Supercurrent anomalies in 4d SCFTs on curved backgrounds

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Ninth Crete Regional Meeting on String Theory

Kolymbari, 15 July 2017

IP arXiv:1703.04299
Ok Song An arXiv:1703.09607

- Supersymmetric QFTs possess a fermionic supercurrent operator $\mathcal{S}^i$ in the same multiplet as the stress tensor $\mathcal{T}^{ij}$.
- Given a Killing spinor ζ the supercurrent defines the conserved supercharges

$$\overline{\mathcal{Q}}[\zeta] = \int d\sigma_i \; \overline{\zeta}\langle \mathcal{S}^i \rangle, \qquad \mathcal{Q}[\zeta] = \int d\sigma_i \; \langle \overline{\mathcal{S}}^i \rangle \zeta.$$

- This talk concerns the transformation of the supercurrent under rigid supersymmetry on general **curved backgrounds** admitting a Killing spinor ζ, i.e.

$$\boxed{\delta_\zeta \mathcal{S}^i = \{\overline{\mathcal{Q}}[\zeta], \mathcal{S}^i\}}$$

Why is this transformation interesting?

- Supersymmetric vacua/states of the theory satisfy

$$\langle \delta_\zeta \mathcal{S}^i \rangle_{\rm susy} = 0$$

which leads to the **BPS condition** between the bosonic conserved charges.

- The transformation $\delta_\zeta \mathcal{S}^i$ determines part of the **superalgebra**, namely the anticommutator

$$\{\overline{\mathcal{Q}}[\zeta], \mathcal{Q}[\zeta]\}$$

- It also determines the dependence of the **supersymmetric partition function** on the moduli of supersymmetric curved backgrounds [Closset, Dumitrescu, Festuccia, Komargodski '13 & '14; Assel, Cassani, Martelli '14].

- Related to **supersymmetric localization** [Pestun '07]: if a local operator $\mathcal{O}$ is $\mathcal{Q}$-exact, i.e.

$$\mathcal{O} = \delta_\zeta \mathcal{O}_F = \{\overline{\mathcal{Q}}[\zeta], \mathcal{O}_F\}$$

for some other local operator $\mathcal{O}_F$, then the theory can be deformed by $\mathcal{O}$ without changing the value of the supersymmetric partition function.

- In 4d flat space

$$(\delta_\zeta \mathcal{S}^i)_{\alpha\dot\beta} = \{\overline{Q}_{\dot\beta}, \mathcal{S}_{i\alpha}\} = \sigma^j_{\alpha\dot\beta}\Big(2\mathcal{T}_{ij} - i\eta_{ij}\partial^k \mathcal{J}_k + i\partial_j \mathcal{J}_i + \frac{1}{2}\epsilon_{ijkl}\partial^k \mathcal{J}^l\Big)$$

where $\mathcal{J}^i$ is the R-current.

- For a 4d curved background admitting a (conformal) Killing spinor ζ_+ satisfying $\mathcal{D}_i\zeta_+ = \Gamma_i\zeta_-$,

$$\{\overline{\mathcal{Q}}[\zeta], \mathcal{S}^i\} = -\frac{1}{2}\mathcal{T}^{ij}\Gamma_j\zeta_+ + \frac{i}{8\sqrt{3}}\Gamma^{ijk}(\Gamma_{kl} - 2g_{(0)kl})\zeta_+ D_j \mathcal{J}^l + \frac{i}{2\sqrt{3}}\Big(\Gamma^i_l - 3\delta^i_l\Big)\zeta_- \mathcal{J}^l + \mathcal{A}^i_\zeta$$

where $\mathcal{A}^i_\zeta$ is a **supercurrent anomaly**.

Outline

Outline

4d $\mathcal{N} = 1$ supercurrent multiplets

- Any supersymmetric field theory possesses a **stress tensor** $\mathcal{T}^{ij}$ and a **supercurrent** $\mathcal{S}^i$.
- Superconformal theories also have a $U(1)_R$ R**-symmetry current** $\mathcal{J}^i$.
- These currents can be organized in a supermultiplet. In terms of $\mathcal{N} = 1$ superspace they appear as components of a real vector superfield

$$\mathbb{S}^i = \mathcal{J}^i + \bar{\theta}\mathcal{S}^i + \overline{\mathcal{S}}^i\theta + 2(\bar{\theta}\sigma_j\theta)\mathcal{T}^{ij} + \cdots$$

- The off-shell supersymmetry multiplets contain a number of **auxiliary fields**.
- There generically exist different ways of packaging the currents $(\mathcal{J}^i, \mathcal{S}^i, \mathcal{T}^{ij})$ into an off-shell supermultiplet, corresponding to a different set of auxiliary fields.
- When more than one such off-shell multiplets exist, they are related through **improvement terms**, which, besides changing the auxiliary fields present, transform the currents as

$$\mathcal{T}_{ij} \to \mathcal{T}'_{ij} = \mathcal{T}_{ij} + (\eta_{ij}\partial^2 - \partial_i\partial_j)t, \qquad \mathcal{S}_{i\alpha} \to \mathcal{S}'_{i\alpha} = \mathcal{S}_{i\alpha} + (\sigma_{ij})_\alpha^\beta \partial^j s_\beta$$

- The most general supercurrent multiplet (most auxiliary fields) is the **S-multiplet** [Komargodski, Seiberg '10]. The S-multiplet always exists and comprises 16+16 off-shell degrees of freedom in the real superfield $\mathbb{S}_{\alpha\dot{\alpha}}$, an auxiliary chiral superfield X, and an auxiliary spinor (chiral fieldstrength) superfield χ_α, satisfying

$$\overline{D}^{\dot{\alpha}}\mathbb{S}_{\alpha\dot{\alpha}} = D_\alpha X + \chi_\alpha, \qquad \overline{D}_{\dot{\alpha}} X = 0, \qquad \overline{D}_{\dot{\alpha}}\chi_\alpha = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} - D^\alpha \chi_\alpha = 0$$

- These defining relations are preserved under the transformations

$$\mathbb{S}_{\alpha\dot{\alpha}} \to \mathbb{S}'_{\alpha\dot{\alpha}} = \mathbb{S}_{\alpha\dot{\alpha}} + [D_\alpha, \overline{D}_{\bar{\alpha}}]U \quad X \to X' = X + \frac{1}{2}\overline{D}^2 U \quad \chi_\alpha \to \chi'_\alpha = \chi_\alpha + \frac{3}{2}\overline{D}^2 D_\alpha U$$

  where U is a real superfield. This shifts the currents by improvement terms.

- The **FZ-multiplet** is obtained by setting $\chi_\alpha = 0$ and comprises 12+12 off-shell degrees of freedom. It exists if there are no FI terms and the Kähler form of the target space is exact.

- The **R-multiplet** is obtained by setting $X = 0$ and contains also 12+12 off-shell degrees of freedom. It exists if there is a $U(1)_R$ symmetry.

Classical Ward identities

- The defining relations

$$\boxed{\overline{D}^{\dot\alpha}\mathbb{S}_{\alpha\dot\alpha} = D_\alpha X + \chi_\alpha\,, \qquad \overline{D}_{\dot\alpha} X = 0, \qquad \overline{D}_{\dot\alpha}\chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha} - D^\alpha\chi_\alpha = 0}$$

  are the **classical**, **flat space** Ward identities, which only impose constraints on the one-point functions of the currents – not on their higher order correlators.

- Turning on an **arbitrary** curved background (not necessarily admitting rigid supersymmetry!) leads to the general **classical**, **curved space** Ward identities, which impose constraints on arbitrary n-point functions of the currents.

- An one-loop calculation on an arbitrary curved background leads to the **quantum** Ward identities, including quantum **anomalies**.

Supercurrent transformation under supersymmetry

- The transformation of the supercurrent $\mathcal{S}^i$ under (flat space) global supersymmetry is given by the anticommutators

$$\{\overline{Q}_{\dot\beta}, \mathcal{S}_{i\alpha}\} = \sigma^j_{\alpha\dot\beta}\Big(2\mathcal{T}_{ij} + \frac{1}{2}\epsilon_{ijkl}F^{kl} - i\eta_{ij}\partial^k\mathcal{J}_k + i\partial_j\mathcal{J}_i + \frac{1}{2}\epsilon_{ijkl}\partial^k\mathcal{J}^l\Big)$$

$$\{Q_\beta, \mathcal{S}_{i\alpha}\} = 2i\epsilon_{\lambda\beta}(\sigma_{ij})^\lambda_\beta\partial^j x^\dagger$$

where F_{ij} is a closed two-form and the complex scalar x is the lowest component of the chiral superfield X.

- The presence of the operators F_{ij} and x depends on the multiplet: x is absent in the R-multiplet while F_{ij} is absent in the FZ-multiplet.

- The main aim of this talk is to generalize these anticommutation relations for **curved backgrounds** admitting rigid supersymmetry.

Coupling to supergravity

- The background supergravity fields reside in a real vector superfield $\mathbb{H}_i$ that to linear order couples to the current superfield as

$$\int d^4\theta \, \mathbb{S}^i \mathbb{H}_i$$

- Gauging the global symmetries amounts to assigning a local gauge transformation to the background superfield

$$\mathbb{H}_{\alpha\dot{\alpha}} \to \mathbb{H}'_{\alpha\dot{\alpha}} = \mathbb{H}_{\alpha\dot{\alpha}} + D_\alpha \overline{L}_{\dot{\alpha}} - \overline{D}_{\dot{\alpha}} L_\alpha$$

  and demanding that the above linear coupling is gauge invariant.

- Given this gauge transformation, the defining relations of the supercurrent multiplet, i.e. the Ward identities, can be derived by imposing gauge-invariance of the linear coupling. This is a generalized **Noether's theorem**, formulated directly in terms of the (composite) current operators.

- The background fields coupling to the physical currents can be extracted by imposing e.g. the Wess-Zumino gauge conditions

$$\mathbb{H}_i| = \mathbb{H}_i|_\theta = \mathbb{H}_i|_{\bar\theta} = \mathbb{H}_i|_{\theta^2} = \mathbb{H}_i|_{\bar\theta^2} = 0$$

- In this gauge the metric h_{ij}, gravitino $\Psi_{i\alpha}$ and $U(1)_R$ gauge field b_i correspond to

$$\mathbb{H}_i|_{\theta\sigma^j\bar\theta} = h_{ij} - h\eta_{ij}, \qquad \mathbb{H}_i|_{\bar\theta^2\theta} = \Psi_{i\alpha} + (\sigma_i\bar\sigma^j\Psi_j)_\alpha, \qquad \mathbb{H}_i|_{\theta^4} = b_i$$

- There is a **residual gauge freedom** that transforms the background fields as

$$\delta h_{ij} = \partial_i\xi_j + \partial_j\xi_i, \qquad \delta\Psi_{i\alpha} = \partial_i\omega_\alpha, \qquad \delta b_i = \partial_i\omega$$

- The precise form of these gauge transformations depends on the choice of supercurrent multiplet, as well as the gauge-fixing condition.

- Coupling different multiplets to supergravity result in different supergravities:
  - FZ-multiplet → **old minimal supergravity** [Stelle, West '78; Ferrara, van Nieuwenhuizen '78; Fradkin, Vasiliev '78]
  - R-multiplet → **new minimal supergravity** [Akulov, Volkov, Soroka '77; Sohnius, West '81],
  - S-multiplet → **16/16 supergravity** [Girardi, Grimm, Muller, Wess '84; Lang, Louis, Ovrut '85; Siegel '86]
  - Conformal multiplet → **conformal supergravity** [Kaku, Townsend, van Nieuwenhuizen '77 & '78; Ferrara, Zumino '78]

Outline

The symplectic space of local QFT observables

- The linear coupling

$$W[\cdots,\mathbb{H}] = \cdots + \int d^4x \int d^4\theta \; \mathbb{S}^i \mathbb{H}_i$$

in the effective action implies that the supercurrent multiplet operators can be defined in the Local Renormalization Group sense [Osborn '94] as

$$\boxed{\mathbb{S}^i = \frac{\delta W}{\delta \mathbb{H}_i}}$$

- This defines the **consistent** current multiplet, which can be coupled to supergravity. The **covariant** current multiplet differs from the consistent one by Bardeen-Zumino terms corresponding to the variation of bulk Chern-Simons terms [Bardeen, Zumino '84].

- The above relation implies that the superfields $\mathbb{S}^i$ and $\mathbb{H}_i$ parameterize a symplectic manifold with Poisson bracket

$$\{,\}_{\rm PB} = \int d^4x \int d^4\theta \Big(\frac{\delta}{\delta \mathbb{H}_i} \frac{\delta}{\delta \mathbb{S}^i} - \frac{\delta}{\delta \mathbb{S}^i} \frac{\delta}{\delta \mathbb{H}_i} \Big)$$

Ward identities as first class constraints

- One of the reasons why the symplectic space $(\mathbb{S}^i, \mathbb{H}_i)$ is useful is that it provides a direct connection between the gauge symmetries and the Ward identities.
- Consider the following functional on the symplectic space $(\mathbb{S}^i, \mathbb{H}_i)$

$$\mathcal{C}[L] = \int d^4x \int d^4\theta \; L^\alpha \left(\overline{D}^{\dot{\alpha}} \mathbb{S}_{\alpha\dot{\alpha}} - D_\alpha X - \chi_\alpha \right) + \text{h.c.}$$

  where L^α is an arbitrary complex superfield.
- The gauge transformation of any symplectic variable can then be obtained through the Poisson bracket. For the background fields we get

$$\{\mathcal{C}[L], \mathbb{H}_i\}_{\text{PB}} = D_\alpha \overline{L}_{\dot{\alpha}} - \overline{D}_{\dot{\alpha}} L_\alpha = \delta_L \mathbb{H}_i$$

- More importantly, the (curved superspace) Ward identities give the transformation of the currents under the gauge symmetries:

$$\boxed{\{\mathcal{C}[L], \mathbb{S}_i\}_{\text{PB}} = \delta_L \mathbb{S}^i = \cdots}$$

- If the quantum Ward identities, i.e. including the anomalies, are used to define the generating function $\mathcal{C}[L]$, the Poisson bracket gives the quantum transformation of the currents, including any anomalous terms.

Killing symmetries and conserved charges

- So far the background fields in $\mathbb{H}_i$ and the gauge parameters in L^α are arbitrary.
- For a given background $\mathbb{H}_i$, the gauge parameters L_o^α that satisfy

$$\boxed{\delta_{L_o}\mathbb{H}_i = D_\alpha \overline{L}_{o\dot{\alpha}} - \overline{D}_{\dot{\alpha}} L_{o\alpha} = 0}$$

  correspond to **Killing symmetries** of the background $\mathbb{H}_i$.
- The Killing spinor of rigid supersymmetry corresponds to a specific component of the superfield L_o^α.
- The conserved charges associated with the Killing symmetries are given by

$$\mathbb{Q}[L_o] = \mathcal{C}[L_o]$$

- The **quantum** transformation of the currents under the Killing symmetries is

$$\boxed{[\mathbb{Q}[L_o], \mathbb{S}^i] = \delta_{L_o}\mathbb{S}^i}$$

  which includes the anticommutators $\{\overline{Q}_{\dot{\beta}}, \mathcal{S}_{i\alpha}\}$ and $\{Q_\beta, \mathcal{S}_{i\alpha}\}$.

Example

- Demanding that the diffemorphisms $z \to z + \epsilon$, $\bar{z} \to \bar{z} + \bar{\epsilon}$ preserve the flat 2d metric leads to the Killing vector equations

$$\bar{\partial}\epsilon = 0, \quad \partial\bar{\epsilon} = 0$$

with solutions $\epsilon(z)$, $\bar{\epsilon}(\bar{z})$.

- The transformation of the stress tensor under the Killing symmetries is given by the restriction of its transformation under general diffeomorphisms to the Killing vectors, namely

$$[Q[\epsilon], \mathcal{T}(z)] = \delta_\epsilon \mathcal{T}(z) = 2\mathcal{T}\partial\epsilon + \epsilon\partial\mathcal{T} - \frac{c}{24\pi}\partial^3\epsilon$$

- This includes the anomalous term due to the central charge!

Derivation of the quantum Ward identities

- The previous discussion implies that if we know the quantum Ward identities for arbitrary background fields we can determine the transformation of the currents under both generic gauge transformations and the Killing symmetries of any given background.
- Determining the quantum superconformal Ward identities requires that:
  - We couple the supercurrent multiplet **non linearly** to arbitrary background fields.
  - We compute the **anomalies** via a one-loop calculation in the presence of general background fields.
- However, the general form of the Ward identities is a consequence of local symmetries and be determined for unspecified values of the anomaly coefficients:
  - Use the **Festuccia-Seiberg argument** [Festuccia, Seiberg '11] to couple the theory to arbitrary background fields and determine the possible anomaly terms through the Wess-Zumino consistency condition (work in progress).
  - Use **holography** to directly obtain the quantum Ward identities for arbitrary background fields, but for $a = c$.

Quantum Ward identities from holography

- Standard 5D minimal gauged supergravity describes holographically the current multiplet in components and without the auxiliary fields.
- The arbitrary sources of the bulk fields

$$e^a_{(0)i}, \qquad \Psi_{(0)+i}, \qquad A_{(0)i}$$

  specify an arbitrary (non-linear) field theory background.

- Note that in the presence of fermionic sources it is imperative to describe the geometry in terms of a vielbein instead of a metric.
- The variation of the renormalized on-shell supergravity action defines the conjugate (consistent) current operators via

$$\delta W = \int d^dx \sqrt{-g_{(0)}} \Big(-\mathcal{T}^i_a \delta e^a_{i\,(0)} + \mathcal{J}^i \delta A_{(0)i} + \overline{\mathcal{S}}^i \delta \Psi_{(0)+i} + \delta \overline{\Psi}_{(0)+i} \mathcal{S}^i \Big)$$

Quantum Ward identities from holography

- Bosonic Ward identities:

$$D_j(e^a_{(0)i}\mathcal{T}^j_a - \overline{\mathcal{S}}^j\Psi_{(0)+i} - \overline{\Psi}_{(0)+i}\mathcal{S}^j) + \overline{\mathcal{S}}^j\mathcal{D}_i\Psi_{(0)+j} + \overline{\Psi}_{(0)+j}\overleftarrow{\mathcal{D}}_i\mathcal{S}^j + F_{(0)ij}\mathcal{J}^j = \mathcal{A}_{Mi},$$

$$D_i\mathcal{J}^i + i\sqrt{3}(\overline{\mathcal{S}}^i\Psi_{(0)+i} - \overline{\Psi}_{(0)+i}\mathcal{S}^i) = \mathcal{A}_R,$$

$$e^a_{(0)i}\mathcal{T}^i_a - \frac{1}{2}\overline{\Psi}_{(0)+i}\mathcal{S}^i - \frac{1}{2}\overline{\mathcal{S}}^i\Psi_{(0)+i} = \mathcal{A}_W,$$

$$e^{i[a}_{(0)}\mathcal{T}^{b]}_i + \frac{1}{4}(\overline{\mathcal{S}}^i\Gamma^{ab}\Psi_{(0)+i} - \overline{\Psi}_{(0)+i}\Gamma^{ab}\mathcal{S}^i) = 0$$

- Fermionic Ward identities:

$$\mathcal{D}_i\mathcal{S}^i + \frac{1}{2}\mathcal{T}^i_a\Gamma^a\Psi_{(0)+i} - \frac{i}{8\sqrt{3}}\mathcal{J}^i(\Gamma_{ij} - 2g_{(0)ij})\Gamma^{jpq}\mathcal{D}_p\Psi_{(0)+q} = \mathcal{A}_S,$$

$$\Gamma_i\mathcal{S}^i - \frac{i\sqrt{3}}{4}\mathcal{J}^i\Psi_{(0)+i} = \mathcal{A}_{sW}$$

- The fermionic anomalies take the form

$$\mathcal{A}_S = \frac{i\ell^2}{18\kappa^2}\,\epsilon^{iskl}F_{(0)sk}A_{(0)l}(\Gamma_{ij}-2g_{(0)ij})\Gamma^{jpq}\mathcal{D}_p\Psi_{(0)+q}$$

$$\mathcal{A}_{sW} = \frac{\ell}{2\kappa^2}\Big[\frac{\ell^2}{4}\Big(R_{ij}-\frac{1}{6}Rg_{(0)ij}\Big)\Gamma^i\Gamma^{jkl}\mathcal{D}_k\Psi_{(0)+l}+\frac{2i\ell}{3}\,\epsilon^{ijkl}F_{(0)jk}A_{(0)l}\Psi_{(0)+i}$$
$$+\frac{i\ell}{4\sqrt{3}}F_{(0)jk}(2\Gamma^{jk}\Gamma^i-3\Gamma^{jki})\Gamma_i{}^{pq}\mathcal{D}_p\Psi_{(0)+q}\Big]$$

- Note that moving the orange terms to the LHS of the fermionic Ward identities shifts the R-current from the consistent to the covariant (and gauge invariant) one:

$$\mathcal{J}^i \to \mathcal{J}^i{}_{\rm cov} = \mathcal{J}^i + \frac{4\ell}{3\sqrt{3}\,\kappa^2}\epsilon^{ijkl}F_{(0)jk}A_{(0)l}$$

Ward identities as generators of local symmetries

- The Ward identities define the generating function (first class constraints) on the symplectic space of local couplings and operators:

$$\mathcal{C}[\sigma,\xi_o,\theta_o,\lambda_o,\epsilon_{o+},\epsilon_{o-}] = \int d^dx\Big(\sigma\mathcal{W}_W + \xi_o^i\mathcal{W}_{Mi} + \theta_o\mathcal{W}_R + \lambda_{o\ ab}\mathcal{W}_L^{ab} + \overline{\epsilon}_{o+}\mathcal{W}_S + \overline{\epsilon}_{o-}\mathcal{W}_{sW} + \overline{\mathcal{W}}_S\epsilon_{o+} + \overline{\mathcal{W}}_{sW}\epsilon_{o-}\Big)$$

- The **quantum** transformation of the symplectic variables under the local symmetries is obtained through the Poisson bracket, e.g.

$$\{\mathcal{C}[\sigma,\xi_o,\theta_o,\lambda_o,\epsilon_{o+},\epsilon_{o-}],\Psi_{(0)+i}\}_{\text{PB}} = -\frac{\delta\mathcal{C}}{\delta\overline{\mathcal{S}}^i} = \delta_{\xi_o,\lambda'_o,\theta'_o,\epsilon_o}\Psi_{(0)+i}$$

$$\{\mathcal{C}[\sigma,\xi_o,\theta_o,\lambda_o,\epsilon_{o+},\epsilon_{o-}],\mathcal{S}^i\}_{\text{PB}} = \frac{\delta\mathcal{C}}{\delta\overline{\Psi}_{(0)+i}} = \delta_{\sigma,\xi_o,\lambda'_o,\theta'_o,\epsilon_{o+},\epsilon_{o-}}\mathcal{S}^i$$

- Since the anomalies are local functions of the sources, they only contribute to the transformation of the currents!

Fermionic transformations of the sources

- Using the explicit expressions for the quantum Ward identities derived holographically one obtains the following local **supersymmetry** and **superWeyl** transformations of the sources, parameterized respectively by the spinors $\epsilon_{o+}(x)$ and $\epsilon_{o-}(x)$:

$$\delta_{\epsilon_{o+},\epsilon_{o-}} e_i^a{}_{(0)} = \frac{1}{2}(\overline{\epsilon}_{o+}\Gamma^a\Psi_{(0)+i} - \overline{\Psi}_{(0)+i}\Gamma^a\epsilon_{o+}),$$

$$\delta_{\epsilon_{o+},\epsilon_{o-}} A_{(0)i} = \frac{i}{4\sqrt{3}}\Big(\overline{\Psi}_{(0)+i}\epsilon_{o-} + \overline{\Psi}_{(2)-i}\epsilon_{o+} - \overline{\epsilon}_{o+}\Psi_{(2)-i} - \overline{\epsilon}_{o-}\Psi_{(0)+i}\Big),$$

$$\delta_{\epsilon_{o+},\epsilon_{o-}} \Psi_{(0)+i} = \mathcal{D}_{(0)i}\epsilon_{o+} - \frac{1}{\ell}\Gamma_{(0)i}\epsilon_{o-}$$

where

$$\Psi_{(2)-i} = -\frac{\ell}{6}(\Gamma_{(0)ij} - 2g_{(0)ij})\Gamma^{jkl}_{(0)}\mathcal{D}_{(0)k}\Psi_{(0)+l}.$$

Fermionic transformations of the supercurrent

- Moreover, the quantum transformations of the supercurrent under local supersymmetry and superWeyl transformations are:

$$\delta_{\epsilon_{o+}}\mathcal{S}^i = -\frac{1}{2}\mathcal{T}^i_a\Gamma^a\epsilon_{o+} + \frac{i\ell}{8\sqrt{3}}\Gamma^{ijk}_{(0)}(\Gamma_{(0)kl} - 2g_{(0)kl})\mathcal{D}_{(0)j}\Big[\Big(\mathcal{J}^l + \frac{4\ell}{3\sqrt{3}\,\kappa^2}\epsilon^{lpqs}F_{(0)pq}A_{(0)s}\Big)\epsilon_{o+}\Big]$$

$$\delta_{\epsilon_{o-}}\mathcal{S}^i = -\frac{i\sqrt{3}}{4}\Big(\mathcal{J}^i + \frac{4\ell}{3\sqrt{3}\,\kappa^2}\epsilon^{lpqs}F_{(0)pq}A_{(0)s}\Big)\epsilon_{o-} - \frac{\ell^2}{8\kappa^2}\Gamma^{ijk}_{(0)}\Gamma^l_{(0)}\mathcal{D}_{(0)j}\Big[\Big(R_{kl}[g_{(0)}] - \frac{1}{6}R[g_{(0)}]g_{(0)kl}\Big)\epsilon_{o-}\Big] - \frac{i\ell}{8\sqrt{3}\,\kappa^2}\Gamma^{ij}_{(0)k}\Big(2\Gamma^k_{(0)}\Gamma^{pq}_{(0)} - 3\Gamma^{kpq}_{(0)}\Big)\mathcal{D}_{(0)j}(F_{(0)pq}\epsilon_{o-})$$

- These transformations are anomalous!

Outline

$\mathcal{N}=1$ supersymmetric backgrounds in 4d

- A general class of $\mathcal{N}=1$ supersymmetric backgrounds in 4d takes the form [Klare, Tomansiello, Zaffaroni '12; Dumitrescu, Festuccia, Seiberg '12]

$$
\begin{aligned}
ds^2_{(0)} &= -\,\mathrm{d}t^2 + \Big(\mathrm{d}\psi + \frac{\mathrm{i}}{2}\partial_{\bar z}\mu \mathrm{d}\bar z - \frac{\mathrm{i}}{2}\partial_z\mu \mathrm{d}z\Big)^2 + 4e^w \mathrm{d}z\mathrm{d}\bar z,\\
A_{(0)} &= -\,\frac{1}{\sqrt{3}}\Big[-\frac{1}{8}e^{-w}\partial_z\partial_{\bar z}\mu \mathrm{d}t + \frac{1}{4}e^{-w}\partial_z\partial_{\bar z}\mu\Big(\mathrm{d}\psi + \frac{\mathrm{i}}{2}\partial_{\bar z}\mu\mathrm{d}\bar z - \frac{\mathrm{i}}{2}\partial_z\mu\mathrm{d}z\Big)\\
&\qquad + \frac{\mathrm{i}}{4}(\partial_{\bar z}w\mathrm{d}\bar z - \partial_z w\mathrm{d}z) + \gamma'\mathrm{d}t + \gamma\mathrm{d}\psi + \mathrm{d}\lambda\Big]
\end{aligned}
$$

  where $w(z,\bar z)$ and $\mu(z,\bar z)$ are arbitrary functions, and γ', γ and $\lambda(z,\bar z)$ are locally pure gauge but contain global information.

- For generic choices of the functions $w(z,\bar z)$ and $\mu(z,\bar z)$ this background admits a single conformal Killing spinor that satisfies

$$
\mathcal{D}_{(0)i}\zeta_+ = \frac{1}{\ell}\Gamma_{(0)i}\zeta_-, \qquad \zeta_- = \frac{\ell}{4}\Gamma^j_{(0)}\mathcal{D}_{(0)j}\zeta_+ \neq 0
$$

Transformation of the supercurrent under global supersymmetry

- Restricting the transformation of the supercurrent under local supersymmetry and superWeyl transformations to the Killing spinor gives the transformation of the supercurrent under **global** supersymmetry:

$$
\begin{aligned}
\delta_\zeta \mathcal{S}^i = & -\frac{1}{2}\mathcal{T}^{ij}\widehat{\Gamma}_{(0)j}\zeta_+ \\
& + \frac{i}{8\sqrt{3}}\Gamma^{ijk}_{(0)}(\Gamma_{(0)kl} - 2g_{(0)kl})\zeta_+ D_{(0)j}\left(\mathcal{J}^l + \frac{4}{3\sqrt{3}\kappa^2}\widehat{\epsilon}^{lpqs}F_{(0)pq}A_{(0)s}\right) \\
& + \frac{i}{2\sqrt{3}}(\Gamma^i_{(0)l} - 3\delta^i_l)\zeta_-\left(\mathcal{J}^l + \frac{4}{3\sqrt{3}\kappa^2}\epsilon^{lpqs}F_{(0)pq}A_{(0)s}\right) \\
& - \frac{\ell^2}{8\kappa^2}\Gamma^{ijk}_{(0)}\Gamma^l_{(0)}\mathcal{D}_{(0)j}\left[\left(R_{kl}[g_{(0)}] - \frac{1}{6}R[g_{(0)}]g_{(0)kl}\right)\zeta_-\right] \\
& - \frac{i}{8\sqrt{3}\kappa^2}\Gamma^{ij}_{(0)\;k}\left(2\Gamma^k_{(0)}\Gamma^{pq}_{(0)} - 3\widehat{\Gamma}^{kpq}_{(0)}\right)\mathcal{D}_{(0)j}(F_{(0)pq}\zeta_-)
\end{aligned}
$$

Outline

w and μ dependence of the partition function

- Under a local deformation of the function $w(z,\bar z)$, keeping $\mu(z,\bar z)$ fixed:

$$\delta_w W = \int d^4x\sqrt{-g_{(0)}}\;\delta w\; \mathrm{i}\sqrt{2}e^{w/2}\Big(\,\delta_\zeta^{\text{anom}}\mathcal{S}^z\Big|_1+\delta_\zeta^{\text{anom}}\mathcal{S}^z\Big|_2\Big)$$
$$=\frac{1}{2^6 3\kappa^2}\int d^4x\sqrt{-g_{(0)}}\;\delta w\Big(-u^2R_{2d}-\frac{1}{2}\Box_{2d}u^2+\frac{19}{32}u^4$$
$$+\frac{8}{9}(\gamma+2\gamma')(2uR_{2d}+2\Box_{2d}u-u^3)\Big)$$

where $u=e^{-w}\partial_z\partial_{\bar z}\mu$ and we have used the fact that $\langle\delta_\zeta\mathcal{S}^i\rangle_{\text{susy}}=0$.

- Under a local deformation of the function $\mu(z,\bar z)$, keeping $w(z,\bar z)$ fixed:

$$\delta_\mu W=\int d^4x\sqrt{-g_{(0)}}\Big\{\sqrt{2}\Big[\frac{\mathrm{i}}{2}\Big(\,\delta_\zeta^{\text{anom}}\mathcal{S}^{\bar z}\Big|_1-\delta_\zeta^{\text{anom}}\mathcal{S}^{\bar z}\Big|_2\Big)$$
$$+\frac{1}{4}e^{-\frac{w}{2}}\Big(\,\delta_\zeta^{\text{anom}}\mathcal{S}^t\Big|_1+\delta_\zeta^{\text{anom}}\mathcal{S}^t\Big|_2\Big)\Big]\partial_{\bar z}\delta\mu+\text{h.c.}\Big\}$$
$$=\frac{1}{2^9 3^2\kappa^2}\int d^4x\sqrt{-g_{(0)}}(e^{-w}\partial_z\partial_{\bar z}\delta\mu)\Big(24uR_{2d}-19u^3$$
$$+\frac{32}{3}(\gamma+2\gamma')(3u^2-4R_{2d})\Big)$$

w and μ dependence of the partition function

- The final expressions precisely match those obtained in [Genoloni, Cassani, Martelli, Sparks '16], but the calculation and the explanation of the result are completely different!
- The derivation here is in fact identical to [Closset, Dumitrescu, Festuccia, Komargodski '13], except that we have included the anomalous transformation of the supercurrent, leading to different conclusions about the dependence of the supersymmetric partition function on the complex structure moduli, i.e. $\mu(z, \bar{z})$, and the hermitian metric, i.e. $w(z, \bar{z})$.

Outline

Casimir charges and the BPS relation

- The conserved electric and Killing charges are respectively:

$$\mathcal{Q}^{\omega}_{\mathrm{e}} = \frac{1}{\sqrt{3}}\int d\sigma_i \left(\langle \mathcal{J}^i\rangle - \omega\frac{2\ell}{3\sqrt{3}\,\kappa^2}\epsilon^{ipqs}F_{(0)pq}A_{(0)s}\right)$$

$$\mathcal{Q}^{\omega}[\mathcal{K}] = -\int d\sigma_i \left[\langle \mathcal{T}^i_j\rangle - \left(\langle \mathcal{J}^i\rangle - \omega\frac{2\ell}{3\sqrt{3}\,\kappa^2}\epsilon^{ipqs}F_{(0)pq}A_{(0)s}\right)A_{(0)j}\right]\mathcal{K}^j$$

where the parameter ω is arbitrary. $\omega = -2$ corresponds to the Maxwell charges and $\omega = 1$ to the Page charges.

- Contracting the identity $\langle \delta_\zeta \mathcal{S}^i\rangle_{\mathrm{susy}} = 0$ with $\mathrm{i}\overline{\zeta}_+$ leads to the BPS relation

$$\boxed{M^{\omega} + J^{\omega} + (\gamma-\gamma')\mathcal{Q}^{\omega}_{\mathrm{e}} = \mathcal{Q}^{\omega}_{\mathrm{anomaly}}.}$$

where

$$M^{\omega} = \mathcal{Q}^{\omega}[-\partial_t], \qquad J^{\omega} = \mathcal{Q}^{\omega}[\partial_\psi],$$

and $\mathcal{Q}^{\omega}_{\mathrm{anomaly}}$ is a non-vanishing **anomaly charge**.

Outline

Conclusions and Outlook

Conclusions:

- The supercurrent transforms **anomalously** under global supersymmetry on 4d curved backgrounds admitting (conformal) Killing spinors!
- The supersymmetric partition function on such backgrounds is **not** invariant under deformations of the complex structure moduli or of the Hermitian metric.
- The BPS relation is **anomalous**!

Outlook:

- Derive the general form of the quantum Ward identities for the various multiplets by classifying all non-trivial cocycles of the Wess-Zumino consistency condition.
- Does the supercurrent anomaly vanish for backgrounds with $\mathcal{N} = 2$ supersymmetry?
- Better understand the current multiplet arising from holography.
- Consequences of anomalous BPS condition for supersymmetric AdS_5 black holes.