Coherent States and String Amplitudes

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based on (published and unublished) work with:

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optimistic objective:

To study non-perturbative objects (e.g., black holes) using perturbative techniques (string perturbation theory)

This would normally be considered "blasphemous", but with appropriate resummations[1] one can hope to get quite far (and the perturbation series is asymptotic[2])

In particular we would like to construct a microscopic picture of quantum black holes, black hole production using scattering experiments (hoop conjecture), understand origin of BH entropy, etc.

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[1] (see e.g. Amati, Ciafaloni and Veneziano, . . .)

[2] E.g., the Stirling approximation of the gamma function is also an asymptotic expansion but is remarkably close to the full result

hints and basic idea:

String scattering amplitudes at high energies (and fixed angle) at higher genus dominated by saddle points [Gross, Mende, Manes]

Such a saddle point has the interpretation of a semiclassical highly excited string (minimising worldsheet area)

Suggestion: instead of searching for saddle points, construct "semiclassical" strings directly (coherent states) and use these to compute string amplitudes at high energies

This might qualitatively capture some higher genus contributions, while possibly providing a handle on non-perturbative string physics (black holes)

Dvali and Gomez (inspired by a corpuscular view of black holes) suggested to view black holes as **condensates of soft gravitons** (classicalisation) at a quantum critical point

This has attracted a lot of attention. E.g., Dvali, Gomez, Isermann, Lüst and Stieberger (2015) set out[3] to make this idea sharp by computing $2 \to N$ tree-level scattering amplitudes (with 2 hard and $N \gg 1$ soft gravitons):

$$|\langle 2|\hat{S}|N\rangle_{\rm pert}|^2 \sim \left(\frac{\lambda}{N}\right)^N N!, \qquad \lambda \leq 1, \qquad \mathrm{BH} \leftrightarrow \lambda = 1$$

They also conjecture to include non-perturbative factor (entropy) "by hand" (with multiplicity e^N) leading to

$$\sum_j |\langle 2|\hat{S}|N\rangle_{\rm pert}|^2 |\langle N|\mathrm{BH}_j\rangle|^2 \sim \lambda^N\Big|_{\lambda=1} \sim 1$$

implying black hole production ($\lambda = 1$) dominates over $\lambda \ll 1$

[3] Using powerful techniques (KLT relations, scattering equations), in both field theory and string theory

However, it is unclear (at least to me) whether one can identify the resulting state in $2 \to N$ as a bound state of N gravitons. Also, I would have expected the result to actually depend on three independent parameters: N, λ and $\xi \equiv \ell_s^2/(4G_4)$.

Then there is the issue of higher loop corrections. These was studied by Addazi, Bianchi and Veneziano (2017), where it was found that **virtual gravitons should actually play a vital role**, leading to a re-interpretation of the Dvali et al results.

One would ideally like to adopt a formalism where the **notion of a bound state is sharp**, and where one can freely speak about size when it is well-defined, as well as mass and charge. What we suggest here[4] is to replace the N graviton final states by a **string coherent state** (which is a bound state and one has full control over its quantum numbers).

[4] Dieter Lüst and DS (in preparation)

Outline:

- String Coherent Vertex Operators
- Superstring Coherent Vertex Operators
- Decay Rates and Infrared EFT description
- 3-Point Amplitudes (2 $\text{gravitons} \to \text{coherent state}$)

Definition of string coherent state (my PhD thesis):

(1) **continuity**: depends on a *set of continuous* quantum numbers $\{\lambda, \bar{\lambda}\}$;

(2) **completeness**: produces a resolution of unity,

$$\int^{\Sigma} = \sum_{(\dots)} \int d\lambda d\bar{\lambda} \int^{\Sigma_1} \mathcal{V}^c(\lambda, \bar{\lambda}; \dots) \int^{\Sigma_2} \mathcal{V}(\lambda, \bar{\lambda}; \dots),$$

so that the $\mathcal{V}(\lambda, \bar{\lambda}; \dots)$ span the string Hilbert space, $\mathcal{H}$. The dots "$\dots$" denote any additional quantum numbers;

(3) **symmetries**: transforms correctly under all (super)string symmetries [5]

Note:

- $\mathcal{V}^c$ denotes the euclidean adjoint of $\mathcal{V}$ (subtle phases..)
- eigenstates of annihilation operators do not exist in covariant or lightcone gauge closed string theory

[5] e.g., $Q_B\mathcal{V}(\lambda, \bar{\lambda}; \dots) = (L_0 - \bar{L}_0)\mathcal{V}(\lambda, \bar{\lambda}; \dots) = (b_0 - \bar{b}_0)\mathcal{V}(\lambda, \bar{\lambda}; \dots) = 0.$

Coherent State Construction (practical approach):

Coherent states satisfying all defining properties can be constructed[6] using DDF operators ($\alpha' = 2$) [7]

$$A_n^i = \frac{1}{2\pi}\oint dz\,\partial_z x^i\, e^{inq\cdot x(z)}, \qquad \bar{A}_n^i = \frac{1}{2\pi}\oint d\bar{z}\,\partial_{\bar{z}} x^i\, e^{inq\cdot x(\bar{z})},$$

with $q^2 = 0$, $q \cdot A_n = 0$ and $[A_n^i, A_m^j] = n\delta^{ij}\delta_{n+m,0}$, in terms of which (including a θ integral for level-matching):

$$\mathcal{V}_{\rm coh}(z,\bar{z}) = C\int_0^{2\pi}\frac{d\theta}{2\pi}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}e^{in\theta}\lambda_n\cdot A_{-n}\right\}$$
$$\times\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m}e^{-im\theta}\bar{\lambda}_m\cdot\bar{A}_{-m}\right\}e^{ip\cdot x(z,\bar{z})},$$

with $p^2 = 2$, $p \cdot q = 1$. (In compact spacetimes there is also an overall phase, $(-)^{iM'_a N^a s}$, and p_L, p_R independent, etc.)

[6] Hindmarsh & Skliros PRL (2011), Skliros, Copeland and Saffin (2017)

[7] Del Giudice, Di Vecchia, Fubini (1972); Ademollo, Del Guidice, Di Vecchia (1974)

Carrying out Wick contractions and contour integrals leads to a complete set of conformal weight $(0,0)$ primaries:

$$\hat{\mathcal{V}}_{\rm coh}(z,\bar{z}) = C \int_0^{2\pi} \frac{d\theta}{2\pi}\, U(z)\tilde{U}(\bar{z}),$$

where, defining $\zeta_{nM} \equiv \lambda^i_n(\delta^i_M - p^i q_M)$, the chiral half reads,

$$U(z) = \,:c(z)\exp\Big(\sum_{n,m>0} \frac{e^{i(n+m)\theta}}{2nm}\zeta_n\cdot\zeta_m\,\mathbb{S}_{n,m}\,e^{-i(n+m)q\cdot x_L(z)} + \sqrt{\frac{2}{\alpha'}}\sum_{n>0}\frac{e^{in\theta}}{n}\zeta_n\cdot D^n_z x_L\,e^{-inq\cdot x_L(z)}\Big)\,e^{ip\cdot x_L(z)}:$$

transversality conditions $\zeta_n\cdot p = \zeta_n\cdot q = 0$, and defined:

$$\begin{cases} D^n_z \equiv \sum_{r=1}^n Z_{n-r}(a_s(n))\frac{i}{(r-1)!}\,\partial^r_z, \\ \mathbb{S}_{m,\ell}(z) \equiv \sum_{r=1}^{\ell} rZ_{m+r}(a_s(m))Z_{\ell-r}(a_s(\ell)). \end{cases}$$

$Z_N(a_s)$ are **cycle index polynomials** and $a_s = -nq\cdot\alpha_{-s}$.

A rest frame only exists in an expectation value sense ($\alpha' = 2$):

$$\langle \hat{p}^\mu \rangle \equiv M\delta_0^\mu, \qquad M^2 = \sum_n |\zeta_n|^2 + \sum_m |\bar{\zeta}_m|^2 - 2, \qquad M^2 \in [-2, \infty)$$

When $\mathcal{V}_{\rm coh}$ has a semiclassical interpretation it has spatial extent, $\mathcal{R} \equiv \sqrt{\langle (\mathbf{X}(z,\bar{z}) - \mathbf{x})^2 \rangle}$:

$$\mathcal{R}^2 = \sum_{n>0} \frac{1}{n^2}\Big(|\zeta_n|^2 + |\bar{\zeta}_n|^2 - 2{\rm Re}\big(\zeta_n \cdot \bar{\zeta}_n e^{-2in\tau_{\rm M}}\big)\Big)$$

for $\mathcal{R} \gg \ell_s$, and to every coherent vertex operator (when a semiclassical interpretation exists) there corresponds a classical (lightcone gauge) trajectory:

$$X^0(z,\bar{z}) = -iM \ln z\bar{z},$$

$$X^i(z,\bar{z}) = \sum_n \frac{i}{n}\left(\lambda_n^i z^{-n} - \lambda_n^{*i} z^n\right) + \sum_m \frac{i}{m}\big(\bar{\lambda}_m^i \bar{z}^{-m} - \bar{\lambda}_m^{*i} \bar{z}^m\big),$$

Notice we can make $\mathcal{R}$ arbitrarily small for any given mass M.

Superstring Generalisation of Coherent Vertex Operators

Superstring Conventions

It is efficient to work in superfield RNS formalism and focus on NS-NS sector

The matter action reads:

$$I = \frac{1}{4\pi} \int d^2 z d^2 \theta E \mathcal{D}_+ X \cdot \mathcal{D}_- X.$$

With appropriate gauge choice, can work in flat superspace (superconformal gauge), where $E = 1$, and x^μ is promoted to a scalar superfield, $X^\mu(\mathsf{z}, \bar{\mathsf{z}}) = X^\mu(\mathsf{z}) + X^\mu(\bar{\mathsf{z}})$, the chiral half of which reads:

$$X^\mu(\mathsf{z}) = x_+^\mu(z) + \theta_z \psi_+^\mu(z),$$

$$\mathcal{D}_+ X = \psi_+ + \theta_z \partial_z x_+, \qquad \text{with} \qquad \mathcal{D}_+^2 = \partial_z$$

Vertex operators take the form:

$$\mathcal{V} = \int d^2\mathbf{z}\,\mathcal{O}_h(\mathbf{z})\bar{\mathcal{O}}_{\bar{h}}(\bar{\mathbf{z}})$$

and will be superconformally invariant (and hence can be inserted into path integrals) provided their weights are $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$.

They should also depend on continuous quantum numbers and there should exist a complete set (defining properties of string coherent states)

Furthermore, super-U(1) invariance requires[8] $\#\mathcal{D}_+ = \#\mathcal{D}_-$ in $\mathcal{V}$ (when $\{\mathcal{D}_+, \mathcal{D}_-\} = 0$), and GSO further restricts to $\#\mathcal{D}_+ = \mathrm{odd}$ (from summing over all spin structures[9]).

Proceeding by analogy with bosonic string, we will be needing super-DDF operators to construct $\mathcal{V}$...

[8] e.g., D'Hoker and Phong (1988)

[9] Seiberg and Witten (1986)

Superstring DDF Operators

The superconformally-invariant (!) Grassmann-even DDF operators read:

$$\boxed{\mathcal{A}^i_n = \frac{1}{2\pi}\oint dz d\theta_z\, \mathcal{D}_+ X^i e^{inq\cdot X}} \qquad (n \in \mathbb{Z})$$

(with $q^2 = q \cdot \mathcal{A}_n = 0$) and there are also Grassmann-odd counterparts:

$$\boxed{\mathcal{B}^i_r = \frac{1}{2\pi}\oint dz d\theta_z\, \mathcal{D}_+ X^i \frac{q \cdot \mathcal{D}_+ X}{(iq \cdot \mathcal{D}_+^2 X)^{\frac{1}{2}}} e^{irq\cdot X}} \qquad (r \in \mathbb{Z} + \tfrac{1}{2})$$

(with $q \cdot \mathcal{B}_r = 0$) and (anti-)commutation relations:

$$[\mathcal{A}^i_n, \mathcal{A}^j_m] = n\delta^{ij}\delta_{n+m}, \qquad \{\mathcal{B}^i_r, \mathcal{B}^j_s\} = \delta^{ij}\delta_{r+s}$$

Superstring Coherent Vertex Operators

Introduce continuous quantum numbers, λ_n^i, ξ_r^i, and define:

$$\mathcal{A} \equiv \sum_{n=1}^{\infty} \frac{1}{n} u^{2n} \lambda_n \cdot \mathcal{A}_{-n}, \qquad \mathcal{B} \equiv \sum_{n=1}^{\infty} u^{2n-1} \xi_{n-\frac{1}{2}} \cdot \mathcal{B}_{-(n-\frac{1}{2})}$$

for some $u \in \mathbb{C}$. The vertex operators satisfying all defining properties are then (for type II superstrings[10]) on $\mathbb{R}^{D-1,1} \times \mathbb{T}^{10-D}$:

$$\boxed{\mathcal{V}(\mathsf{z}, \bar{\mathsf{z}}) = \oint_0 \frac{du}{2\pi i u} \left(e^{\mathcal{A}} \sinh \mathcal{B}\right)\left(e^{\bar{\mathcal{A}}} \sinh \bar{\mathcal{B}}\right) e^{ip\cdot X(\mathsf{z},\bar{\mathsf{z}})}}$$

(up to overall normalisation) and correspond to a complete set of coherent state superstring vertex operators, with $p^2 = 1$.

- $\oint du$ enforces super-U(1) invariance, $\#\mathcal{D}_+ = \#\mathcal{D}_-$
- $\sinh \mathcal{B}$ enforces GSO projection
- in presence of KK and winding charges include $u^{M_a' N^a}$ factor and take p_L, p_R independent

[10] for Heterotic string replace anti-chiral half by bosonic result

The momentum expectation value of these superstring coherent vertex operators is (recall $p^2 = 1$, $p \cdot q = 1$ and $q^2 = 0$):

$$\langle \hat{\mathbb{P}}^\mu \rangle = p^\mu - N q^\mu, \qquad \text{with} \qquad N \equiv \sum_{n \in \mathbb{N}}^{\infty} |\lambda_n|^2 + \sum_{r \in \mathbb{N} - \frac{1}{2}}^{\infty} r |\xi_r|^2,$$

The corresponding mass expectation value is:

$$M^2 = 2N - 1,$$

in precise agreement with what expected for NS sector mass eigenstates, but here N is a continuous quantum number. (There is no "tachyon" if[11] $N \geq 1/2$.)

We can also enforce "classical level matching" $N = \bar{N}$

In what follows we switch back to bosonic string for simplicity

[11] i.e., $M^2 \geq 0$ if $\sum_r r|\xi_r|^2 \geq \frac{1}{2}$

Before proceeding further we should check that there is agreement with low energy effective theory at low energies:[12]

$$S_{\rm eff} = \frac{1}{16\pi G_D}\int d^Dx\sqrt{-G}\, e^{-2\Phi}\Big(R_{(D)} + 4(\nabla\Phi)^2 - \frac{1}{12}H_{(3)}^2 + \dots\Big)$$
$$- \frac{1}{2\pi\alpha'}\int_{S^2}\partial X^\mu \wedge \bar\partial X^\nu\,(G_{\mu\nu} + B_{\mu\nu}) + \dots$$

and the coherent state corresponds to evaluating the source with:

$$X^0(z,\bar z) = -iM\ln z\bar z,$$
$$X^i(z,\bar z) = \sum_n \frac{i}{n}\,(\lambda^i_n\, z^{-n} - \lambda^{*i}_n\, z^n) + \sum_m \frac{i}{m}(\bar\lambda^i_m\,\bar z^{-m} - \bar\lambda^{*i}_m\,\bar z^m),$$

We can check agreement noting that at large distances from the source gravity should be weak and there will be massless radiation

$\Rightarrow$

[12] where Φ, $G_{\mu\nu}$ and $H_{(3)}$ are the dilaton, spacetime metric and 3-form field strength, $H = dB$, respectively

The decay rate into massless radiation can be computed from the imaginary part of the coherent state 2-point (1-ploop) amplitude using the optical theorem:

$$\mathcal{A}_{1\to 1^c} = \frac{1}{2}\int d^D\mathbb{P}\int_{\mathcal{F}_1} d^2\tau \int \mathcal{D}(bcX)\, e^{-I}|(\mu, b)|^2\delta^D\big(\mathbb{P}^\mu - \hat{\mathbb{P}}^\mu\big)\mathcal{V}^c\,\hat{\mathcal{V}}$$

The b, c are the $\mathrm{Diff}(\Sigma)$ ghosts, $\tau, \bar{\tau}$ is the modular parameter of the torus and $\mathbb{P}$ the loop momentum[13]. The result in the IR is:

$$\frac{d\Gamma}{d\Omega_{S^{D-2}}} = \sum_{\omega_N} \frac{16\pi G_D}{(2\pi)^{D-4}(2\pi\alpha')^2}\,\omega_N^{D-4-\delta} N^2$$

$$\Big[J'^2_N + \Big(\frac{1}{z^2} - 1\Big)J_N^2 + \dots\Big]\Big[\bar{J_N}'^2 + \Big(\frac{1}{\bar{z}^2} - 1\Big)\bar{J_N}^2 + \dots\Big]$$

where $\delta = 1$ yields a decay rate, $\delta = 0$ a power, and the frequency of emitted radiation,[14]

$$\omega_N = \frac{4\pi N}{L}, \qquad \text{with} \qquad N = 1, 2, \dots, \qquad \text{and} \quad L = 2\pi\alpha' M$$

[13] D'Hoker and Phong (1989); DS, Copeland, Saffin (2016)

[14] Here $z = \bar{z} = \sin\theta$, the $J_N = J_N(Nz)$, $\bar{J_N} = J_N(N\bar{z})$ are Bessel.

Carrying out the analogous computation in the EFT (power into $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ radiation) leads to precise agreement[15]

Having gained confidence that our coherent states have a sensible low energy limit we can now start to explore the UV where EFT is expected to breakdown while also probing for black hole signatures

In particular, inspired by the $2 \to N$ graviton scattering computation of Dvali, Gomez, Isermann, Luest and Stieberger (2015), I finally mention some preliminary results for $2 \to CS$ (2 graviton to coherent state) scattering amplitudes

[15] DS, Copeland, Saffin (2013)

Writing $\hat{V}_{\rm gr}(j)$ for graviton vertex operators ($j = 1, 2$) of momenta k_j and polarisations ζ^j, and $\hat{\mathcal{V}}_{\rm coh}$ the most general coherent vertex operator allowed by symmetries and charge conservation; the amplitude of interest is:

$$\boxed{\mathcal{A}_{2\to{\rm coh}} = e^{-2\Phi}\big\langle \hat{V}_{\rm gr}(1)\hat{V}_{\rm gr}(2)\hat{\mathcal{V}}^c_{\rm coh}(3)\big\rangle_{S_2}}$$

For illustration purposes, the simplest interesting case is where the coherent state has a single harmonic (n) excited with mass expectation value M and size $\mathcal{R}$ (when well-defined). Write:

$$X \equiv |\zeta_n|^2 = \frac{\ell_s^2}{4}M^2, \qquad n = \frac{\ell_s^2}{2}\frac{M}{\mathcal{R}}, \qquad \xi \equiv \frac{\ell_s^2}{4G_4}$$

The correspondence with the Dvali *et al* computation is:

$$X = \xi^2\Big(\frac{\mathcal{R}_s}{\ell_s}\Big)^2 = \xi N$$

$$n = \xi\Big(\frac{\mathcal{R}_s}{\mathcal{R}}\Big) = \xi\sqrt{\lambda}$$

The result for the full amplitude is[16]

$$\mathcal{A}_{2\to\mathrm{coh}} = i(2\pi)^4\delta^4(k)\big(\tfrac{\sqrt{32\pi}}{\ell_s\xi}\big)\, I_0\big(\tfrac{2X}{n}\big)^{-1/2}\sum_{a=1}^{29}\mathcal{Z}_a\bar{\mathcal{Z}}_a,$$

where, writing $\gamma_\pm = \frac{1}{2}(1 \pm \frac{\alpha'}{2}k_{12}\cdot q)$, e.g.,

$$\mathcal{Z}_1 = (\zeta^1\cdot\zeta^2)\big(\tfrac{1}{4}B\big)^{\frac{X}{2n}} H_{\frac{X}{n}}\big(\tfrac{A}{\sqrt{B}}\big)\Gamma\big(\tfrac{X}{n}+1\big)^{-1}, \quad \mathcal{Z}_2 = \dots,$$

The $H_n(x)$ are Hermite polynomials and $I_0(x)$ modified Bessel functions. We have defined:

$$A = \sqrt{\frac{\alpha'}{2}}\frac{1}{2n}(k_{12}\cdot\zeta_n)\mathcal{F}_n, \qquad B = \frac{1}{n}(\zeta_n\cdot\zeta_n)\gamma_+\gamma_-\mathcal{F}_n^2$$

$$\mathcal{F}_n(k_{12}\cdot q) = (-)^{n-1}\frac{\sin(\pi n\gamma_-)}{\pi\Gamma(n)}\Gamma(n\gamma_+)\Gamma(n\gamma_-)$$

Asymptotics pending ...

[16] Dieter Lüst and DS (in preparation)

Conclusions

- I have presented the first covariant construction of string coherent vertex operators in bosonic, type II and heterotic (super)string theories
- These vertex operators have well defined mass and momentum expectation values, and (when sufficiently macroscopic) size. In latter case they're in direct correspondence with semiclassical trajectories but extend fully into quantum regime
- Computed decay rates and power into massless radiation (in IR!) using both EFT and string amplitudes with coherent states finding precise agreement; (after chiral splitting results resum into Bessel functions)
- Preliminary results for a $2 \to CS$ (two gravitons to most general coherent state allowed by symmetries); For n^{th} harmonics the full tree-level result is given in terms of Hermite polynomials and Gamma functions. Is there evidence for black hole production?

EXTRA SLIDES

Coherent States in QM

Consider harmonic oscillator Hamiltonian,

$$\hat{H} = \omega\Big(a^\dagger a + \frac{1}{2}\Big), \qquad \text{with} \qquad [a, a^\dagger] = 1 \quad \text{and} \quad a|0\rangle = 0,$$

$a^\dagger$, a are creation and annihilation operators. Coherent states usually defined as eigenstates of the annihilation operator, a,

$$a|\lambda\rangle = \lambda|\lambda\rangle, \qquad \text{with} \qquad |\lambda\rangle = \exp\big(\lambda a^\dagger - \lambda^* a\big)|0\rangle,$$

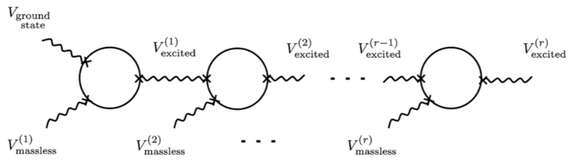
which therefore lead to classical evolution of expectation values, e.g.,

$$\frac{d^2}{dt^2}\langle x(t)\rangle = -\omega^2\langle x(t)\rangle, \qquad \text{with} \qquad \langle x(t)\rangle = \frac{1}{\sqrt{2}}\big(\lambda^* e^{i\omega t} + \lambda e^{-i\omega t}\big).$$

This procedure does not work in string theory (for a variety of reasons) and we need a more general definition of coherent states

Coherent State Construction (intuitive approach):

Excite ground state string (of momentum p_L, p_R) with r massless vertex operators (of momenta $-n_j q$, $-\bar{n}_j q$, $j = 1, \ldots, r$, $n_j \in \mathbb{Z}^+$):



set $\sum_i n_i = \sum_i \bar{n}_i$ and ressum ($V^{(j)}_{\text{massless}}$ are bosons)[17]:

$$\boxed{\mathcal{V}_{\text{coh}} = \sum_{r=0}^{\infty} \frac{1}{r!} V^{(r)}_{\text{excited}} = :\exp\left(V_{\text{excited}}\right):}$$

Then promote massless polarisation tensors to (renormalised) **continuous quantum numbers**, $\lambda_n, \bar{\lambda}_n$ and normalise:

$$\mathcal{V}^c_{\text{coh}}(z, \bar{z})\, \mathcal{V}_{\text{coh}}(w, \bar{w}) \simeq \frac{g_D^2}{|z-w|^4} + \ldots$$

[17] Taking into account factorisation and conformal invariance

This procedure produces physical vertex operators, $\mathcal{V}_{\rm coh}$, that depend on continuous quantum numbers.

To obtain a *complete set* relax level matching in, $V^{(j)}_{\rm massless}$, (i.e. take $-n_j q$ and $-\bar{n}_j q$ independent) and project onto $\mathcal{V}_{\rm coh}$ satisfying,

$$(L_0 - \bar{L}_0)\,\mathcal{V}_{\rm coh} \simeq 0$$

$\rightarrow$This procedure produces coherent states,

$$\mathcal{V}_{\rm coh}(z,\bar{z}) = \,:\exp\big(V_{\rm excited}(z,\bar{z})\big):$$

that satisfy **all** defining coherent state properties

(a) One can think of this as a **"coherent state of gravitons"**
. . . **when** a special choice of polarisation tensors is made
(b) $\mathcal{V}_{\rm coh}$ are **not eigenstates of annihilation operators**!

The cycle index polynomials, $Z_N(a_s)$, are defined by:

$$Z_N(a_s) = \oint_0 \frac{du}{2\pi i u} u^{-N} \exp \sum_{s=1}^{N} \frac{1}{s} a_s u^s$$
$$= \sum_{k_1+2k_2+\cdots+Nk_N=N} \frac{1}{k_1!}\left(\frac{a_1}{1}\right)^{k_1} \cdots \frac{1}{k_N!}\left(\frac{a_N}{N}\right)^{k_N}$$

and for vertex operators: $a_s = (-nq) \cdot \frac{i}{(s-1)!}\partial_z^s x_L$. A number of useful properties are:

$$Z_N(b^s a_s) = b^N Z_N(a_s) \qquad \text{(scaling relation)}$$

$$Z_N(a_s) = \frac{1}{N}\sum_{m=1}^{N} a_m Z_{N-m}(a_s) \qquad \text{(recursion relation)}$$

$$Z_N(a_s + b_s) = \sum_{m=0}^{N} Z_{N-m}(a_s) Z_m(b_s) \qquad \text{(multiplication theorem)}$$

$$Z_N(a/b^s) = \frac{1}{N} b^{-N} \frac{1}{B(a,N)} \qquad \text{(beta function relation)}$$

Superconformal transformations generated by the super-stress tensor, $T(\mathbf{z})$,[18]

$$T(\mathbf{z}) = -\frac{1}{2}\mathcal{D}_+X \cdot \mathcal{D}_+^2 X$$

Consider a superfield $\mathcal{V}(\mathbf{z}) = \mathcal{V}[X(\mathbf{z})]$ of conformal weight h. That is, Under general superconformal transformation

$$(z,\theta) \to (z+\delta z, \theta+\delta\theta) = \left(z + V - \tfrac{1}{2}\theta_z \mathcal{D}_+ V, \theta + \tfrac{1}{2}\mathcal{D}_+ V\right),$$

parametrised by infinitesimal superfield $W(\mathbf{z})$

$$\delta_W \mathcal{V}(\mathbf{w}) = \frac{1}{2\pi i}\oint dz d\theta_z W T(\mathbf{z})\mathcal{V}(\mathbf{w}),$$

with poles in contour integral generated from OPE:

$$T(\mathbf{z})\mathcal{V}(\mathbf{w}) = \left(h\frac{\delta\theta}{Y^2} + \frac{\frac{1}{2}}{Y}\mathcal{D}_+ + \frac{\delta\theta}{Y}\mathcal{D}_+^2 + \dots\right)\mathcal{V}(\mathbf{w})$$

with $Y \equiv z - w - \theta_z\theta_w$, $\delta\theta \equiv \theta_z - \theta_w$. Contractions carried out with:

$$\langle X(\mathbf{z})X(\mathbf{w})\rangle = -\ln\left(z - w - \theta_z\theta_w\right)$$

[18]Note that $\mathbf{z} = (z, \theta_z)$ and $\mathbf{w} = (w, \theta_w)$.

Superstring Vertex Operator Vacuum

Again, by analogy with bosonic string, act with DDF's on a vacuum state, e.g.,

$$e^{ip\cdot X(\mathbf{z})}$$

This will be physical (i.e. a weight $h = \frac{1}{2}$ superfield) when $p^2 = 1$.

BUT! by GSO we know that physical vacuum requires:
$\#\mathcal{D}_+ = \mathrm{odd}\ldots$ (eliminating the tachyon, etc.)

DDF operators $\mathcal{A}^i_{-n}$ and $\mathcal{B}^i_{-\frac{1}{2}(2n-1)}$ contribute generically:

$$\begin{array}{ccc} \mathcal{A}^i_{-n} & \rightarrow & \#\mathcal{D}_+ = 2n \\ \mathcal{B}^i_{-\frac{1}{2}(2n-1)} & \rightarrow & \#\mathcal{D}_+ = 2n-1, \qquad \text{with} \qquad n = 1, 2, \ldots \end{array}$$

to the corresponding vertex operators.
$\rightarrow$ So need ODD number of $\mathcal{B}^i_{-r}$'s acting on $e^{ip\cdot X(\mathbf{z})}$.

Annihilation Operator Eigenvalues

Construct Grassmann-even polarisation tensors, λ^i_n, and Grassmann-odd counterparts, ξ^i_r. Then, define:

$$\mathcal{A} \equiv \sum_{n=1}^{\infty} \frac{1}{n} u^{2n} \lambda_n \cdot \mathcal{A}_{-n}, \qquad \mathcal{B} \equiv \sum_{n=1}^{\infty} u^{2n-1} \xi_{n-\frac{1}{2}} \cdot \mathcal{B}_{-(n-\frac{1}{2})}$$

for some $u \in \mathbb{C}$. Then consider the quantity:

$$e^{\mathcal{A}+\mathcal{B}} e^{ip\cdot X(z)}$$

This is an eigenstate of $\mathcal{A}^i_n$ and $\mathcal{B}^i_{n-\frac{1}{2}}$ (for $n = 1, 2, \dots$):

$$\mathcal{A}^i_n \left(e^{\mathcal{A}+\mathcal{B}} e^{ip\cdot X} \right) = u^{2n} \lambda^i_n \left(e^{\mathcal{A}+\mathcal{B}} e^{ip\cdot X} \right)$$

$$\mathcal{B}^i_{n-\frac{1}{2}} \left(e^{\mathcal{A}+\mathcal{B}} e^{ip\cdot X} \right) = u^{2n-1} \xi^i_{n-\frac{1}{2}} \left(e^{\mathcal{A}+\mathcal{B}} e^{ip\cdot X} \right)$$

so is a candidate (chiral half of a) coherent superstring vertex operator. However, GSO condition *not* satisfied . . .

GSO implies $\#\mathcal{B} =$odd, so we should decompose $e^{\mathcal{B}}$ into even and odd $\#\mathcal{B}$'s, and drop even pieces, i.e.,

$$e^{\mathcal{B}} \to \sinh \mathcal{B}$$

Consider therefore quantity:

$$\boxed{\mathcal{O}_u(\mathbf{z}) \equiv e^{\mathcal{A}} \sinh \mathcal{B}\, e^{ip\cdot X(\mathbf{z})}}$$

This is an eigenstate of $\mathcal{A}^i_n$ (for $n = 1, 2, \dots$):

$$\mathcal{A}^i_n \Big(e^{\mathcal{A}} \sinh \mathcal{B}\, e^{ip\cdot X} \Big) = u^{2n} \lambda^i_n \Big(e^{\mathcal{A}} \sinh \mathcal{B}\, e^{ip\cdot X} \Big)$$

but *not* of $\mathcal{B}^i_{n-\frac{1}{2}}$,

$$\mathcal{B}^i_{n-\frac{1}{2}} \Big(e^{\mathcal{A}} \sinh \mathcal{B}\, e^{ip\cdot X} \Big) = u^{2n-1} \xi^i_{n-\frac{1}{2}} \Big(e^{\mathcal{A}} \cosh \mathcal{B}\, e^{ip\cdot X} \Big)$$

It is rather an eigenstate of two $\mathcal{B}_r$'s, with Grassmass-even eigenvalues. (This is seemingly forced upon us)

Picture Changing

In the bosonic string the position of the vertex operator $(z, \bar{z})$ may correspond to a symmetry, and so we should in that case not integrate over its position. Instead we replace:

$$\int d^2 z \mathcal{O}(z, \bar{z}) \to \delta(c)\delta(\bar{c})\mathcal{O}(z, \bar{z}),$$

with $c, \bar{c}$ (ghosts) associated to translations generated by CKV's (the number of which depends on topology and is constrained by index theorems).

$$\int d^2 z \mathcal{O}(z, \bar{z}), \qquad \text{and} \qquad \delta(c)\delta(\bar{c})\mathcal{O}(z, \bar{z})$$

correspond to different pictures of the same state.

Picture Changing

Similarly, in the superstring, either locations of vertex operators $(z, \bar{z})$ or $(\theta_z, \bar{\theta}_{\bar{z}})$ (or both) may correspond to a symmetry, and so we should in that case not integrate over these positions. Instead we replace, e.g.,

$$\int d^2z d^2\theta \mathcal{V}(z, \bar{z}; \theta, \bar{\theta}) \to \int d^2z \delta(\gamma)\delta(\bar{\gamma})\mathcal{V}(z, \bar{z}; 0, 0),$$

with zero modes of superghosts, $C(\mathbf{z}) = c(z) + \theta\gamma(z)$, associated to translations generated by sCKV's. Focusing on the chiral half of $\mathcal{O}(z, \bar{z}; \theta, \bar{\theta}) = \mathcal{O}(z, \theta)\mathcal{O}(\bar{z}, \bar{\theta})$, we say that:

$$\int d\theta\, \mathcal{V}(z, \theta), \qquad \text{has picture number } 0$$

$$\delta(\gamma)\mathcal{V}(z, 0), \qquad \text{has picture number } -1$$

$\rightarrow$ Above superstring vertex operators contain both pictures.

Example

As an example, consider the chiral vertex operator:

$$\begin{aligned}\mathcal{O}(\mathbf{w}) &= \xi^i \mathcal{B}^i_{-\frac{1}{2}} e^{ip\cdot X(\mathbf{w})} \\ &= \zeta_\mu \mathcal{D}_+ X^\mu e^{ik\cdot X(\mathbf{w})},\end{aligned}$$

with $\zeta \cdot k = k^2 = 0$. In terms of components, $X = x + \theta\psi$,

$$\mathcal{O}(w,\theta) = \zeta \cdot \psi^\mu e^{ik\cdot x} + \theta\zeta_\mu\big(\partial_w x^\mu - k \cdot \psi\psi^\mu\big) e^{ik\cdot x},$$

and we recognise immediately the two (-1 and 0) pictures respectively:

$$\delta(\gamma)\zeta_\mu\psi^\mu e^{ik\cdot x}, \qquad \text{and} \qquad \zeta_\mu\big(\partial x^\mu - ik \cdot \psi_+\psi^\mu\big) e^{ik\cdot x}.$$

Similar remarks hold for the full superstring coherent vertex operators above