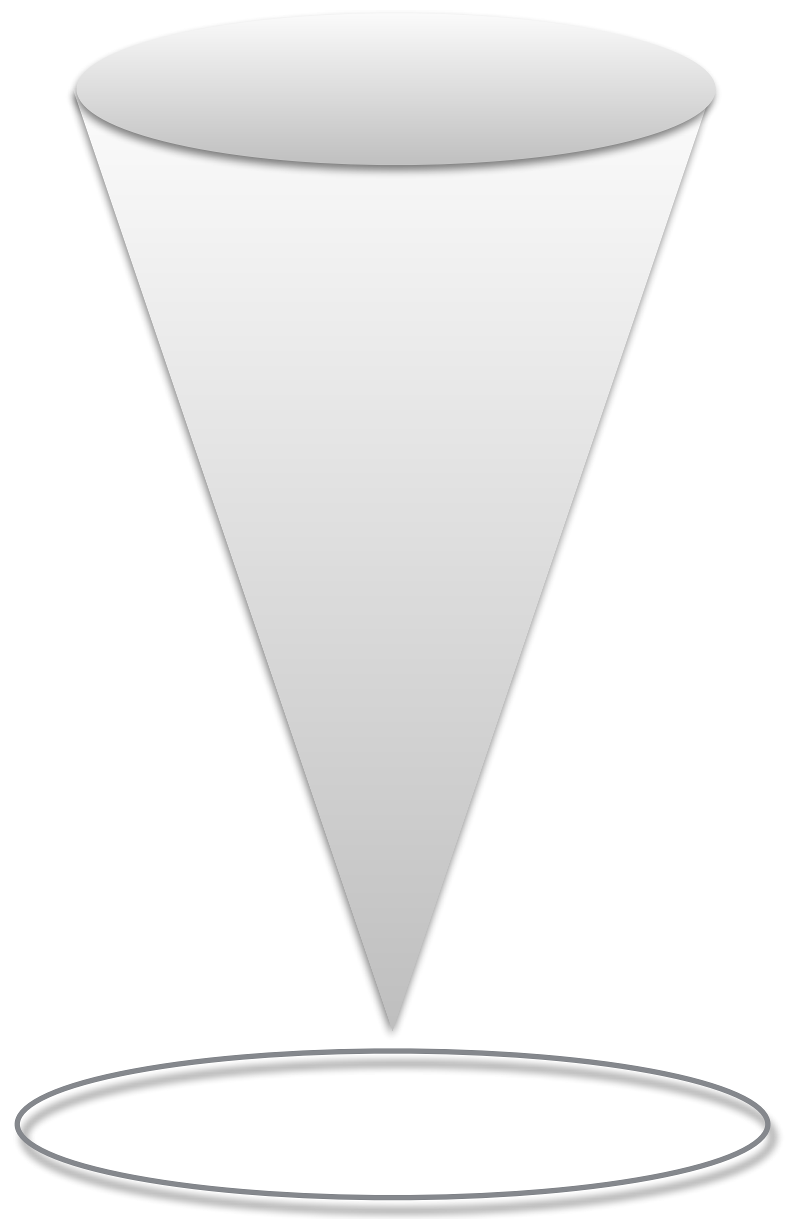
A type I realisation of 4d Abelian $\mathcal{N}$=2* SYM

Requirements:

i. Abelian theory (i.e. only one D-brane)

ii. a tuneable parameter to control the field theory limit

iii. a consistent open string construction



The simplest solution:

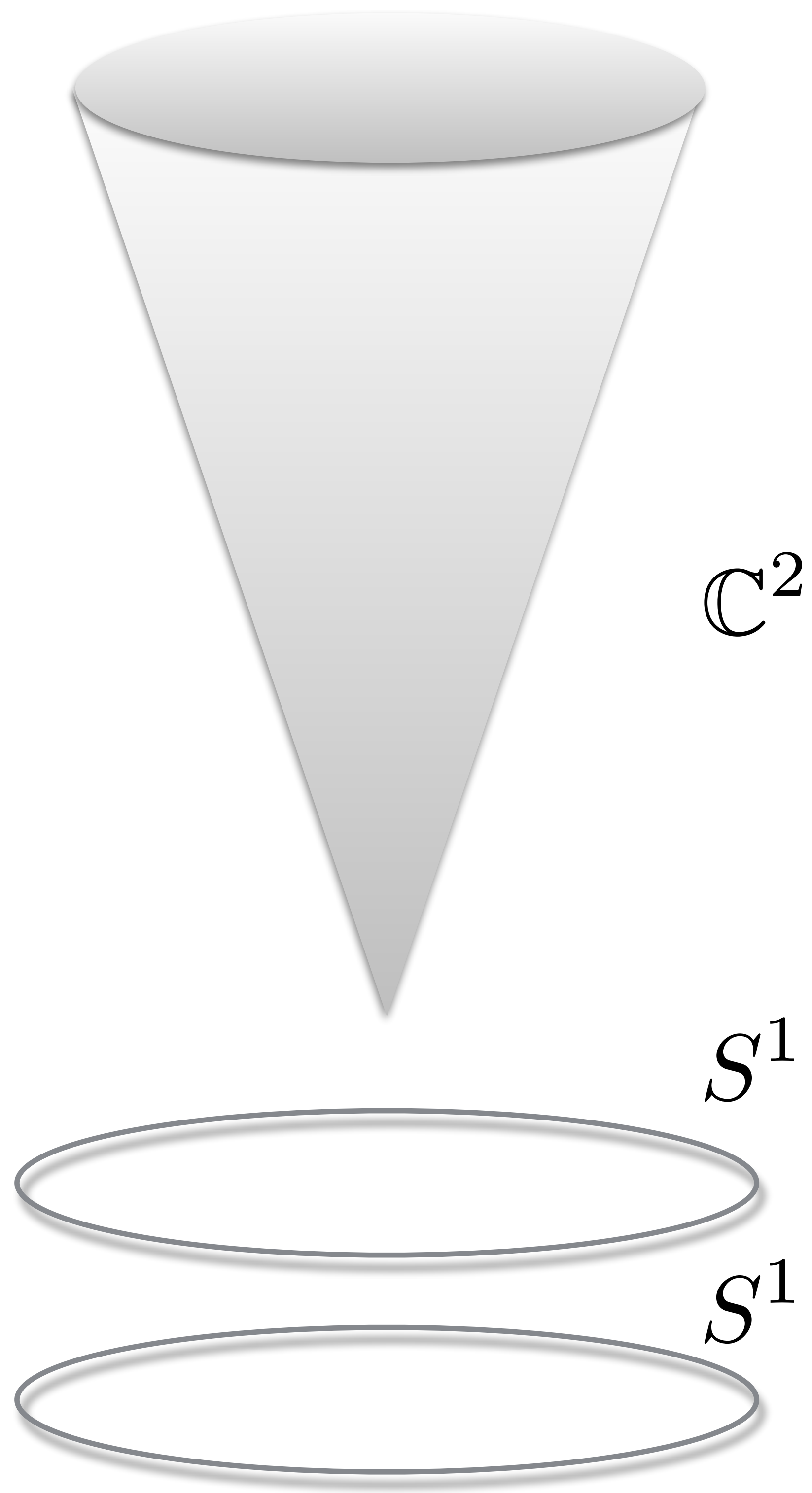
A single D5 brane sitting on top of a $\mathbb{C}^2/\mathbb{Z}_N$ "singularity"

A type I realisation of 4d Abelian $\mathcal{N}=2^*$ SYM

$$\frac{\mathcal{M}_{1,3} \times S^1(R) \times S^1(1/m) \times \mathbb{C}^2}{\mathbb{Z}_N}$$

$$\mathbb{Z}_N \ : \qquad \begin{cases} (z_1, z_2) & \to (e^{2i\pi/N} z_1, e^{-2i\pi/N} z_2) \\ \qquad y & \to y + \frac{2\pi}{Nm} \end{cases}$$

Note: The (freely acting) orbifold does not act on the Chan-Paton factors



A type I realisation of 4d Abelian $\mathcal{N}=2^*$ SYM

The annulus partition function then reads

$$\mathscr{A} = \frac{1}{N}\sum_{\ell=0}^{N-1}\sum_{r,s\in\mathbb{Z}}\left(V_4 O_4^{(\ell)} + O_4 V_4^{(\ell)} - S_4 S_4^{(\ell)} - C_4 C_4^{(\ell)}\right) e^{2i\pi\ell r/N} P_r(1/m) P_s(R)$$

At the massless level

$$\mathscr{A}_0 = \sum_{\ell=0}^{N-1}\sum_{r,s\in\mathbb{Z}} \left[(V_4 - C_4)\, P_{rN}(1/m) + (4O_4 - S_4)\, P_{rN\pm1}(1/m)\right]\, P_s(R)$$

massless vector multiplet

massive hypermultiplet

Legenda

$$O_4^{(\ell)} = \frac{\theta[{}^{0}_{\ell/N}]\theta[{}^{0}_{-\ell/N}] + \theta[{}^{0}_{1/2+\ell/N}]\theta[{}^{0}_{1/2-\ell/N}]}{[\theta[{}^{1/2}_{1/2+\ell/N}]/\sin(\pi\ell/N)]^2}$$

$$V_4^{(\ell)} = \frac{\theta[{}^{0}_{\ell/N}]\theta[{}^{0}_{-\ell/N}] - \theta[{}^{0}_{1/2+\ell/N}]\theta[{}^{0}_{1/2-\ell/N}]}{[\theta[{}^{1/2}_{1/2+\ell/N}]/\sin(\pi\ell/N)]^2}$$

$$S_4^{(\ell)} = \frac{\theta[{}^{1/2}_{\ell/N}]\theta[{}^{1/2}_{-\ell/N}] - \theta[{}^{1/2}_{1/2+\ell/N}]\theta[{}^{1/2}_{1/2-\ell/N}]}{[\theta[{}^{1/2}_{1/2+\ell/N}]/\sin(\pi\ell/N)]^2}$$

$$C_4^{(\ell)} = \frac{\theta[{}^{1/2}_{\ell/N}]\theta[{}^{1/2}_{-\ell/N}] + \theta[{}^{1/2}_{1/2+\ell/N}]\theta[{}^{1/2}_{1/2-\ell/N}]}{[\theta[{}^{1/2}_{1/2+\ell/N}]/\sin(\pi\ell/N)]^2}$$

$$P_m(\rho) = q^{\frac{1}{2}(m/\rho)^2}$$

A type I realisation of 4d Abelian $\mathcal{N}=2^*$ SYM

Tadpole conditions

No need to impose tadpole conditions for the untwisted sector since the space transverse to the brane is non-compact

In the twisted sector no massless tadpoles arise because of the Scherk-Schwarz shift along the circle

$$\sum_{r\in\mathbb{Z}} e^{2i\pi\ell r/N} e^{-\pi t m^2 r^2} \to \sum_{r'\in\mathbb{Z}} e^{-\pi t'(r'-\ell/N)^2/m^2}$$

It is then consistent not to act on the Chan-Paton factors

A type I realisation of 4d Abelian $\mathcal{N}=2^*$ SYM

Note

The non-compactness of the space transverse to the D5 branes is crucial in order to be able to build a (consistent) Abelian theory. **(I)**

Had we considered the space compact, consistency would have called for orientifold planes and extra D-branes impinging on the existence of a consistent Abelian theory

A non-compact space allows us to consider order-N twists, with arbitrarily large N. This is crucial in order to take the field theory limit **(II)**

The type I topological amplitude: the one parameter case

We can now move to compute the amplitude

$$\mathscr{A}_g = \left\langle (V^{+}_{\rm grav})^2\, (V^{-}_{\rm grav})^2\, V^{2g-2}_{\rm gph} \right\rangle$$

on the vacuum we have just discussed,
and on the world-sheet with the topology of a annulus

$$V^{\pm}_{\rm grav}(\xi_\mu \alpha, p) = \xi_{\mu\alpha} e^{-\varphi/2} S^\alpha e^{i\phi_3/2} \sigma^{\pm} \left(\bar\partial Z^\mu + i(p\cdot\tilde\chi)\tilde\chi^\mu\right) e^{ip\cdot Z},$$

$$V_{\rm gph}(\epsilon, p) = \epsilon_\mu \Big[(\partial X + i(p\cdot\chi)\psi)(\bar\partial Z^\mu + i(p\cdot\tilde\chi)\tilde\chi^\mu)$$
$$- e^{-(\varphi+\tilde\varphi)/2} p_\nu S^\alpha (\sigma^{\mu\nu})_\alpha{}^\beta \tilde S_\beta\, e^{i(\phi_3+\tilde\phi_3)/2} \Sigma^+ \tilde\Sigma^- \Big] e^{ip\cdot Z} + (\text{left} \leftrightarrow \text{right})$$

$(Z^1, Z^2) \in \mathscr{M}_{1,3}$

$(Z^4, Z^5) \in \mathbb{C}^2$

$X \in T^2 = S^1(1/m) \times S^1(R)$

$\chi^{1,2}, \psi, \chi^{4,5}$ fermions

$S, e^{\pm i\phi_3/2}, \Sigma$ spin fields

The type I topological amplitude: the one parameter case

$$\mathscr{A}_g = \left\langle (V^+_{\rm grav})^2\, (V^-_{\rm grav})^2\, V^{2g-2}_{\rm gph} \right\rangle$$

To compute this amplitude it is convenient to make a convenient choice for the polarisations of the scattered particles.

In this way, upon summation over the spin structures, the amplitude reduces to correlators in the odd spin structure involving only fields along the space-time directions.

"Twisted" coordinates along the "internal" $\mathbb{C}^2$ do not contribute

Moreover, in the vertex operator for the anti-self-dual graviphotons only the holomorphic X coordinate is present, and thus contributes to the amplitude only through its zero-mode momentum

The type I topological amplitude: the one parameter case

$$\mathscr{A}_g = \left\langle (V^+_{\text{grav}})^2 \, (V^-_{\text{grav}})^2 \, V_{\text{gph}}^{2g-2} \right\rangle$$

The amplitude can then be extracted from the generating functions

$$G_{\text{Bose}}(\hbar) = \left\langle \exp \frac{\hat{\hbar}}{t} \int d^2\sigma \left[Z^1 (\bar{\partial} - \partial) Z^2 + \bar{Z}^1 (\bar{\partial} - \partial) \bar{Z}^2 \right] \right\rangle = \frac{(\pi \hat{\hbar})^2}{\sin^2(\pi \hat{\hbar})} \left[H_1(\hat{\hbar}; t/2) \right]^{-2} ,$$

$$G_{\text{Fermi}}(\hbar) = \left\langle \exp \frac{\hat{\hbar}}{t} \int d^2\sigma \left[(\chi^1 - \tilde{\chi}^1)(\chi^2 - \tilde{\chi}^2) + (\bar{\chi}^1 - \tilde{\bar{\chi}}^1)(\bar{\chi}^2 - \tilde{\bar{\chi}}^2) \right] \right\rangle = \left[H_1(\hat{\hbar}; t/2) \right]^2 ,$$

$$\hat{\hbar} = 2\hbar t \, (r m - i s / R)$$

is the dressed polarisation of the anti-self-dual graviphotons, and

$$H_1(z;\tau) = \frac{\theta_1(z|\tau)}{2 \sin(\pi z) \, \eta^3(\tau)} \prod_{\substack{m \in \mathbb{Z} \\ n > 0}} \left(1 - \frac{z^2}{|m + z - n\tau|^2} \right) .$$

The type I topological amplitude: the one parameter case

$$\mathscr{A}_g = \left\langle (V^+_{\rm grav})^2\,(V^-_{\rm grav})^2\,V^{2g-2}_{\rm gph}\right\rangle$$

Taking into account the various contributions one gets the string amplitude

$$\mathscr{F}(\hbar) = -\frac{4}{N}\sum_{\ell=1}^{N-1}\sum_{r,s\in\mathbb{Z}}\sin^2\left(\frac{\pi\ell}{N}\right)e^{2i\pi r\ell/N}\int_0^\infty\frac{dt}{t}\,\frac{(\pi\hbar)^2}{\sin^2(\pi\hat{\hbar})}\,P_r(1/m)\,P_s(R)\,.$$

or, alternatively,

$$\mathscr{F}(\hbar) = \left[-2\sum_{\substack{r=0\,{\rm mod}\,N\\ s\in\mathbb{Z}}} + \sum_{\substack{r=\pm1\,{\rm mod}\,N\\ s\in\mathbb{Z}}}\right]\int_0^\infty\frac{dt}{t}\,\frac{(\pi\hbar)^2}{\sin^2(\pi\hat{\hbar})}\,P_r(1/m)\,P_s(R)$$

$$P_n(\rho) = e^{-\pi t(n/\rho)^2}$$

$$\hat{\hbar} = 2\hbar t(rm - is/R)$$

The type I topological amplitude: the one parameter case

The field theory limit is simple in this case,
since no string states contribute to the amplitude

$$\mathscr{F}(\hbar) = \left[-2 \sum_{\substack{r=0 \bmod N \\ s\in\mathbb{Z}}} + \sum_{\substack{r=\pm 1 \bmod N \\ s\in\mathbb{Z}}} \right] \int_0^\infty \frac{dt}{t} \, \frac{(\pi\hbar)^2}{\sin^2(\pi\hat{\hbar})} \, P_r(1/m)\, P_s(R)$$

It is thus enough to take the large *N* limit
to decouple the KK excitations along the Scherk-Schwarz direction

$$\hbar^{-2}\mathscr{F}(\hbar) = \pi^2(-2\delta_{r,0} + \delta_{r,1}, +\delta_{r,-1}) \int_0^\infty \frac{dt}{t}\, e^{-\pi t (mr)^2} \sum_{s\in\mathbb{Z}} \frac{P_s(R)}{\sin^2(\pi\hat{\hbar})}$$

$$\equiv \pi^2 \sum_{\mu=0,\pm m} d(\mu)\, F(\hbar,\mu)$$

$P_n(\rho) = e^{-\pi t (n/\rho)^2}$

$\hat{\hbar} = 2\hbar t (rm - is/R)$

The type I topological amplitude: the one parameter case

$$\hbar^{-2}\,\mathscr{F}(\hbar) = \pi^2 \sum_{\mu=0,\pm m} d(\mu)\,F(\hbar,\mu)$$

Taken independently, each contribution to the free energy is UV divergent.
We cure this divergence by deforming the integral to the Hankel contour, and get

$$\begin{aligned} F(\hbar,\mu) = &-\frac{2}{(4\hbar R)^2}\left(\zeta'(-2;-i\mu R)+\zeta'(-2;1+i\mu R)\right)+2\left(\zeta'(0;-i\mu R)+\zeta'(0;1+i\mu R)\right) \\ &+4\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}\,(4R\hbar)^{2g-2}\left(\zeta(2g-2;-i\mu R)+\zeta(2g-2;1+i\mu R)\right) \end{aligned}$$

$$\zeta(s;a) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{(0^+)} dt\,(-t)^{s-1}\,\frac{e^{-at}}{1-e^{-t}}$$

is the (analytically continued) Hurwitz zeta function

The type I topological amplitude: the one parameter case

Upon proper redefinition of the 5D radius and of the coupling constant,
and the link of Hurwitz function to polylogarithms,
one finds perfect agreement with Nekrasov-Okounkov

$$
\begin{aligned}
\hbar^{-2}\mathscr{F}(\hbar) = \frac{8}{R^2}&\left[\zeta(3) - \mathrm{Li}_3(e^{-mR}) + \tfrac{1}{6}(mR)^3 - i\pi(mR)^2 - \tfrac{1}{3}\pi^2 mR\right] \\
&+ \tfrac{2}{3}\left[\log(\Lambda R) - \log\left(2\sinh\left(\frac{mR}{2}\right)\right)\right] \\
&+ 8\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)!}(R\hbar)^{2g-2}\left[\mathrm{Li}_{3-2g}(e^{-mR}) - \zeta(3-2g)\right]
\end{aligned}
$$

in the 4D limit $\quad \rightarrow \tfrac{2}{3}\log\left(\frac{\Lambda}{m}\right) + 8\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}\left(\frac{\hbar}{m}\right)^{2g-2}$

[See also Florakis, Zein Assi (2015)]

The type I topological amplitude: the two parameters case

$$\mathscr{A}_{g,n} = \left\langle (V^{+}_{\text{grav}})^2 \, (V^{-}_{\text{grav}})^2 \, V^{2g-2}_{\text{gph}} \, V^{2n}_{S'} \right\rangle$$

How to determine the new insertion,
i.e. the refinement of the topological string?

include the unrefined case **(I)**

admit an exact world-sheet description **(II)**

reproduce the correct field theory limit **(III)**

The type I topological amplitude: the two parameters case

$$\mathscr{A}_{g,n} = \left\langle (V^{+}_{\text{grav}})^2 \, (V^{-}_{\text{grav}})^2 \, V^{2g-2}_{\text{gph}} \, V^{2n}_{S'} \right\rangle$$

Antoniadis, Florakis, Hohenegger, Narain, Zein Assi found the correct topological amplitude involving the vertex operator of the self-dual vector, partner of the (complex conjugate of the) D5 gauge coupling

$$V_{S'} = \epsilon_\mu \Big[(\partial X + i(p\cdot\chi)\psi)(\bar{Z}^\mu + i(p\cdot\tilde{\chi})\tilde{\chi}^\mu) + e^{-\frac{1}{2}(\varphi+\tilde{\varphi})} p_\nu S_{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \tilde{S}^{\dot{\beta}} \, e^{\frac{i}{2}(\phi_3+\tilde{\phi}_3)} \hat{\Sigma}^+ \hat{\tilde{\sigma}}^- \Big] e^{ip\cdot Z} + (\text{left} \leftrightarrow \text{right})$$

[Antoniadis, Florakis, Hohenegger, Narain, Zein Assi 2013]

The type I topological amplitude: the two parameters case

$$\mathscr{A}_{g,n} = \left\langle (V^{+}_{\mathrm{grav}})^2\, (V^{-}_{\mathrm{grav}})^2\, V^{2g-2}_{\mathrm{gph}}\, V^{2n}_{S'} \right\rangle$$

The (generating function of the) amplitude reduces to "simple" gaussian integration. Special care is needed though in properly regularising the determinants since the naive ζ function regularisations fails

A modular invariant regularisation is via the analytic continuation of Selberg-Poincaré series

[Antoniadis, Florakis, Hohenegger, Narain, Zein Assi 2013]

The type I topological amplitude: the two parameters case

$$\mathscr{A}_{g,n} = \left\langle (V^{+}_{\mathrm{grav}})^2 \, (V^{-}_{\mathrm{grav}})^2 \, V^{2g-2}_{\mathrm{gph}} \, V^{2n}_{S'} \right\rangle$$

In our case, one finds

$$\begin{aligned}
\mathscr{A}_{g,n} &= \frac{1}{N} \sum_{\ell=0}^{N-1} \mathscr{A}_{g,n}\left[\begin{smallmatrix} 0 \\ \ell \end{smallmatrix}\right] \\
&= -4 \sin^2\left(\frac{\pi \ell}{N}\right) \int_0^\infty \frac{dt}{t} \sum_{r,s\in\mathbb{Z}} \left(\cos^2(\pi\hat{\epsilon}_+) - \cot\left(\frac{\pi\ell}{N}\right) \sin^2(\pi\hat{\epsilon}_+)\right) P_r(1/m) P_s(R) \, \mathbb{Z}_{\mathrm{K3}}\left[\begin{smallmatrix} 0 \\ \ell \end{smallmatrix}\right] \\
&\quad \times \frac{\pi^2(\epsilon_- - \epsilon_+)(\epsilon_- + \epsilon_+)}{\sin\pi(\hat{\epsilon}_- - \hat{\epsilon}_+) \, \sin\pi(\hat{\epsilon}_- + \hat{\epsilon}_+)} \left[H_1\left(\frac{\hat{\epsilon}_-}{2}; 0; \frac{t}{2}\right)\right]^2 \frac{H_1\left(\frac{\hat{\epsilon}_+}{2}; \frac{\ell}{N}; \frac{t}{2}\right) H_1\left(\frac{\hat{\epsilon}_+}{2}; -\frac{\ell}{N}; \frac{t}{2}\right)}{H_1\left(\frac{\hat{\epsilon}_- - \hat{\epsilon}_+}{2}; 0; \frac{t}{2}\right) H_1\left(\frac{\hat{\epsilon}_- + \hat{\epsilon}_+}{2}; 0; \frac{t}{2}\right)}
\end{aligned}$$

The type I topological amplitude: the two parameters case

$$\mathscr{A}_{g,n} = \left\langle (V^+_{\text{grav}})^2 \, (V^-_{\text{grav}})^2 \, V^{2g-2}_{\text{gph}} \, V^{2n}_{S'} \right\rangle$$

The field theory limit is now more subtle: both string modes and KK excitations along the Scherk-Schwarz direction need to be properly decoupled

$$\mathscr{F}(\epsilon_1,\epsilon_2) = \lim_{t,N\to\infty} \frac{1}{N} \sum_{\ell=0}^{N} \mathscr{A}_{g,n}\left[{0 \atop \ell}\right] \equiv \sum_{\mu=0,\pm m} d(\mu)\, F(\epsilon_1,\epsilon_2;\mu)$$

with

$$F(\epsilon_1,\epsilon_2;0) = \int_0^\infty \frac{dt}{t} \sum_{s\in\mathbb{Z}} \frac{\pi(\epsilon_- - \epsilon_+)}{\sin\pi(\hat\epsilon_- - \hat\epsilon_+)} \frac{\pi(\epsilon_- + \epsilon_+)}{\sin\pi(\hat\epsilon_- + \hat\epsilon_+)} \cos(2\pi\hat\epsilon_+)\, e^{-\pi t (s/R)^2}$$

$$F(\epsilon_1,\epsilon_2;m) = \int_0^\infty \frac{dt}{t} \sum_{s\in\mathbb{Z}} \frac{\pi(\epsilon_- - \epsilon_+)}{\sin\pi(\hat\epsilon_- - \hat\epsilon_+)} \frac{\pi(\epsilon_- + \epsilon_+)}{\sin\pi(\hat\epsilon_- + \hat\epsilon_+)}\, e^{-\pi t (m^2 + (s/R)^2)}$$

The type I topological amplitude: the two parameters case

$$\mathscr{A}_{g,n} = \left\langle (V^+_{\mathrm{grav}})^2\,(V^-_{\mathrm{grav}})^2\,V^{2g-2}_{\mathrm{gph}}\,V^{2n}_{S'}\right\rangle$$

As before, these integral can be regularised in the UV by deforming the integration domain to the Hankel contour, to get

$$F(\epsilon_1,\epsilon_2;\mu) = \sum_{g_1,g_2=0}^{\infty} F_{g_1,g_2}(\epsilon_1,\epsilon_2;\mu)$$

with

$$F(\epsilon_1,\epsilon_2;0) = -16\pi^2\epsilon_1\epsilon_2\,\frac{B_{g_1}}{g_1!}\,\frac{B_{g_2}}{g_2!}\,(R\epsilon_1)^{g_1-1}\,(R\epsilon_2)^{g_2-1}\,(1+(-1)^{g_1+g_2})\,\zeta(3-g_1-g_2)$$

$$F(\epsilon_1,\epsilon_2;m) = -16\pi^2\epsilon_1\epsilon_2\,\frac{B_{g_1}}{g_1!}\,\frac{B_{g_2}}{g_2!}\,(R\epsilon_1)^{g_1-1}\,(R\epsilon_2)^{g_2-1}\,\mathrm{Li}_{3-g_1-g_2}\left(e^{-mR+R\epsilon_+}\right)$$

The type I topological amplitude: the two parameters case

$$\mathscr{A}_{g,n} = \left\langle (V^{+}_{\text{grav}})^2 \, (V^{-}_{\text{grav}})^2 \, V^{2g-2}_{\text{gph}} \, V^{2n}_{S'} \right\rangle$$

These expressions reproduce the Nekrasov-Okounkov result and, upon expanding in ε_+ and ε_-, yield the $F_{g,n}$ topological couplings.

The non-Abelian extension

Introduce *M* D5 branes to get a U(*M*) gauge theory

$$\hbar^{-2}\,\mathscr{F}(\hbar) = \pi^2 M^2 \sum_{\mu=0,\pm m} d(\mu)\, F(\hbar,\mu)$$

or

$$\hbar^{-2}\,\mathscr{F}(\hbar) = \pi^2 \sum_{\mu=0,\pm m}\sum_{i,j=1}^{M} d(\mu)\, F(\hbar,\mu + a_i - a_j)$$

upon introducing Wilson lines along the Scherk-Schwarz direction.

Similar expressions also hold for the two-parameter *Ω* background

Outlook

Actually, can one directly realise the Ω background on the world-sheet?

From its very first definition, the Ω background involves *simple* rotations on the four-dimensional Minkowski space-time, possibly coupled to rotations in the R-symmetry group (together with shifts along a compact direction).

$$ds^2 = A\,dz\,d\bar{z} + g_{IJ}\left(dx^I + \Omega^I{}_K x^K dz + \bar{\Omega}^I{}_K x^K d\bar{z}\right)\left(dx^J + \Omega^J{}_L x^L dz + \bar{\Omega}^J{}_L x^L d\bar{z}\right)$$

Simple, isn't it?

This definition goes under various names

Freely acting orbifolds

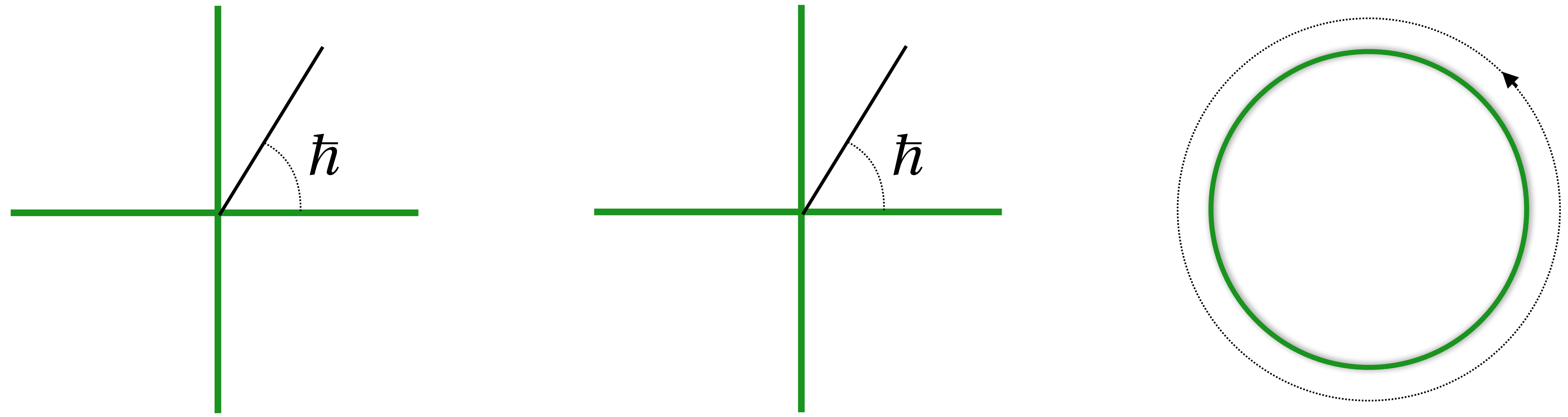
Fluxbranes

Coordinate dependent compactifications

Scherk-Schwarz reduction

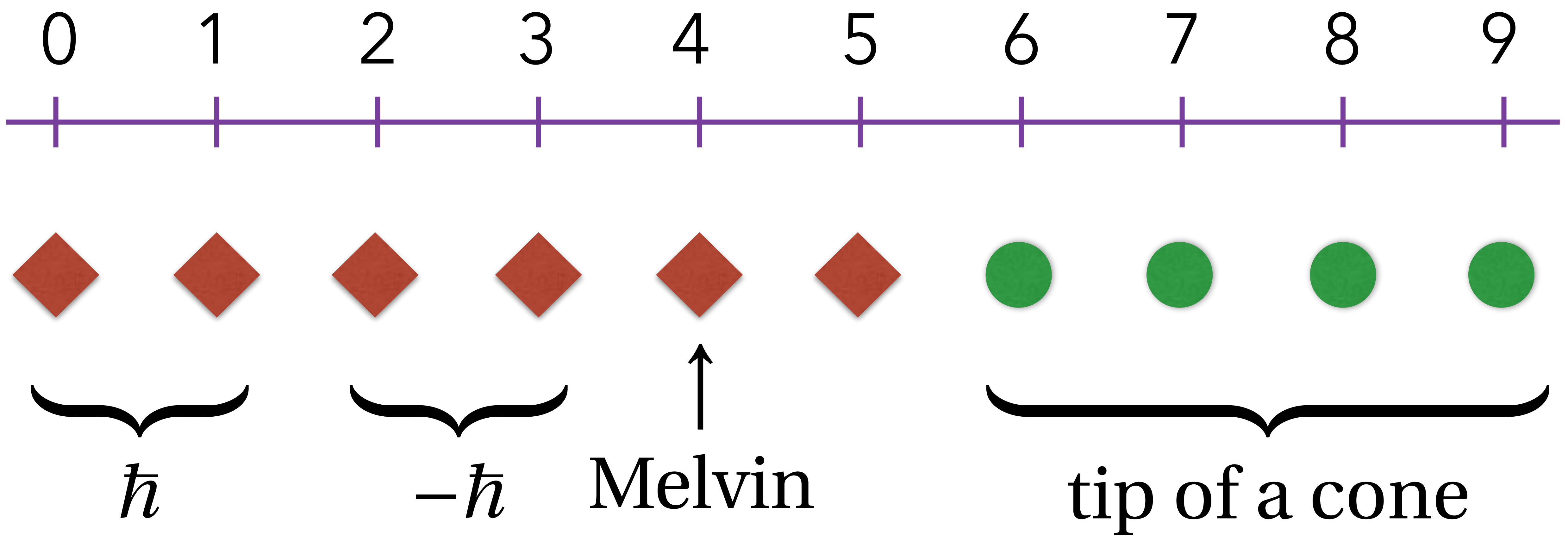
Melvin backgrounds

In the simplest instance of a single-parameter deformation



$$ds^2 = \underset{01}{|dz_1 + \hbar z_1\, dy|^2} + \underset{23}{|dz_2 + \hbar z_2\, dy|^2} + \underset{4}{R^2 dy^2} + \underset{56789}{d\boldsymbol{x}^2}$$

To represent the gauge theory on the Ω background put D5 branes as



Upon T duality along the Melvin direction, a DBI reduction on this background yields the Ω deformed SYM action

[Hellerman, Orlando, Reffert 2012]

Can we perform string computations?

YES: The Melvin background corresponds to an exact CFT

In fact many papers have already appeared, though in different contexts

[Tseytlin 1995]

[Takayanagi, Useugi 2001]

[Dudas, Mourad 2002]

[Angelantonj, Dudas, Mourad 2002]

[.........]

What is (if any) the connection between the topological amplitudes and the Melvin background?

Thank you!