Hydrodynamic Modes of Incoherent Black Holes

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Based on work in collaboration with A. Donos, J. Gauntlett
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Transport in Inhomogeneous Media

- Conductivity matrix characterizes linear response of system to external sources.
- Apply *constant, time-dependent* electric field E_j and thermal gradient ζ_j and read off the $U(1)$ and heat currents:

$$\left(\begin{array}{c} J^i \\ Q^i \end{array} \right) = \left(\begin{array}{cc} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{array} \right) \left(\begin{array}{c} E_j \\ \zeta_j \end{array} \right)$$

- Translation invariance $\Rightarrow$ divergent DC conductivities $\sim \left(\delta(\omega) + \frac{i}{\omega}\right)$ when $\omega \to 0$.
- For physical, finite conductivities, we need a mechanism for momentum to dissipate.
- Explicit breaking of translational invariance by introduction of spatially dependent sources.

Transport in Inhomogeneous Media

Strongly broken translational invariance $\Rightarrow$ hydrodynamics dominated by energy and charge diffusion.

Holography is useful:

- Hydrodynamic modes in incoherent setups.
- New ground states (Q-Lattices, Helical lattices,...),
- DC Conductivity from black hole horizons,[1] ...

Proposed bounds on diffusion:[2] $D \gtrsim v^2\tau$

- Einstein relations $D \sim \bar{\kappa}/c$ hold in translational invariant case,[3] need to investigate inhomogeneous case.

[1] [A. Donos and J. Gauntlett '15], [E. Banks, A. Donos and J. Gauntlett '15]
[2] [Hartnoll '15]
[3] [Forster '90]

Green's Functions

Consider a finite temperature QFT with a conserved current J^μ, preserving translational invariance.[3]
Can Fourier transform the retarded two point functions

$$G_{AB}(t,\mathbf{x};t',\mathbf{x}') \equiv -i\theta(t-t')\,\langle\,\left[A(t,\mathbf{x}),B(t',\mathbf{x}')\right]\,\rangle \rightsquigarrow G_{AB}(\omega,\mathbf{k})$$

Define the charge susceptibility and the AC conductivity

$$\chi(\varepsilon\,\mathbf{k}) \equiv \lim_{\omega\to 0} G_{J^t\,J^t}(\omega,\varepsilon\,\mathbf{k})\,, \qquad \sigma^{ij}(\omega) \equiv \lim_{\varepsilon\to 0}\frac{1}{i\omega}G_{J^i\,J^j}(\omega,\varepsilon\,\mathbf{k})$$

We write

$$G_{J^t\,J^t}(\omega,\varepsilon\,\mathbf{k}) = \frac{N(\omega,\varepsilon\,\mathbf{k})}{-i\omega+N(\omega,\varepsilon\,\mathbf{k})}\chi(\varepsilon\,\mathbf{k})$$

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[3] [Forster '90]

Green's Functions

$N(\omega, \varepsilon\,\mathbf{k})$ is an analytic function of ω away from the real axis.
The Ward identity $\partial_\mu \langle\, J^\mu \,\rangle = 0$ and time reversal invariance imply that

$$N = \varepsilon^2 \frac{k_i k_j \sigma^{ij}(\omega)}{\chi(\mathbf{0})} + \mathcal{O}(\varepsilon^3)$$

Assume that N has no poles at $\omega = 0$ and that DC conductivity is finite

$$\sigma^{ij}_{DC} = \lim_{\omega \to 0} \sigma^{ij}(\omega)$$

Then we find a diffusive pole of $G_{J^t J^t}$ at

$$\omega = -i\varepsilon^2 D(k) + \dots\,, \qquad D(k) = \frac{k_i k_j \sigma^{ij}_{DC}}{\chi(\mathbf{0})}$$

Can also obtain Einstein relations for multiple conserved currents.

Green's Functions

Let's place our theory on a space with lattice symmetry $\mathbf{x} \to \mathbf{x} + \mathbf{L}$.

We can perform a continuous and a discrete Fourier transform

$$G_{AB}(t, \mathbf{x}; t', \mathbf{x}') \rightsquigarrow G_{AB}(\omega, \mathbf{k}, \{n_j\})$$

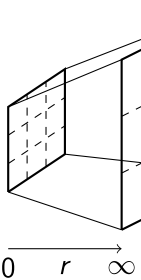
and focus on

$$G_{AB}(\omega, \mathbf{k}) \equiv G_{AB}(\omega, \mathbf{k}, \{0\}) = \oint d\mathbf{x} \int d\mathbf{x}' \, G_{AB}(\omega, \mathbf{x}, \mathbf{x}') e^{i\mathbf{k}(\mathbf{x}' - \mathbf{x})}$$

Then we can repeat the previous analysis.

Holographic Setup

Bulk action: Einstein, but can generalise to include Maxwell and coupling to scalars.[1]



Background static black holes:

$$ds^2 = -U(r)G\, dt^2 + \frac{F}{U(r)}\, dr^2 + g_{ij} dx^i dx^j$$

[1] [A. Donos and J. Gauntlett '15], [E. Banks, A. Donos and J. Gauntlett '15]

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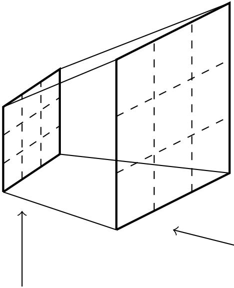

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$$ds^2 = -U(r)G\,dt^2 + \frac{F}{U(r)}\,dr^2 + g_{ij}dx^i dx^j$$

AdS boundary asymptotics

$$g_{ij}(r,x) = r^2\bar{g}_{ij}(x) + \dots$$
$$U = r^2 + \dots$$

Regular horizon

$$g_{ij} = g_{ij}^{(0)}(x) + \dots$$
$$U = 4\pi T\,r + \dots$$

[1] [A. Donos and J. Gauntlett '15], [E. Banks, A. Donos and J. Gauntlett '15]

DC Conductivity from Black Hole Horizon

Within linear response, the thermal conductivity relates the heat current *flux* to a *constant* thermal gradient source

$$\bar{Q}^i = T\bar{\kappa}^{ij}\bar{\zeta}_j$$

Holographic calculation of thermal DC conductivity: apply time-independent source $\delta g_{tj} \leftrightarrow \bar{\zeta}_j$ on the boundary.

Find (static) black hole quasinormal modes by solving equations of motion for bulk perturbations $\delta g_{\mu\nu}(r,x)$ with infalling boundary conditions at the horizon.

Define a subset of bulk modes by the near horizon expansions

$$\delta g_{ti} = -v_i(x) + \dots, \qquad \delta g_{tr} = p(x)/(4\pi T) + \dots$$

DC Conductivity from Black Hole Horizon

Local current on the horizon given by

$$Q^i_{(0)} = 4\pi T \sqrt{g^{(0)}} v^i$$

In general the *local* horizon current differs from the *local* boundary current, but their *fluxes* are equal

$$\int_H Q^i_{(0)} = \bar{Q}^i$$

On horizon, obtain linearised (charged, forced) Navier-Stokes:

$$\nabla_i v^i = 0\,, \qquad -2\,\nabla^i \nabla_{(i}\, v_{j)} + \nabla_j\, p = 4\pi T\, \bar{\zeta}_j$$

We interpret as v^i as velocity and p as pressure of auxiliary horizon fluid.

Realisation of membrane paradigm, but no hydrodynamic expansion.

Diffusive Mode

We first note that there is a zero mode solution corresponding to perturbing the temperature T by a constant amount δT

$$\delta g_{\mu\nu}^{RT} = \frac{\partial g_{\mu\nu}}{\partial T} \delta T$$

Does not introduce boundary sources and is regular at horizon.

To construct a bulk diffusive mode, we need to consider a time-dependent *sourceless* perturbation. We take

$$\delta g_{\mu\nu} = e^{-i\omega t} e^{i\varepsilon k_i x^i} \left[\delta g_{\mu\nu}^{RT} + \varepsilon \delta g_{\mu\nu}^{\{1\}} + \dots \right]$$

The functions $\delta g_{\mu\nu}^{\{\alpha\}}(x)$ are periodic on the lattice $\Rightarrow$ Bloch decomposition.

Diffusive Mode

In particular

$$v_i = e^{i\varepsilon k_i x^i}\left(\varepsilon\, v_i^{(1)} + \varepsilon^2\, v_i^{(2)} + \ldots\right), \qquad \delta g_{ij}^{(0)} = e^{i\varepsilon k_i x^i}\,\frac{\partial g_{ij}^{(0)}}{\partial T}\,\delta T + \ldots$$

$$p = e^{i\varepsilon k_i x^i}\left(4\pi\delta T + \varepsilon\, p^{(1)} + \ldots\right), \qquad \omega = \varepsilon\,\omega^{(1)} + \varepsilon^2\,\omega^{(2)} + \ldots$$

For time-dependent perturbations with infalling conditions, we find a non-closed system of constraints on the horizon.

At order $\mathcal{O}(\varepsilon)$ we get $\omega^{(1)} = 0$ and

$$\nabla_i v^{(1)i} = 0\,, \qquad -2\,\nabla^j\nabla_{(j} v_{i)}^{(1)} + \nabla_i p^{(1)} = -\,ik_i 4\pi\,\delta T$$

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But this is the DC problem with source $\bar{\zeta}_i \leftrightarrow -ik_i\delta T/T$! So we know that

$$4\pi T\int_H \sqrt{g^{(0)}}\,v^{(1)i} = -i\bar{\kappa}^{ij}k_j\delta T$$

Diffusive Mode

Finally, at order $\mathcal{O}(\varepsilon^2)$ we have

$$\nabla_i v^{(2)i} + ik_i v^{(1)i} - \frac{i\omega^{(2)}}{2}(g^{(0)})^{ij}\,\frac{\partial g^{(0)}_{ij}}{\partial T}\,\delta T = 0$$

Note that we can write the thermodynamic susceptibility as a horizon integral from

$$c \equiv T\frac{\delta s}{\delta T}\,, \qquad s = 4\pi\int_H \sqrt{g^{(0)}}$$

After integrating we obtain the dispersion relation

$$\omega = -i\varepsilon^2\frac{\bar{\kappa}^{ij}k_i k_j}{c} + \dots$$

satisfying the Einstein relation.

Outlook

Comments:

- Everything generalises to finite charge density case, resulting in the generalised Einstein relations
- Thermodynamic instability $\Rightarrow$ dynamical instability

Outlook:

- Analogous construction of diffusive modes within hydrodynamics on curved manifolds
- Hydrodynamic modes in models with spontaneous symmetry breaking?
- Bounds on diffusion from chaos?

Thank you for your attention!

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