Aspects of Berry phase in QFT

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based on work with Marco Baggio and Kyriakos Papadodimas

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Berry phase in QFT

- Berry phase is a basic property of Quantum Mechanics

  ☞ lack of holonomy in adiabatic variation of parameters

- QFTs depend on parameters: masses, general couplings etc…

  Expect a rich pattern of Berry-like properties

Towards a systematic exploration of Berry phase in (continuum) QFT

Potential lessons

- geometry of parameter-spaces (space-of-theories)
- (non-perturbative) relations between correlation functions
- new experimentally observable predictions from QFT

…

Technical approach

Berry phase by quantizing QFT in the Hamiltonian framework

Obvious issues:

- UV divergences:

  ☞ renormalize

- IR issues: e.g. continuous spectra

  ☞ put theory on a hypercylinder : $\mathbb{R} \times \mathcal{Q}$, $\mathcal{Q}$ compact

Main messages of this talk

1. Geometric phases can be non-trivial even in trivial QFTs !
(Berry phase of photons)

2. Connection & parallel transport in Conformal Perturbation Theory is Berry phase in disguise

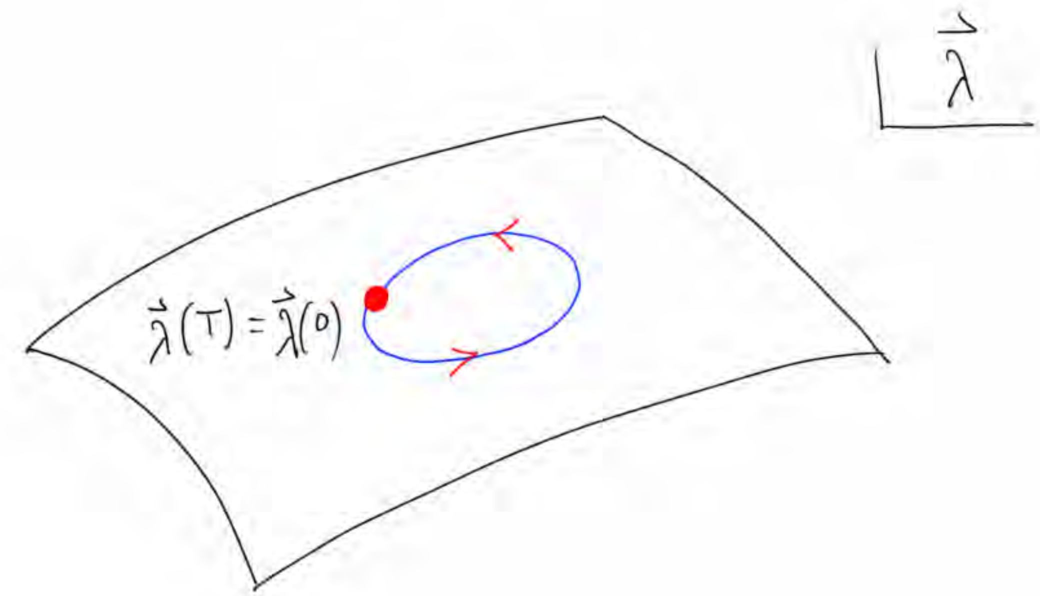
3. Exact, non-perturbative computations of Berry phase in interacting QFTs are possible
(supersymmetry, *tt** geometry)

Spectral formula for Berry curvature

Berry phase in Quantum Mechanics

Adiabatic variation of coupling constants in a quantum mechanics

Hamiltonian $$H = H(\vec{\lambda})\ ,\quad \vec{\lambda} = (\lambda_1, \lambda_2, \ldots)$$



$$|\Psi(\vec{\lambda})\rangle_T = e^{i\gamma}\, e^{-\frac{i}{\hbar}\int_0^T dt\, E_\Psi(t)}\ |\Psi(\vec{\lambda})\rangle_0$$

Berry phase

- **Berry phase** is an intrinsic property of the quantum system, it depends only on path *C*

$$\gamma = i \oint_C d\vec{\lambda}\, \langle \Psi(\vec{\lambda}) | \partial_{\vec{\lambda}} | \Psi(\vec{\lambda}) \rangle$$

or

$$\gamma = i \oint_C d\vec{\lambda}\, \mathcal{A}(\vec{\lambda})$$

Berry connection

connection on a vector bundle of spaces of states (Hilbert spaces)

- For *D* degenerate states the *U(1)* phase upgrades to a *U(D)* transform
⇒ Berry connection is **non-abelian**

- this connection has curvature

$$F_{\mu\nu} = \frac{\partial \mathcal{A}_\nu}{\partial \lambda^\mu} - \frac{\partial \mathcal{A}_\mu}{\partial \lambda^\nu} + [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

- elementary derivation of a general spectral formula for the curvature

$$\left(F^{(n)}_{\mu\nu}\right)_{ab} = \sum_{m\neq n} \sum_{c,d} \frac{1}{(E_n - E_m)^2} \langle n, b|\partial_\mu H|m, c\rangle g^{cd}_{(m)} \langle m, d|\partial_\nu H|n, a\rangle - (\mu \leftrightarrow \nu)$$

in sector with energy E_n and degenerate states $\left|n, a\right\rangle$

$$g_{ab} = \langle a|b\rangle$$

Non-trivial Berry phases in a simple QFT

1 Berry phase of photons

- Consider electromagnetism with a θ interaction

e.g. Wilczek, PRL '89

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{64\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

- θ has non-trivial implications if it varies in spacetime, or if there are boundaries/walls

- interested in adiabatic changes of θ in time

- this theory is free: as we change *e* and θ adiabatically the spectrum is unchanged, but the energy eigenstates can rotate

- Hamiltonian

$$H = \frac{1}{2}\int d^3x \left[e^2 \left(\vec{\pi} + \frac{\theta}{8\pi^2}\vec{B} \right)^2 + \frac{1}{e^2}\vec{B}^2 + \vec{\pi}\cdot\vec{\nabla}A_0 \right]$$

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \partial_t A_i} = \frac{1}{e^2}E_i - \frac{\theta}{8\pi^2}B_i$$

$$\partial_{e^2} H = \frac{1}{e^4}\int d^3x \left(\vec{E}^2 - \vec{B}^2 \right) \ , \quad \partial_\theta H = \frac{1}{8\pi^2}\int d^3x\, \vec{E}\cdot\vec{B}$$

- Place on $\mathbb{R} \times \mathbb{T}^3$ and quantize in Coulomb gauge

$$\vec{A}(t,\vec{x}) = \sum_{\vec{k}} \sum_{\epsilon=\pm} \sqrt{\frac{\hbar e^2}{2\omega_k V}} \left(\vec{e}_\epsilon(\vec{k}) a_{\vec{k},\epsilon} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + \bar{\vec{e}}_\epsilon(\vec{k}) a^\dagger_{\vec{k},\epsilon} e^{i\omega_k t - i\vec{k}\cdot\vec{x}} \right)$$

$$\omega_k = c|\vec{k}| \ , \quad k_i = \frac{2\pi n_i}{R} \ , \quad n_i \in \mathbb{Z} \ , \quad V = R^3$$

- Evaluate the spectral sum in the formula for Berry curvature

  The curvature has non-vanishing components only for identical external states

- for a general multi-photon state $\left|\mathfrak{n}_+, \mathfrak{n}_-\right\rangle$ with
  $\mathfrak{n}_+$ positive helicity photons, $\mathfrak{n}_-$ negative helicity photons

$$(F_{\tau\bar{\tau}})_{(\mathfrak{n}_+,\mathfrak{n}_-)} = \frac{\mathfrak{n}_+ - \mathfrak{n}_-}{8}\frac{1}{(\mathrm{Im}\tau)^2}$$

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$$

☞ a non-trivial Berry phase for photons

☞ independent of momentum

☞ Implications:

linearly polarized light changes polarization under (e,θ) variation

$$|p_z, \hat{x}\rangle = \frac{1}{\sqrt{2}}\left[|p_z, +\rangle + |p_z, -\rangle\right]$$

$$\downarrow$$

$$|p_z, \hat{\phi}\rangle = \frac{1}{\sqrt{2}}\left[e^{i\phi}|p_z, +\rangle + e^{-i\phi}|p_z, -\rangle\right]$$

☞ experimentally visible? (e.g. topological insulators…)

e.g. Essin-Moore-Vanderbilt, PRL '09

Conformal Perturbation Theory

vs

Berry phase

CFT in *d*+1 dimensions with a non-trivial **conformal manifold**

(continuous family of CFTs connected by exactly marginal deformations)

$$\delta S = \int d^{d+1}x \, \lambda^i \mathcal{O}_i(x)$$

Natural notions of geometry on the space of $\{ \lambda^i \}$

Zamolodchikov metric: $$g_{ij} = \langle \mathcal{O}_i(\infty) \mathcal{O}_j(0) \rangle$$

Zamolodchikov ‘86

but also notions of **parallel transport, connection**

on the **vector bundle of operators** over the conformal manifold:

how one compares operators at near-by CFTs

Kutasov ’89, Sonoda et al ’90s,…

- in conformal perturbation theory a covariant derivative incorporates regularization prescriptions

$$\nabla_\mu \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle \sim \left[\int d^{d+1}x \, \langle \mathcal{O}_\mu(x) \varphi_1(z_1) \dots \varphi_n(z_n) \rangle \right]_{regularised}$$

- the curvature of this connection involves integrated correlation functions

  Roughly:

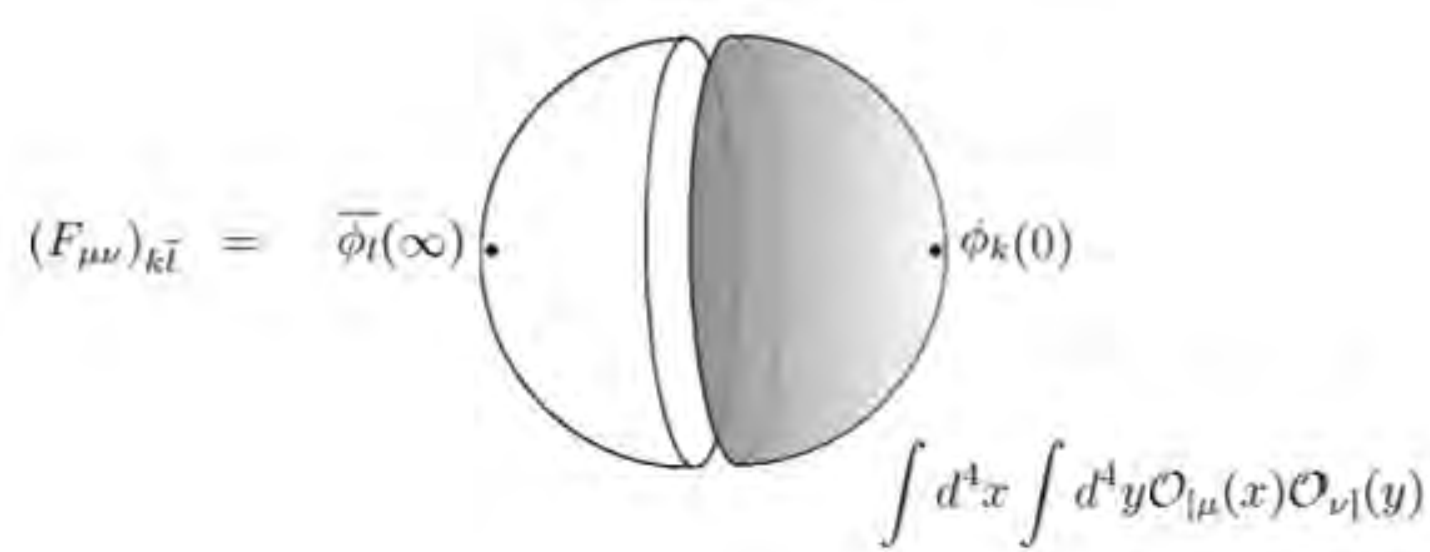
$$[\nabla_\mu, \nabla_\nu]_{12} \sim \int d^{d+1}x \int d^{d+1}y \, \langle \mathcal{O}_{[\mu}(x)\mathcal{O}_{\nu]}(y)\varphi_1(z_1)\varphi_2(z_2)\rangle$$

antisymmetry

- integrations lead to divergences
  regularisation leads to non-vanishing commutators ⇒ curvature

- **Ranganathan-Sonoda-Zwiebach** ('93) prescription
  (cut out small balls around operators, do not allow collisions, remove divergent pieces)

- 4-point operator formula for the curvature



$on\ \mathbb{R}^4$

$$(F_{\mu\nu})_{kl} = \frac{1}{(2\pi)^4} \int_{|x|\leq 1} d^4x \int_{|y|\leq 1} d^4y \, \langle \varphi_l(\infty) \mathcal{O}_{[\mu}(x) \mathcal{O}_{\nu]}(y) \varphi_k(0) \rangle$$

- In CFT there is a natural correspondence between states and operators

OPERATOR-STATE correspondence

in radial quantization or on $\mathbb{R} \times S^d$

$$|I\rangle \sim \mathcal{O}_I(0)|0\rangle$$

- The **Berry connection for states** should map to a natural **connection for operators in Conformal Perturbation Theory**

- Two seemingly different expressions for curvature

Berry

double d-dim integral

$$(F_{\mu\nu})_{IJ} = \sum_{n \notin \mathcal{H}_I} \sum_{a,b \in \mathcal{H}_n} \frac{1}{(\Delta_I - \Delta_n)^2} \langle J | \partial_\mu H | n, a \rangle g^{ab}_{(n)} \langle n, b | \partial_\nu H | I \rangle - (\mu \leftrightarrow \nu)$$

CFT

$$(F_{\mu\nu})_{IJ} = \int_{|x| \leq 1} d^{d+1}x \int_{|y| \leq 1} d^{d+1}y \langle \mathcal{O}_J(\infty) \mathcal{O}_\mu(x) \mathcal{O}_\nu(y) \mathcal{O}_I(0) \rangle - (\mu \leftrightarrow \nu)$$

double (d+1)-dim integral

Claim: these expressions compute the same object !

Non-perturbative computation in SCFTs

*tt** equations

4d N=2 superconformal manifolds

The **curvature of N=2 chiral primary vector bundles** in 4d N=2 superconformal manifolds is computable <u>non-perturbatively</u>

Previously computed by **Papadodimas'09** in conformal perturbation theory

The QM derivation of Berry curvature in the 1/2-BPS sector of N=2 chiral primary states reproduces the same result

- 4d N=2 has 8 supercharges $Q^i_\alpha, \bar{Q}_{i,\dot{\alpha}}$ $(i = 1, 2 \quad \alpha = \pm)$

  & superconformal partners $S^\alpha_i, \bar{S}^{i,\dot{\alpha}}$

- N=2 chiral primaries: $[\bar{Q}^i_{\dot{\alpha}}, \phi_I] = 0$ (+ complex conjugate)

$$\Delta = \frac{R}{2}$$

chiral ring under the Operator Product Expansion (OPE)

$$\phi_I(x)\phi_J(0) = C^K_{IJ}\,\phi_K(0) + \dots$$

Exactly marginal interactions are descendants of N=2 chiral primaries

$$\mathcal{O}_i = \int d^4\theta\,\phi_i\ ,\quad \bar{\mathcal{O}}_j = \int d^4\bar{\theta}\,\bar{\phi}_j$$

Berry curvature in sector of N=2 chiral primary states

$$(F_{\mu\nu})_{IJ} = \sum_{n \notin \mathcal{H}_I} \sum_{a,b \in \mathcal{H}_n} \frac{1}{(\Delta_I - \Delta_n)^2} \langle J|\delta_\mu H|n,a\rangle g^{ab}_{(n)} \langle n,b|\delta_\nu H|I\rangle - (\mu \leftrightarrow \nu)$$

N=2 superconformal deformations

conveniently recast as

$$(F_{\mu\nu})_{I\bar{J}} = \lim_{x \to 0} \left(\tilde{F}_{\mu\nu}\right)_{I\bar{J}}$$

$$\left(\tilde{F}_{\mu\nu}\right)_{IJ} = \langle J|\delta_\mu H (H + \hat{R} - x)^{-2} \delta_\nu H|I\rangle - (\mu \leftrightarrow \nu)$$

Insert expressions for *δH* and compute…

Dolan-Osborn conventions

$$H + \hat{R} = H - \frac{R}{2}$$

After a few elementary steps using SCA relations

$$\left(\tilde{F}_{k\bar{l}}\right)_{I\bar{J}} = \left[\langle \bar{J}| \oint \phi_k \frac{(H+\hat{R})^4}{(H+\hat{R}-x)^2} \oint \bar{\phi}_{\bar{l}} |I\rangle - \langle \bar{J}| \oint \bar{\phi}_{\bar{l}} \frac{(H+\hat{R})^4}{(H+\hat{R}-x)^2} \oint \phi_k |I\rangle \right]$$
$$-4 \left[\langle \bar{J}| \oint \phi_k \frac{(H+\hat{R})^2}{(H+\hat{R}-x)^2} \oint \bar{\phi}_{\bar{l}} |I\rangle - \langle \bar{J}| \oint \bar{\phi}_{\bar{l}} \frac{(H+\hat{R})^2}{(H+\hat{R}-x)^2} \oint \phi_k |I\rangle \right]$$

1

in disguise the `localized' 4-point function formula in **Papadodimas '09**

guarantees we get the same result as in Conformal Perturbation Theory

2

direct computation requires careful separation of ϕ insertions in time

final result is the same as in Conf. Pert. Theory from Papadodimas '09

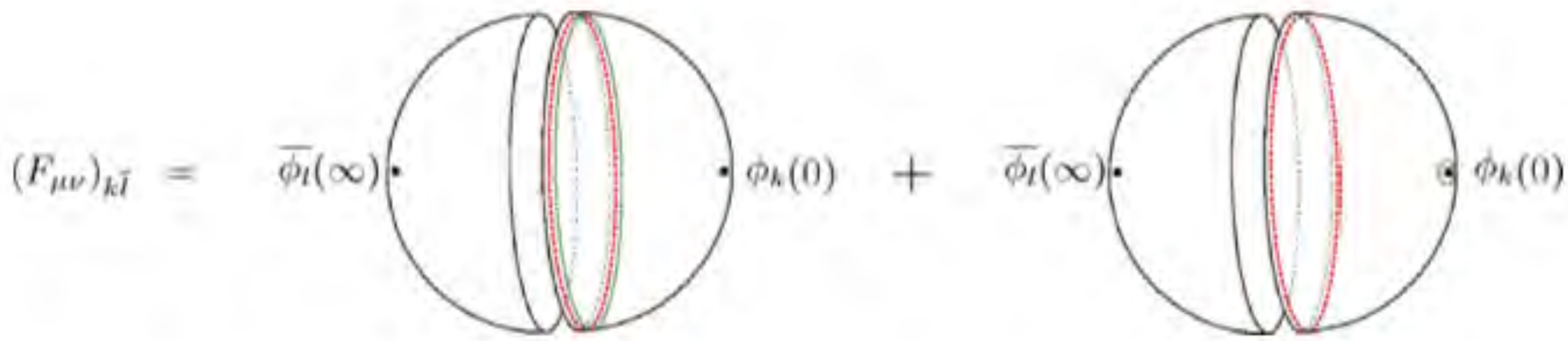


Figure 2: The colored dotted lines denote surface integrals of chiral primary operators and their conjugates.

$$
\begin{aligned}
(F_{i\bar{j}})_{k\bar{l}} = \; & \frac{1}{(2\pi)^4} \lim_{r\to 1^-} \int_{|x|=r} d\Omega_3^x \int_{|y|=1} d\Omega_3^y |x|^2 |y|^2 (y\cdot\partial_y)(x\cdot\partial_x) \\
& \left(\frac{|y|^2}{|x|^2} \langle \bar{\phi}_l(\infty)\Phi_i(x)\overline{\Phi}_j(y)\phi_k(0)\rangle - \frac{|x|^2}{|y|^2}\langle \bar{\phi}_l(\infty)\Phi_i(y)\overline{\Phi}_j(x)\phi_k(0)\rangle \right) \\
& - \frac{1}{(2\pi)^4} \lim_{r\to 0} \int_{|x|=r} d\Omega_3^x \int_{|y|=1} d\Omega_3^y |x|^2 |y|^2 (y\cdot\partial_y)(x\cdot\partial_x) \\
& \left(\frac{|y|^2}{|x|^2} \langle \bar{\phi}_l(\infty)\Phi_i(x)\overline{\Phi}_j(y)\phi_k(0)\rangle - \frac{|x|^2}{|y|^2}\langle \bar{\phi}_l(\infty)\Phi_i(y)\overline{\Phi}_j(x)\phi_k(0)\rangle \right)
\end{aligned}
$$

this is the expression we get almost automatically from QM !!

Interesting technical point

QM streamlines the computation

bypasses non-trivial superconformal Ward identities

needed to localize on a double 3-dimensional integral one has to use conformal Ward identities of the form

$$\partial_x^2\partial_x^2\langle\bar{\phi}_l(\infty)\mathcal{O}_i(x)\bar{\mathcal{O}}_j(y)\phi_k(0)\rangle = \partial_y^2\partial_y^2\left(\frac{|y|^4}{|x|^4}\langle\bar{\phi}_l(\infty)\mathcal{O}_i(x)\bar{\mathcal{O}}_j(y)\phi_k(0)\rangle\right)$$

$$\partial_y^2\langle\bar{\phi}_l(\infty)\mathcal{O}_i(y)\bar{\mathcal{O}}_j(x)\phi_k(0)\rangle = \partial_x^2\left(\frac{|x|^2}{|y|^2}\langle\bar{\phi}_l(\infty)\mathcal{O}_i(y)\bar{\mathcal{O}}_j(x)\phi_k(0)\rangle\right)$$

$$\int_{|x|=\mathrm{const}} d\Omega_3^x\partial_y^2\left(|y|^4\langle\bar{\phi}_l(\infty)\mathcal{O}_i(x)\bar{\mathcal{O}}_j(y)\phi_k(0)\rangle\right)$$
$$= \int_{|x|=\mathrm{const}} d\Omega_3^x|y|^2(x\cdot\partial_x)\left(|x|^2(x\cdot\partial_x)\left(\frac{1}{|x|^2}\langle\bar{\phi}_l(\infty)\mathcal{O}_i(x)\bar{\mathcal{O}}_j(y)\phi_k(0)\rangle\right)\right)$$

to finally localize on a double 3d integral

- The rest can be evaluated using OPEs. Only a few terms contribute

- The end result is a nice simple formula that relates the curvature with the chiral ring OPE coefficients and the chiral ring 2-point functions

$$\left(F_{i\bar{j}}\right)_{K\bar{L}} = -[C_i, \bar{C}_j]_{K\bar{L}} + g_{i\bar{j}} g_{K\bar{L}} \left(1 + \frac{R}{c}\right)$$

$$g_{K\bar{L}} = \langle \bar{\phi}_L(\infty) \phi_K(0) \rangle$$

☞ ***tt* equations***

similar to Cecotti-Vafa in 2d N=(2,2) with topological twist, '91

powerful combined with **SUSY localization** on S^4 partition functions

Baggio-VN-Papadodimas '14, '15, '16
Gerschkovitz-Gomis-Komargodski '14
Gerschkovitz et al. '16

Example: SU(2) N=2 SCQCD

Baggio-VN-Papadodimas '14

chiral primary operators $\phi_{2n} \propto \left(\mathrm{Tr}\left[\varphi^2\right]\right)^n$

in normalization $\phi_2(x)\phi_{2n}(0) = \phi_{2n+2}(0) + \ldots$

non-trivial info in 2-point functions $\langle \phi_{2n}(x)\bar{\phi}_{2n}(0)\rangle = \frac{g_{2n}(\tau,\bar{\tau})}{|x|^{4n}}$

tt* equations $\partial_\tau \partial_{\bar{\tau}} \log g_{2n} = \frac{g_{2n+2}}{g_{2n}} - \frac{g_{2n}}{g_{2n-2}} - g_2$

$$g_0 = 1\ , \quad n = 1, 2, \ldots$$

semi-infinite Toda

solution recursively from $g_2 = \partial_\tau \partial_{\bar{\tau}} Z_{S^4}$ from localization on S^4

Gerschkovitz-Gomis-Komargodski '14

Baggio-VN-Papadodimas '14

SU(2) N=2 SCQCD

$$\langle \left(\mathrm{Tr}\varphi^2\right)^{n_1} \left(\mathrm{Tr}\varphi^2\right)^{n_2} \left(\mathrm{Tr}\bar{\varphi}^2\right)^{n_1+n_2} \rangle$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}}$$

$$\hat{C}^{(0)}_{224} = \sqrt{\frac{10}{3}}\left(1 - \frac{9\,\zeta(3)}{2\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{525\,\zeta(5)}{8\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \dots\right),$$

$$\hat{C}^{(0)}_{246} = \sqrt{7}\left(1 - \frac{9\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{675\,\zeta(5)}{4\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \dots\right),$$

$$\hat{C}^{(0)}_{268} = 2\sqrt{3}\left(1 - \frac{27\,\zeta(3)}{2\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{2475\,\zeta(5)}{8\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \dots\right),$$

$$\hat{C}^{(0)}_{2810} = \sqrt{\frac{55}{3}}\left(1 - \frac{18\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{975\,\zeta(5)}{2\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \dots\right),$$

$$\hat{C}^{(0)}_{448} = 3\sqrt{\frac{14}{5}}\left(1 - \frac{18\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{825\,\zeta(5)}{2\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \dots\right),$$

$$\hat{C}^{(0)}_{4610} = \sqrt{66}\left(1 - \frac{27\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{2925\,\zeta(5)}{4\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \dots\right).$$

+ ALL instanton corrections

Large-N 't Hooft limit
SU(N) N=2 SCQCD

$$\langle \mathrm{Tr}\varphi^{n_1}\mathrm{Tr}\varphi^{n_2}\mathrm{Tr}\bar{\varphi}^{n_1+n_2} \rangle$$

$$\langle 2,2,\overline{4}\rangle_n = \frac{4}{N}\left(1 - \frac{3\zeta(3)}{64\pi^4}\lambda^2 + \frac{45\zeta(5)}{512\pi^6}\lambda^3 + \frac{3(72\zeta(3)^2 - 1085\zeta(7))}{32768\pi^8}\lambda^4 + \dots\right),$$

$$\langle 2,4,\overline{6}\rangle_n = \frac{4\sqrt{3}}{N}\left(1 - \frac{3\zeta(3)}{128\pi^4}\lambda^2 + \frac{15\zeta(5)}{256\pi^6}\lambda^3 + \frac{99\zeta(3)^2 - 2275\zeta(7)}{32768\pi^8}\lambda^4 + \dots\right),$$

$$\langle 4,4,\overline{8}\rangle_n = \frac{8\sqrt{2}}{N}\left(1 + \frac{15\zeta(5)}{512\pi^6}\lambda^3 - \frac{665\zeta(7)}{16384\pi^8}\lambda^4 + \dots\right),$$

$$\langle 4,6,\overline{10}\rangle_n = \frac{4\sqrt{15}}{N}\left(1 + \frac{15\zeta(5)}{512\pi^6}\lambda^3 - \frac{35(263520\zeta(3)^2 - 501551\zeta(7))}{32768\pi^8}\lambda^4 + \dots\right),$$

$\dots$

Baggio-VN-Papadodimas '16
(also Gomez-Russo '16, '17)

This study led to the first **exact non-perturbative** computation of non-trivial 3-point functions in 4d QFTs

Baggio-VN-Papadodimas ’14

Supersymmetric localization now allows the complete solution of extremal N-point functions in the N=2 chiral ring

Gerschkovitz-Gomis-Ishtiaque-Karasik-Komargodski-Pufu ’16

Main messages - outlook

- Non-trivial Berry phases appear quite generically in QFT (examples in this talk: E&M, CFTs)

  Possibly new physically interesting effects await to be discovered

- Evaluating the Berry phase of QFTs by putting them on different curved manifolds appears to be a useful strategy

- The Berry phase of 1/2-BPS states in **4d N=2** & **2d N=(2,2)** SCFTs is a non-trivial example where non-perturbative computations are possible

- Very interesting to extend these results to:
  - lower SUSY,
  - non-conformal theories…

- Ultimate goals (work in progress):

  ⊙ **non-perturbative relations between correlation functions**

  ⊙ **geometry of parameter spaces (space-of-theories)…**