Thermofield Dynamics & Chern-Simons Gravity

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Introduction

- Fuzzy spaces (noncommutative spaces) provide approximations to a differential manifold in terms of finite-dimensional matrices.
- They can be modeled by the lowest Landau level of a quantum Hall system.
- Thermofield dynamics gives a way of discussing mixed states in terms of a pure state description. Presumably they are important for gravity. (Israel; Maldacena; Jacobson; + *others*)
- We want to put these together to analyze gravity on odd dimensional noncommutative spacetimes, an approach different from the one based on the spectral action principle. (Connes, Chamseddine, ...)

Some of the material is joint work with D. Karabali; Lei Jiusi.

- Thermofield dynamics can be expressed in terms of a field theory for a quantum Hall system, with a particular limit to be taken at the end.
- For a fuzzy space, gauge fields enter as a way of defining the large N limit.
- Double the Hilbert space modeling the fuzzy space to $\mathcal{H}_N \otimes \tilde{\mathcal{H}}_N$, with left chirality gravitational fields ($SO(3)_L$ in 2+1) on $\mathcal{H}_N$ and right chirality fields ($SO(3)_R$) on $\tilde{\mathcal{H}}_N$.
- This leads to

$$\begin{aligned} S &= -\frac{1}{4\pi}\int \left[\mathrm{Tr}\left(A\,dA + \tfrac{2}{3}A^3\right)_L - \mathrm{Tr}\left(A\,dA + \tfrac{2}{3}A^3\right)_R\right] \\ &= \text{Einstein - Hilbert action} \end{aligned}$$

- Particle dynamics can be described via the Einstein-Infeld-Hoffmann method.

- For a system with Hilbert space $\mathcal{H}$, the expectation value of observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho\, \mathcal{O}) = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} \mathcal{O}\right), \qquad Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

- We double the Hilbert space to $\mathcal{H} \otimes \tilde{\mathcal{H}}$ and introduce the pure state (called thermofield vacuum)

$$|\Omega\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{1}{2}\beta E_n} |n, \tilde{n}\rangle$$

- Then we get

$$\langle \Omega | \, \mathcal{O} \, | \Omega \rangle = \frac{1}{Z} \sum_{m,n} e^{-\frac{1}{2}\beta(E_n + E_m)} \langle m | \mathcal{O} | n \rangle \langle \tilde{m} | \tilde{n} \rangle = \mathrm{Tr}(\rho\, \mathcal{O})$$

- The Hamiltonian is taken as

$$\check{H} = H + \tilde{H} = H \otimes \mathbb{1} - \mathbb{1} \otimes H, \qquad \Longrightarrow \quad \check{H} \, |\Omega\rangle = 0$$

- The formalism of thermofield dynamics is very useful for considering time-dependent (nonequilibrium) effects at finite temperature.

- For a quantum system, the density matrix evolves by the Liouville equation

$$i\,\frac{\partial \rho}{\partial t} = H\,\rho - \rho\,H$$

- We can write an “action" for this,

$$S = \int dt\,\mathrm{Tr}\left[\rho_0\,\left(U^\dagger\,i\partial_t U - U^\dagger\,H\,U\right)\right]$$

  where U's are to be varied, and $\rho = U\,\rho_0\,U^\dagger$.

- Our first step is to write thermofield dynamics as a field theory functional integral with an action similar to this.

- The transition amplitude for observables B, C is given by

$$\langle T\, B_{t_1}\, C_{t_2} \rangle = \sum (\sqrt{\rho})_{\tilde{m}m} \langle m|\, T\left[B_{t_1}\, C_{t_2}\, e^{-iHt}\right] |n\rangle\; (\sqrt{\rho})_{n\tilde{n}}\, \langle \tilde{n}|\, e^{iH^T t}\, |\tilde{m}\rangle$$

- For the tilde part, we have $H \to -H^T$ as expected for conjugation for unitary matrices.

- Introduce coherent states for some suitable orbit G/H of some Lie group G, $f_n(z)$, $h_n(w)$ such that

$$\int_{\mathcal{M}} d\mu(\bar{z}, z)\; f_n^*\, f_m = \delta_{nm}, \qquad \int_{\mathcal{M}} d\mu(\bar{w}, w)\;\; h_n^*\, h_m = \delta_{nm}$$

$\mathcal{M} = G/H$

- There are many choices for the space of $z, \bar{z}$ (and $w, \bar{w}$); the simplest is to use $\mathbb{CP}^{N-1} \sim SU(N)/U(N-1)$.
- The states can be taken for this case as

$$f_N = \frac{1}{\sqrt{1 + \bar{z} \cdot z}}, \quad f_i = \frac{z_i}{\sqrt{1 + \bar{z} \cdot z}}, \quad i = 1, 2, \cdots, (N-1)$$

- Another choice could be coherent states for $\mathbb{CP}^1 \sim SU(2)/U(1)$. We can use the rank r representation with

$$f_n(z, \bar{z}) = \left[\frac{(r+1)!}{n!\,(r-n)!} \right]^{\frac{1}{2}} \frac{z^n}{(1 + \bar{z}z)^{r/2}}, \qquad n = 0, 1, \cdots, r$$

- One can use similar formulae for $h_n(w, \bar{w})$.

- The thermofield vacuum is given by $\Omega = \sum h_m^* \sqrt{\rho}_{mn} f_n$.
- There is a way of using C^*-algebras to formulate thermofield dynamics.
- Key result: $\exists$ an antilinear operation J, called 'modular conjugation', and a 'modular operator' Δ.
- These are easily and explicitly realized in terms of the coherent states as

$$J \cdot f_n = h_n^*, \qquad J \cdot h_n = f_n^*, \qquad J \cdot \lambda f_n = \lambda^* h_n^*$$
$$J \cdot \Omega = \Omega$$

- For the thermal state $\Delta = e^{-\beta \check{H}}$.

- The coherent states are of the form $\langle n|U|N\rangle$ for a highest weight state $|N\rangle$ (which has H-invariance) and U is a unitary representation of G.
- The amplitude in thermofield dynamics is given by

$$F = \int [\mathcal{D}U]\, B_{t_1}\, C_{t_2}\, \Omega^*(t)\, \Omega(0)\, \exp\left(i\, S\right)$$

$$S = \int dt\, \left[\left(i\, U^{(1)\dagger} \dot{U}^{(1)} - U^{(1)\dagger} H\, U^{(1)} \right)_{NN} - \left(i\, U^{(2)\dagger} \dot{U}^{(2)} - U^{(2)\dagger} H\, U^{(2)} \right)_{NN} \right]$$

- We can also write this as a field theory for ease of computation.

- The diagonal coherent state representation of operators also allows us to introduce $A_0(z,\bar{z}) = H(z,\bar{z})$ such that

$$H_{kl} = \int_{\mathcal{M}} d\mu(z,\bar{z})\, f_k^* \, A_0(z,\bar{z})\, f_l$$

and similarly for the tilde part (and for other operators).

- Introduce fermionic fields ψ, χ on $\mathcal{M}$,

$$\psi(z,\bar{z},t) = \sum_k a_k f_k, \qquad \psi^*(z,\bar{z},t) = \sum_k a_k^* f_k^*$$
$$\chi(w,\bar{w},t) = \sum_k b_k\, h_k, \qquad \chi^*(w,\bar{w},t) = \sum_k b_k^*\, h_k^*$$

- The transition amplitude can now be written as

$$
\begin{aligned}
F &= \mathcal{N} \int [d\psi d\psi^* \, d\chi d\chi^*] \, e^{iS} \, B_{t_1} \, C_{t_2} \, \Omega^*(t) \, \Omega(0) \\
\Omega(\psi^*, \chi^*) &= \int_{\mathcal{M}} d\mu_z d\mu_w \;\; \psi^*(z) \, \chi^*(w) \;\; (z_k \sqrt{\rho}_{kl} \, w_l) \\
B_{t_1} &= \int d\mu \;\; \psi^* B(z, \bar{z}, t_1) \psi
\end{aligned}
$$

- The action is given by

$$
\begin{aligned}
S = & \int dt \int_{\mathcal{M}} \psi^* \left(i \, \partial_0 - A_0(z, \bar{z}) + \frac{D^2 + E_0}{2m} \right) \psi \\
& + \int dt \int_{\tilde{\mathcal{M}}} \chi^* \left(i \, \partial_0 - A_0(w, \bar{w}) + \frac{D^2 + E_0}{2m} \right) \chi
\end{aligned}
$$

$\tilde{\mathcal{M}}$ has orientation opposite to $\mathcal{M}$.

- We consider general fields by regarding the holomorphic ones as the lowest Landau level of a mock quantum Hall system, with $m \to 0$ eventually.
- We can generalize to many copies of these fields. Take the states to be of the form $|k\rangle = |\alpha\,\mathrm{I}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and define a set of fermion fields $\psi_{\mathrm{I}} = \sum_\alpha a_{\alpha\mathrm{I}}\, z_\alpha$.
- We then write

$$
\begin{aligned}
S \;=\; & \int dt \int_{\mathcal{M}} \psi^*_{\mathrm{I}} \left(i\,\partial_0 \delta_{\mathrm{IJ}} - \,(A_0(z,\bar{z})_{\mathrm{IJ}} + \frac{D^2 + E_0}{2m}\delta_{\mathrm{IJ}} \right) \psi_{\mathrm{J}} \\
& + \int dt \int_{\tilde{\mathcal{M}}} \chi^*_{\mathrm{I}} \left(i\,\partial_0 \delta_{\mathrm{IJ}} - \,(A_0(w,\bar{w}))_{\mathrm{IJ}} + \frac{D^2 + E_0}{2m}\delta_{\mathrm{IJ}} \right) \chi_{\mathrm{J}}
\end{aligned}
$$

I, J label some internal symmetry or degrees of freedom.

- Start with a large physical system and consider creation of particles by a perturbation $\sum_i a_i^\dagger \, C_i^\dagger$ within the system.
- Amplitude of interest is

$$\begin{aligned} \mathcal{A} &= \sum \langle a_{i_1} \, C_{i_1} a_{i_2} \, C_{i_2} \cdots U(t,t_1) \, B \, U(t_1) \cdots a_{j_1}^\dagger \, C_{j_1}^\dagger a_{j_2}^\dagger \, C_{j_2}^\dagger \rangle \\ &\sim \sum \langle a_{i_1} \, a_{i_2} \, \cdots U(t,t_1) \, B \, U(t_1) \cdots a_{j_1}^\dagger \, a_{j_2}^\dagger \, \rangle \, \langle C_{i_1} \, C_{i_2} \cdots C_{j_1}^\dagger \, C_{j_2}^\dagger \rangle \end{aligned}$$

- $a \, a \cdots \sim$ particles we study, $B \sim$ some measurement
- We need, at least approximately, this factorization/decomposition of amplitudes to isolate the subsystem under study.
- If a_i evolves as $U_{ik}(t) \, a_k$, then C_i must evolve with $U_{ik}^* \, C_k$ so that the perturbation $a_i C_i$ corresponds to zero energy change.

- This means that we can regard

a-part of $\mathcal{A}$ $\leftarrow$evolution in $\mathcal{H}$ of the subsytem

C-part of $\mathcal{A}$ $\rightarrow$evolution in $\tilde{\mathcal{H}} \sim \mathcal{H}^*$

- Cs interact with many things (environment), so generally

$$\langle C_{i_1}\, C_{i_2} \cdots C^\dagger_{j_1}\, C^\dagger_{j_2} \rangle \sim (\sqrt{\rho})^*_{\alpha\tilde{\alpha}}\, \langle \tilde{\alpha}| e^{iH^T t} |\tilde{\beta}\rangle\, (\sqrt{\rho})_{\beta\tilde{\beta}}$$

$$\alpha = (i_1 i_2 \cdots), \qquad \beta = (j_1 j_2 \cdots).$$

- $\mathcal{A}$ reduces to the thermofield amplitude.

- If the apparatus is large enough, we may ignore correlations for the Cs and they may be taken as c-numbers ($\sim \eta, \bar{\eta}$). The amplitude reduces to

$$\mathcal{A} \sim \langle 0|(\bar{\eta}a)\cdots(\bar{\eta}a)\, e^{-iH(t-t_1)}\, B\, e^{-iHt_1}(a^\dagger\eta)\cdots(a^\dagger\eta)|0\rangle$$

$\eta,\ \bar{\eta} \sim$ sources.

- The key point is that the particle "creation operators" are of the form

$$a^\dagger C^\dagger \sim \int \bar{\psi}\chi \qquad \text{(of net zero energy)}$$

- This implies there is parity reversal between $\mathcal{H}$ and $\tilde{\mathcal{H}}$.

- Fuzzy spaces can be defined by the triple $(\mathcal{H}_N, Mat_N, \Delta_N)$.
  - $\mathcal{H}_N = N$-dimensional Hilbert space
  - $Mat_N =$ matrix algebra of $N \times N$ matrices which act as linear transformations on $\mathcal{H}_N$
  - $\Delta_N =$ matrix analog of the Laplacian.
- In the large N approximation
  - $\mathcal{H}_N \longrightarrow$ Phase space $\mathcal{M}$
  - $Mat_N \longrightarrow$ Algebra of functions on $\mathcal{M}$
  - $\Delta_N \longrightarrow$ needed to define metrical and geometrical properties.
- $\mathcal{M}_F \equiv (\mathcal{H}_N, Mat_N, \Delta_N)$ defines a noncommutative and finite mode approximation to $\mathcal{M}$.

- For quantum Hall effect on a compact space $\mathcal{M}$, the lowest Landau level defines a Hilbert space $\mathcal{H}_N$.
- Observables restricted to the lowest Landau level $\in Mat_N$.
- So the lowest Landau level of QHE can be used to model a fuzzy space.
- Phase spaces with symplectic structure ω and $\omega + dA$ correspond to the same Hilbert space,

$$\int \left(\frac{\omega}{2\pi}\right)^k = \int \left(\frac{\omega + dA}{2\pi}\right)^k$$

- There is ambiguity in which phase space we obtain as $N \to \infty$.

- Starting from $\mathcal{H}_N$, this shows up in the wave functions used to take the large N limit via the symbols

$$O(x,t) = \frac{1}{N} \sum_{m,l} \Psi_m(x)\, \hat{O}_{mn}(t)\, \Psi_n^*(x)$$

The wave functions $\Psi_l^*(x)$ are sensitive to A.

- The spatial components of the gauge fields characterize how the large N limit is taken.

- Further, for a space G/H, the “magnetic" fields for QHE are in H, which is part of the isometry group (G) of the space.

Spatial components of gauge fields $\sim$ Gravitational perturbations

- For gravity on a noncommutative space ($even + 1$ dimensional)
  - Use lowest Landau level of QHE to model the space.
  - Use thermofield dynamics for amplitude calculations, because the state describing space itself is highly entangled.
  - Gravitational fields couple to $\mathcal{H}$ and $\tilde{\mathcal{H}}$ with parity reversal, so we model $\mathcal{H}$ by left chiral fermions, $\tilde{\mathcal{H}}$ by right chiral fermions.
- The Chamseddine-Connes prescription is to use the Wodzicki residue (Dixmier trace) of (a suitable power of) the Dirac operator D as the action,

$$\begin{aligned} \text{Dixmier trace of } D^{-2} &\sim \text{Coefficient of } \log\text{- divergence in } \mathrm{Tr}\left[\frac{1}{D^2}\right] \\ &\sim \text{Einstein-Hilbert action} \end{aligned}$$

Gravitational Action in $2+1$

- For the gravitational part of $\mathcal{H} \otimes \tilde{\mathcal{H}}$, $SO(3)_L$ fields couple to $\mathcal{H}$ while $SO(3)_R$ fields couple to $\tilde{\mathcal{H}}_R$. i.e., $A_L \sim SO(3)_L, A_R \sim SO(3)_R$.

- The large N action is

$$S = k(C.S._L - C.S._R) = -\frac{k}{4\pi l}\int d^3x \; \det e \; \left[R - \frac{2}{l^2}\right] + \text{total derivative}$$

$$A^a_{L,R} = \left(-\tfrac{1}{2}\epsilon^a{}_{bc}\,\omega^{bc} \pm (e^a/l)\right), \qquad k = (l/4G) = 1$$

- A_i are auxiliary fields introduced for simplicity of representing the transformation. It is also not clear what A_0 should be for gravity.

- So we could try to "optimize" the large N limit by eliminating them via equations of motion.

Optimization of large N limit = Field equations for gravity

Gravitational Action in 4+1

- We can do a similar analysis in 4+ 1 dimensions to obtain the effective action

$$S = k\,(C.S._L - C.S._R) = -i\frac{k}{24\pi^2\, l}\int \mathrm{Tr}\left[3\,e\,R^2 + \frac{2}{l^2}\,e^3\,R + \frac{3}{5\,l^4}e^5 + \frac{e\,De\,De}{l^2}\right]$$

- This is, of course, not Einstein gravity.

- Regard point-particles as singularities of the solutions for the gravitational field as in Einstein-Infeld-Hoffmann.

- General solution is of the form $A = g^{-1}dg$ where g can have point-like singularities at $\vec{x}_\alpha$ (nonsingular on $\mathcal{M} - \{\vec{x}_\alpha\}$).

$$A = g^{-1}\, a\, g + g^{-1}\, dg, \qquad da = \sum_{\alpha=1}^{N} q_\alpha\, \delta^{(2)}(x - x_\alpha), \qquad a_0 = 0$$

- The action reduces to

$$S \;=\; -\frac{k}{4\pi}\int dt \sum_\alpha \Big[q_{L\alpha} \mathrm{Tr}(M_0\, g_{L\alpha}^{-1}\, \dot{g}_{L\alpha}) - q_{R\alpha} \mathrm{Tr}(N_0\, g_{R\alpha}^{-1}\, \dot{g}_{R\alpha}) \Big]$$

M_0, N_0 = diagonal generators of $SO(3)_L$, $SO(3)_R$.

- This gives multiparticle dynamics as representations of the isometry group with

$$\begin{aligned} \text{mass} &= m = (k/8\pi l)\,(q_L + q_R) = (q_R + q_L)/32\pi G \\ \text{spin} &= s = (k/4)\,(q_L - q_R) \end{aligned}$$

- In 4+1 dimensions, we need to consider point-like instantons for particle dynamics. For canonical embedding of $SU(2)$ in the $SO(4,2)$, we get the co-adjoint orbit action with

$$m = \frac{k}{2l}(Q_\alpha^{(1)} - Q_\alpha^{(2)})$$

$Q_\alpha^{(1)}$, $Q_\alpha^{(2)}$ = instanton numbers.

COMMENTS

- A similar analysis can be done for any $even + 1$) dimensions, although it is not Einstein gravity.
- The level number is 1 so far, we need multiplicity ($l/8G$) for a large level number.
- Continuation to Minkowski space seems possible using the field theory representation for the TFD.
- One can use the $SL(2,\mathbb{R})$ orbits of the Virasoro group to carry out a similar construction. (Large-central-charge limit needed to simplify the action; may connect to WITTEN, MALONEY, + *others*)
- Point-particles with nontrivial dynamics or coupling of matter fields is being explored.

Thank you