Entanglement Entropy In Gauge Theories

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Outline

- Introduction

- Extended Hilbert Space Definition :
  a) Properties
  b) Abelian, Non-Abelian Theories
  c) Extractable Part

- U(1) in 3+1 Dim. Continuum Limit

- Mutual Information etc in Continuum Limit

- Conclusions

Introduction

Entanglement Entropy in Gauge Theories is non-trivial to define.

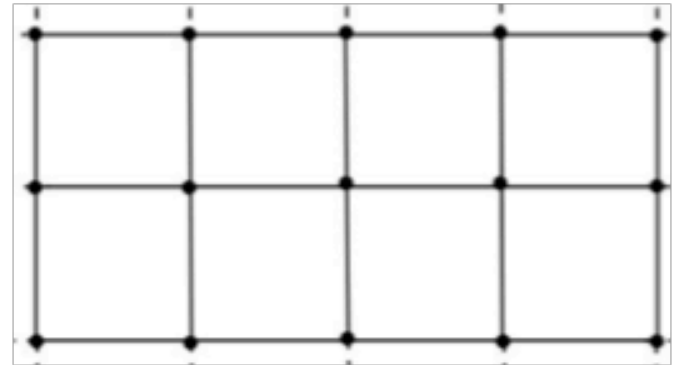Interesting: Gauge Theories are ubiquitous.

Non-Trivial: because there are Non-local excitations, e.g. Wilson and 'tHooft Loops.

Introduction

In a system with local degrees of freedom, like a spin system, the entanglement entropy is straight forward to define.

Introduction

E.g. in a spin system

Z_2 Spin System: single qubit at each site $|\pm>$

2- dim Hilbert space at site i: $\mathcal{H}_i$

Interested in the entanglement between a subset of spins, called the ``inside'' with the rest, the ``outside''

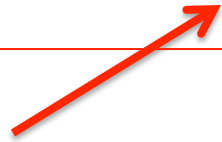Full Hilbert space:

$$\mathcal{H} = \bigotimes \mathcal{H}_i$$

Admits a tensor product decomposition

$$\mathcal{H} = \mathcal{H}_{in} \otimes \mathcal{H}_{out}$$

$$\rho = Tr_{\mathcal{H}_{out}} |\psi><\psi|$$

$$\rho = Tr_{\mathcal{H}_{out}} |\psi><\psi|$$

$$S_{EE} = S_{vN} = -Tr_{\mathcal{H}_{in}} \rho \log \rho$$

Von Neumann entropy

Entanglement In A Gauge Theory

- Not as simple to define.

- Because there are non-local degrees of freedom, e.g., Wilson loops, or loops of electric flux.

- Physical Hilbert space of states does not admit a tensor product decomposition between $\mathcal{H}_{in}, \mathcal{H}_{out}$
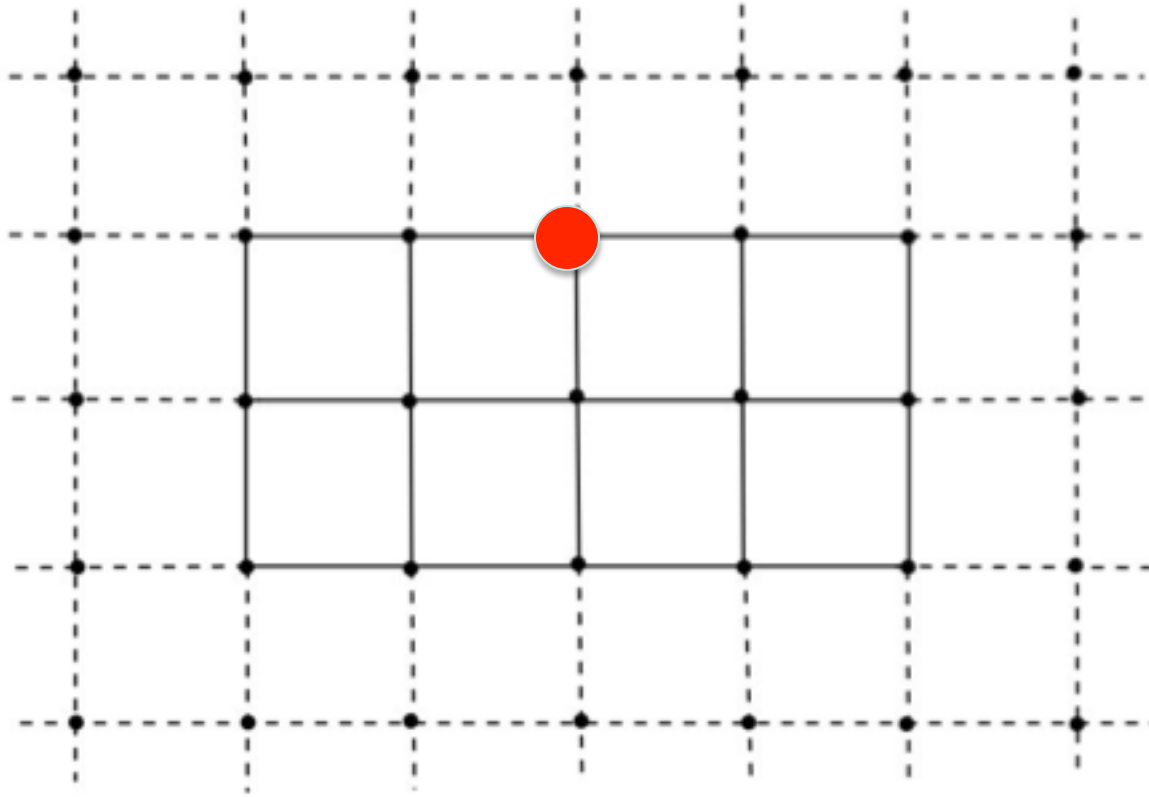
Entanglement Entropy In Gauge Theory

- Lattice Gauge Theory

- Hamiltonian Framework: time continuous, Spatial lattice

- Discussion applies in d+1 dimensions.

- (Diagrams in 2 spatial dimensions)

Entanglement Entropy In A Gauge Theory :
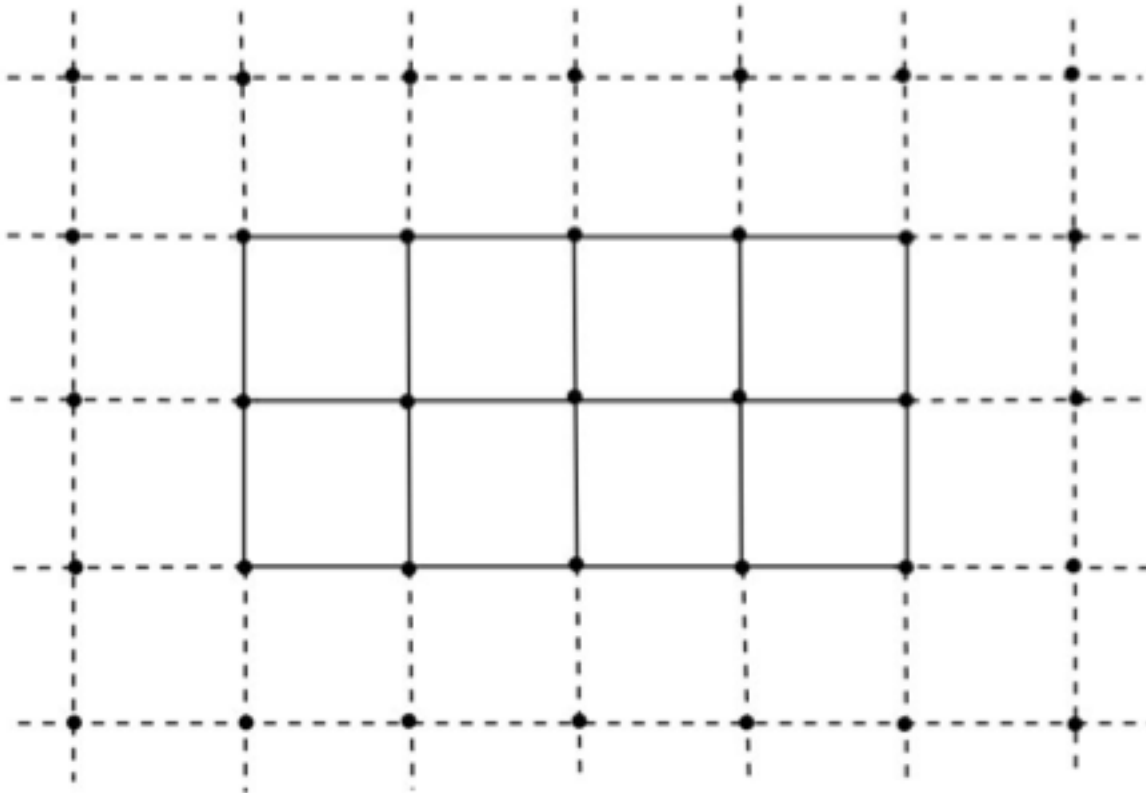
Entanglement in Gauge Theories

- The Gauss Law constraint means the inside and outside links ending on boundary vertices are not independent.

- Resulting in a lack of a tensor product decomposition.

- General feature: Discrete gauge theories, Abelian, Non-Abelian

The Gauss law constraint must apply at every boundary vertex, e.g. the red one in this figure.

This is the essential obstruction or complication in defining the entanglement entropy for gauge theories.

Gauge Fixing in different ways leads to dependent answers which are therefore gauge dependent.

References

- Casini, Huerta, Rosabal (CHR): Phys. Rev. D. 89, 085012 (2014) proposed a definition in the Abelian case.

- Identified the presence of a non-trivial centre in the algebra of operators as the essential complication.

- Different choices of centre gives different definitions.

On The Entanglement Entropy For Gauge Theories,
Sudip Ghosh, Ronak Soni and ST,
arXiv: 1501.2593

Aspects of Entanglement Entropy For Gauge Theories,
Ronak Soni, ST, arXiv:1510.07455

Entanglement Entropy in U(1) Gauge Theory, Ronak Soni, ST, arXiv: 1608.00353

References :

Casini, Huerta, Rosabal: Phys. Rev. D. 89, 085012 (2014)

Casini, Huerta, Phys. Rev. D 90, 105013 (2014)

Cassini, Huerta: arXiv: 1512.06182

References :

D. Radicevic, arXiv:1404.1391, 1509.08478

S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba, H. Tasaki, arXiv: 1502.04267

P. V. Buividovich and M. Polikarpov, Phys. Lett. B 670 (2008), 141.

References:

W. Donnelly, Phys. Rev. D 77, 104006 (2008)

W. Donnelly, Phys. Rev. D 85, 085004 (2012)

W. Donnelly and A. C. Wall, Phys. Rev. D 86, 064042 (2012)

W. Donnelly, Class. Quant. Grav. 31, no. 21, 214003 (2014)

References

- Pretko and Senthil, 1510.03863

- K. van Acoleyen, N. Bultnick, J. Haegeman, M. Marien, V. B. Scholz, F. Verstraete, 1511.04369

Extended Hilbert Space Definition

Work in an extended Hilbert Space $\mathcal{H}$
Obtained by taking the tensor product
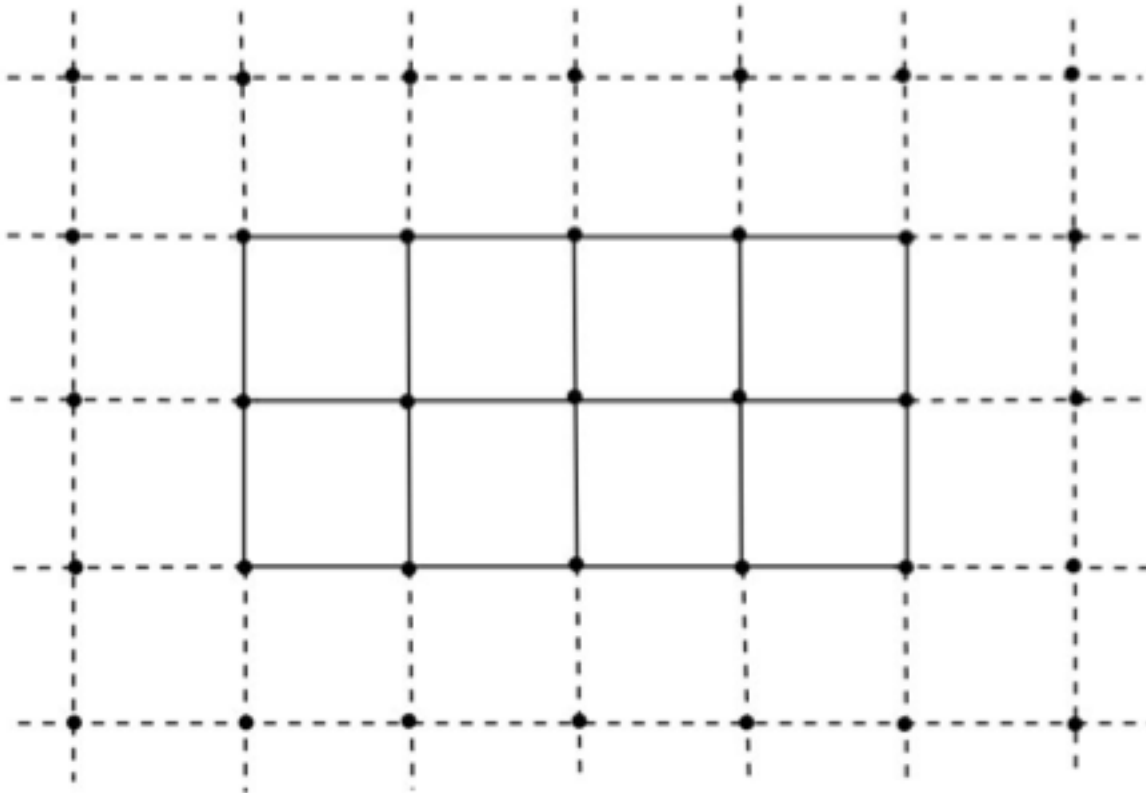of the Hilbert spaces on each link.

$$\mathcal{H} = \bigotimes \mathcal{H}_{ij}$$

Ghosh, Soni, ST, arXiv: 1501.2593
S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N.
Shiba, H. Tasaki, arXiv: 1502.04267

Entanglement Entropy In A Gauge Theory :

$\mathcal{H}$ admits a tensor product decomposition.

$$\mathcal{H}_{in} = \bigotimes_{(ij)\in in} \mathcal{H}_{ij}$$

$$\mathcal{H}_{out} = \bigotimes_{(ij)\in out} \mathcal{H}_{ij}$$

$$\mathcal{H} = \mathcal{H}_{in} \bigotimes \mathcal{H}_{out}$$

Embed State $|\psi >$ in $\mathcal{H}$ as follows:

$$\mathcal{H}_{ginv} \subset \mathcal{H}$$

$$\mathcal{H} = \mathcal{H}_{ginv} \oplus \mathcal{H}^{\perp}_{ginv}$$

Orthogonal
complement

$$|\psi > \in \mathcal{H}_{ginv}$$

Extended Hilbert Space Definition

Construct $\rho_{in} = Tr_{\mathcal{H}_{out}} |\psi><\psi|$

$$S_{EE} = -Tr_{\mathcal{H}_{in}}(\rho_{in} \log \rho_{in})$$

Renyi Entropies Can be Defined similarly.

Properties of Extended Hilbert Space Definition

- Definition Unambiguous.

- Gauge Invariant

- Meets Strong Subadditivity property

Properties of Extended Hilbert Space Definition

Simple, can be easily generalised:

- Discrete and Continuous,

- Abelian, Non-Abelian Groups.

- Also, with matter.

Properties of Extended Hilbert Space Definition

Definition unambiguous.

$|\psi>$ Gives rise to a unique state in $\mathcal{H}$

And after tracing out over $\mathcal{H}_{out}$ a unique ρ

Properties:

$\mathcal{H}$ Endowed with a natural inner product from that on $\mathcal{H}_{ij}$. Meets positivity condition.

Properties

Thus S_{EE} meets the strong subadditivity condition

A, B, C three sets of links that do not share any links in common

$$S_{A \cup B} + S_{B \cup C} \geq S_{A \cup B \cup c} + S_B$$

Gauge Invariant Characterisation Of The Extended Hilbert Space Definition

- Gauge Invariant: Essentially because

- $$|\psi> \in \mathcal{H}_{ginv}$$

-

- With no component along $\mathcal{H}^{\perp}$

Gauge Invariant Characterisation Of The Extended Hilbert Space Definition

- Let us understand the gauge invariance of the definition some more.

- How can the resulting answer be expressed in terms of gauge invariant data?

Gauge Invariant Characterisation

For simplicity we consider a theory without matter.

Any state can be decomposed into different sectors.

Each sector has a fixed amount of electric flux coming into the inside region.

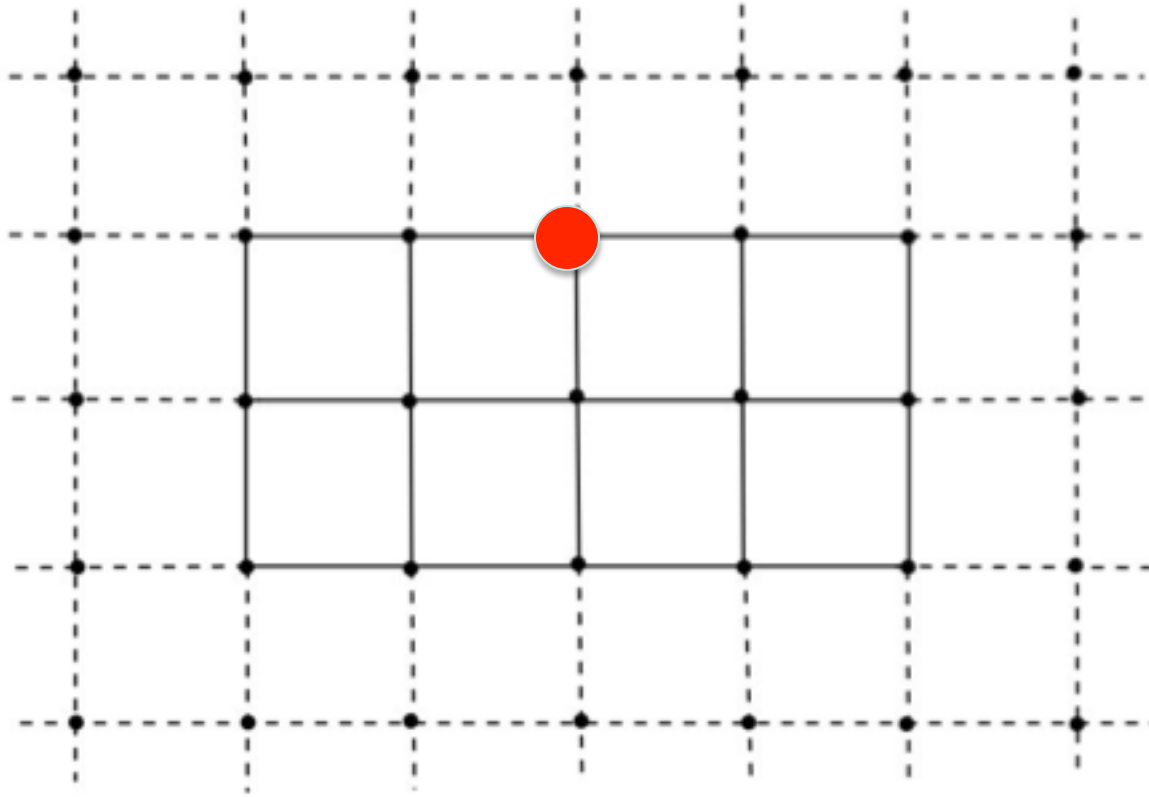Let the boundary vertices be labelled as

$$V_1, V_2, \cdots V_N$$

And let the Electric flux coming in be
$$\mathbf{k} = (k_1, k_2, \cdots k_N)$$

Then each sector is specified by
A choice of the electric flux vector $\mathbf{k}$.

$k_a, a = 1, \cdots N$ integer in the Abelian
case (compact U(1)).

Red dot location of one boundary vertex

Result (Abelian Case)

$$S_{EE} = -\sum_i p_i \log p_i - \sum_i p_i(\bar{\rho}_i \log \bar{\rho}_i)$$
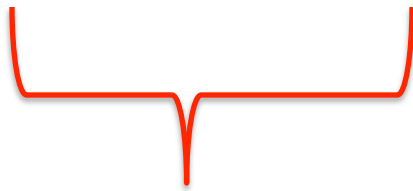
`i': labels sectors of different electric flux

p_i : probability to be in sector i

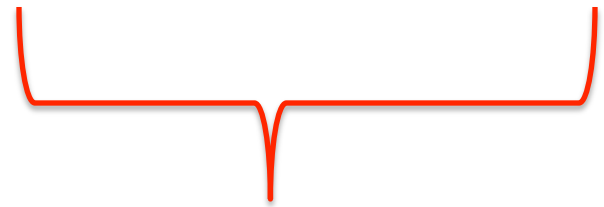$\bar{\rho}_i$ normalised density matrix in sector `i' $Tr_i(\bar{\rho}_i) = 1$

Result

$$S_{EE} = -\sum_i p_i \log p_i - \sum_i p_i (\bar{\rho} \log \bar{\rho}_i)$$

``Classical Term''

``Quantum Term''

The Extended Hilbert Space definition, in Abelian Case, can be shown to be equivalent to the electric center prescription of Cassini, Huerta and Rosabal.

Non-Abelian Theory

An extra contribution arises in the Non-Abelian case.

Tied to the fact that irreducible representations have dimension greater than unity.
(Donnelley)

Non-Abelian Theory

Let the total flux going inside at a boundary vertex transform in an irreducible representation R_a of the group.

Then the total flux going out at this vertex must also be in the same R_a irreducible representation.

And together they must pair to form a singlet under the gauge group (Gauss' Law)

Result (Non-Abelian Case)

The extra entanglement arises due to this pairing.

In the `i'th sector let the dimensions of the representations at the boundary vertices be $\qquad\qquad V_1, V_2, \cdots V_N$
$(d^i_1, d^i_2 \cdots d^i_N)$

Result (Non-Abelian Case)

Then

$$S_{EE} = -\sum_i p_i \log p_i - \sum_i Tr_i p_i \bar{\rho}_i \log \bar{\rho}_i$$
$$+ \sum_i p_i (\underbrace{\sum_a \log d_a^i})$$

Extra piece

Superselection Sectors

 The different sectors, with differing electric flux, are actually different super selection sectors.

Operations involving Gauge invariant operators in the inside or outside cannot change these sectors, or the probabilities p_i and dimensions d_a^i

Superselection Sectors

- This prevents some of the entanglement entropy from being ``extracted'' through local operations acting only on the inside or outside.

Extractable Part of Entanglement

How much of the entanglement we have defined can be actually used for quantum information processing?

Quantum Information Theory

- Entanglement quantified by comparison with a reference system of N Bell pairs.

- Comparison done using entanglement distillation or entanglement dilution.
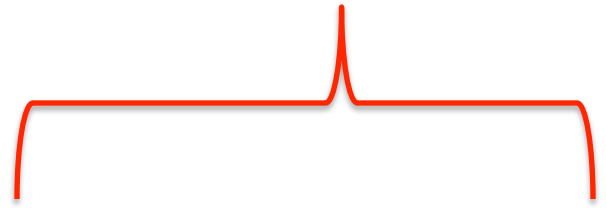
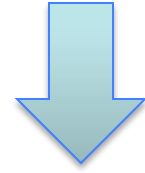# Extracting The Entanglement

So how well can we do?

The maximum entanglement which can be extracted is

$$\Delta S_{EE} = - \sum_i Tr_i \; p_i \bar{\rho}_i \log(\bar{\rho}_i)$$
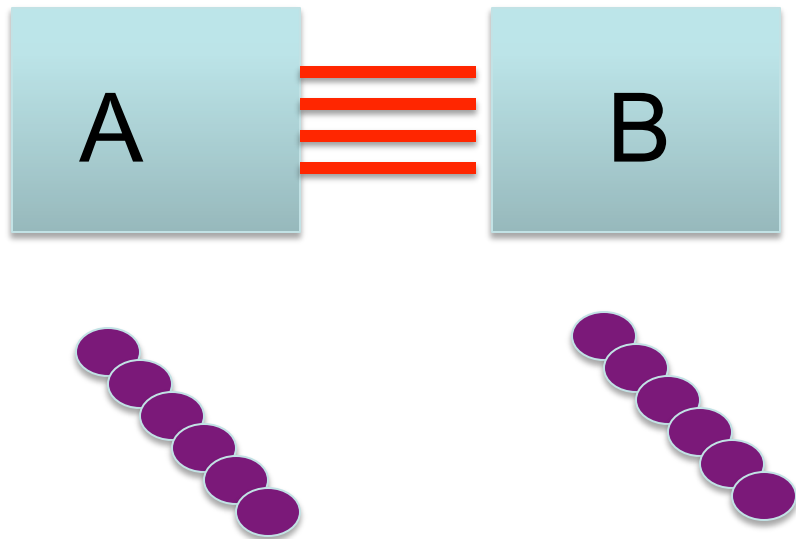
R. Soni and ST, 1510.07455

K. van Acoleyen, N. Bultnick, J. Haegeman, M. Marien, V. B. Scholz, F. Verstraete, 1511.04369
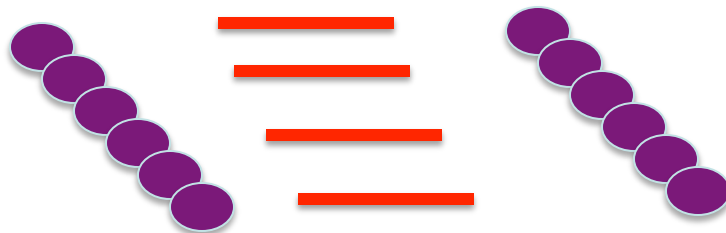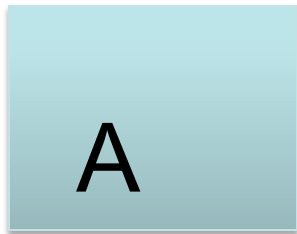
Only this term can be extracted

$$S_{EE} = -\sum_i p_i \log p_i - \overbrace{\sum_i Tr_i p_i \bar{\rho}_i \log \bar{\rho}_i}$$

$$+ \sum_i p_i \left(\sum_a \log d_a^i \right)$$

Entanglement Distillation :

A

B

2N unentangled qubits

To finally arrive at the situation:

A   B

N entangled Bell
pairs

Entanglement Distillation

Carry Out Transformations involving Local Operations and Classical Communication
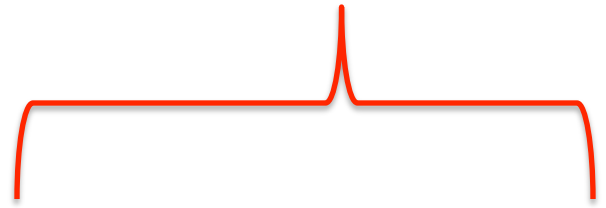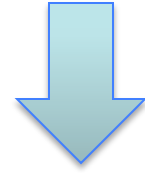
Local operations act only in A and one set of N qubits. Or B and the other set. Correspond to Gauge Invariant Operators.

Let N be the maximum number of Bell pairs we can produce.

Then $S_{EE} = N \log(2)$

(Actually in an asymptotic sense with $\mathcal{N}$ copies of the system in the $\mathcal{N} \to \infty$ Limit.)

Only this term can be extracted

$$S_{EE} = -\sum_i p_i \log p_i - \sum_i Tr_i p_i \bar{\rho}_i \log \bar{\rho}_i$$

$$+ \sum_i p_i \left(\sum_a \log d_a^i \right)$$

Extracting The Entanglement

We have given explicit protocols showing that this is the maximum bound on the extractable entanglement.

R. Soni and ST, 1510.07455

K. van Acoleyen, N. Bultnick, J. Haegeman, M. Marien, V. B. Scholz, F. Verstraete, 1511.04369

Extractable Part of Entanglement

The Gauge Theory may arise at energy
$$E \ll \Lambda$$

The extractable part is then what can be recovered using operations with energy $E \ll \Lambda$.

Using operations at higher energy more, sometimes even the full, entanglement can be recovered.

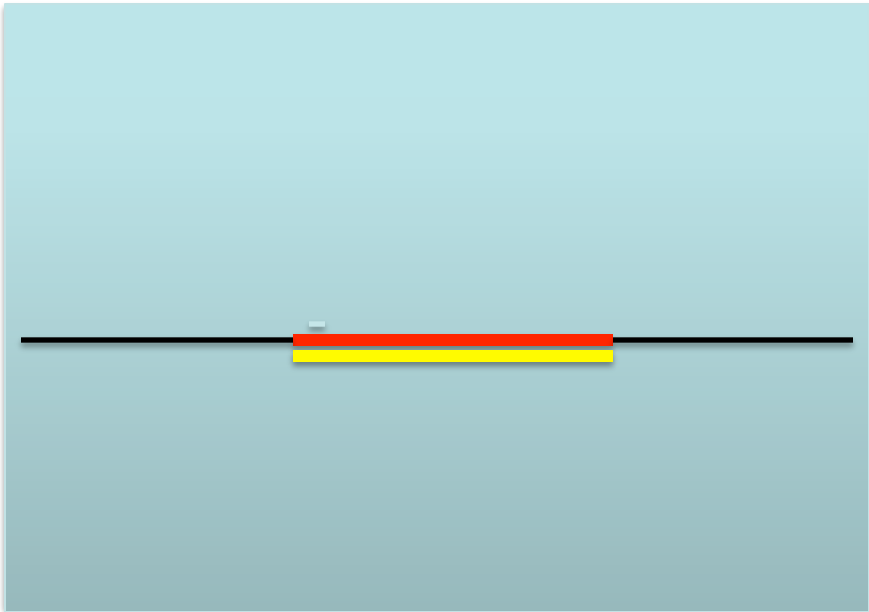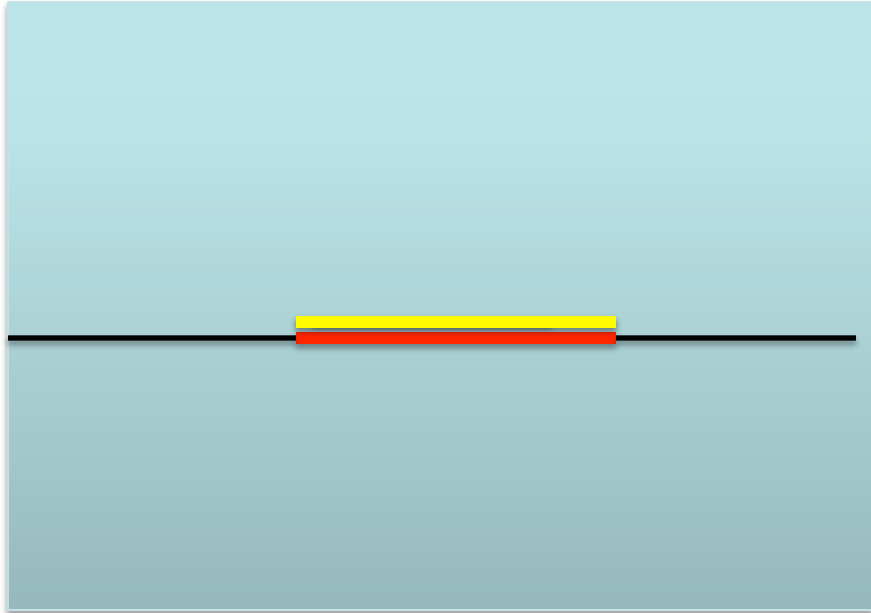# Properties of Extended Hilbert Space Definition

- Agrees with the Replica Trick Path Integral (on the lattice).

Replica Trick

- Essentially because each link variable is independent in the path integral.

- The path integral automatically gives $|\psi>$ embedded in the extended Hilbert space

$$|\psi> = \int_{-\infty}^{t=0^-} DU_{ij} \ e^{-S}$$

$$<\psi| = \int_{0^+}^{t=\infty} DU_{ij} \ e^{-S}$$

Continuum Limit

(R. Soni and ST, arXiv 1608.00353)

Start with Pure Gauge Theory (without matter) at weak coupling on the lattice in vacuum state.

And take the continuum limit.

This gives the replica trick integral with the standard Fadeev Popov Gauge fixing (plus a caveat   since the path integral needs to be regulated in continuum):

Continuum Limit

$$\int [DA^1] e^{-S} \delta(f(A^a)) |det(\frac{\delta f(A^a)}{\delta \omega^b})|$$

M is the singular n-fold cover of R^{d+1} obtained by introducing a branch cut along the boundary

Even Bigger Hilbert Space

- Result is gauge invariant.

- Suggests that we can work with an even bigger Hilbert space to define entanglement.

- Obtained by Including ghosts. This extended Hilbert space has negative norm states.

Even Bigger Hilbert Space

However physical states correspond to cohomology of the BRST operator.

$$Q_{BRST}|\psi> = 0$$

$$|\psi> \simeq |\psi> + Q_{BRST}|\chi>$$

Thus the result of the entanglement entropy so obtained should agree with the Extended Hilbert space definition.

Even Bigger Hilbert Space

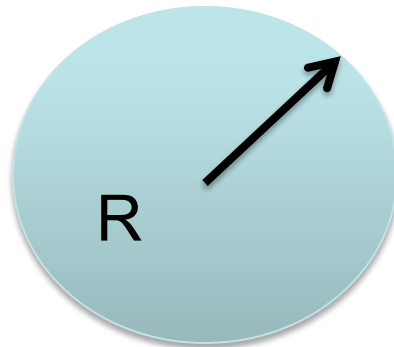This is true and can be made precise.

(Abelian case; Soni and SPT in prep.)

U(1) Theory 3+1 dimensions

A cutoff needs to be introduced to make the Path Integral well defined.

Take 3+1 dim and the spatial region whose entanglement we seek to be a sphere of radius R

R

U(1) In 3+1 Dim

On general grounds we expect:

$$S_{EE} = A/\epsilon^2 + C \log(R/\epsilon) + \text{finite}$$

Cut-off
independent

U(1) Theory in 3+1 Dim

C gets related to the A anomaly coefficient

$$C = -\frac{31}{45}$$

$$T^{\mu}_{\mu} = aE_4 + bW^2$$

$$C = a$$

Fursayev, Patrushev, Solodukhin, 1306.400, ...

U(1) Theory in 3+1 Dim

How much of this is extractable?

The ``classical piece'' turns out to be:

$$\Delta S = -\sum_i p_i \log(p_i)$$
$$= -\frac{1}{3} \log(R/\epsilon)$$

Related to 1+1 Free scalar CFT

R. Soni and ST, arXiv 1608.00353;
Donnelley and Wall, PRL, 2015, 114, 111603.

U(1) in 3+1 Dim

The resulting extractable piece is then

$$\Delta S_{extractable} = -\frac{16}{45}\log(R/\epsilon)$$

U(1) in 3+1 Dim

The extractable part agrees with the result for the full entanglement recently obtained by Casini and Huerta, using a trivial center definition.

And also earlier by D'Howker.

The confusion about why this answers did not agree with the anomaly coefficient can now be understood.

Note the full answer and the extractable piece are both different from that for two scalar fields:
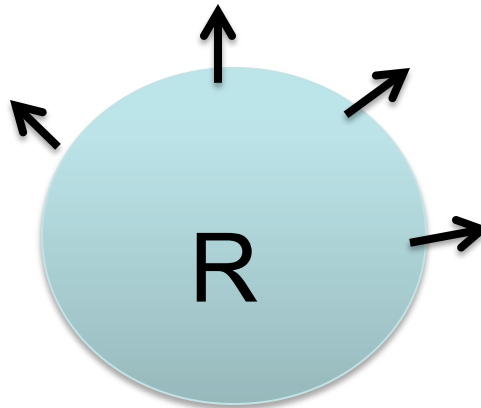
C= $2 \times -(1/90) = -1/45$

(Due to the presence of Kabat Terms)

U(1) Theory: Classical Term

Classical piece:

$$S_{class} = -\sum_i p_i \log p_i$$

p_i: probability for normal component of electric field to take some value

U(1) Theory: Classical Term

This probability is dominated by the behaviour of modes with wavelength of order the cut –off scale transverse to the boundary.

$$< E_{n,l} E_{n,l} > \sim l(l+1) \log(\frac{\Delta}{R})$$

Soni, SPT: to appear

U(1) Theory

Planar Boundary:

$$< E_n(\vec{k}_\perp) E_n(\vec{k}'_\perp) > \sim \delta^2(k_\perp - k'_\perp) k_\perp^2 \log(k_\perp \Delta)$$

Log dependence on cutoff Δ also true in 2+1 dimensions.

U(1) Theory: Classical term

As a result one can show that this term remains the same if we consider states differing from vacuum at finite wavelength
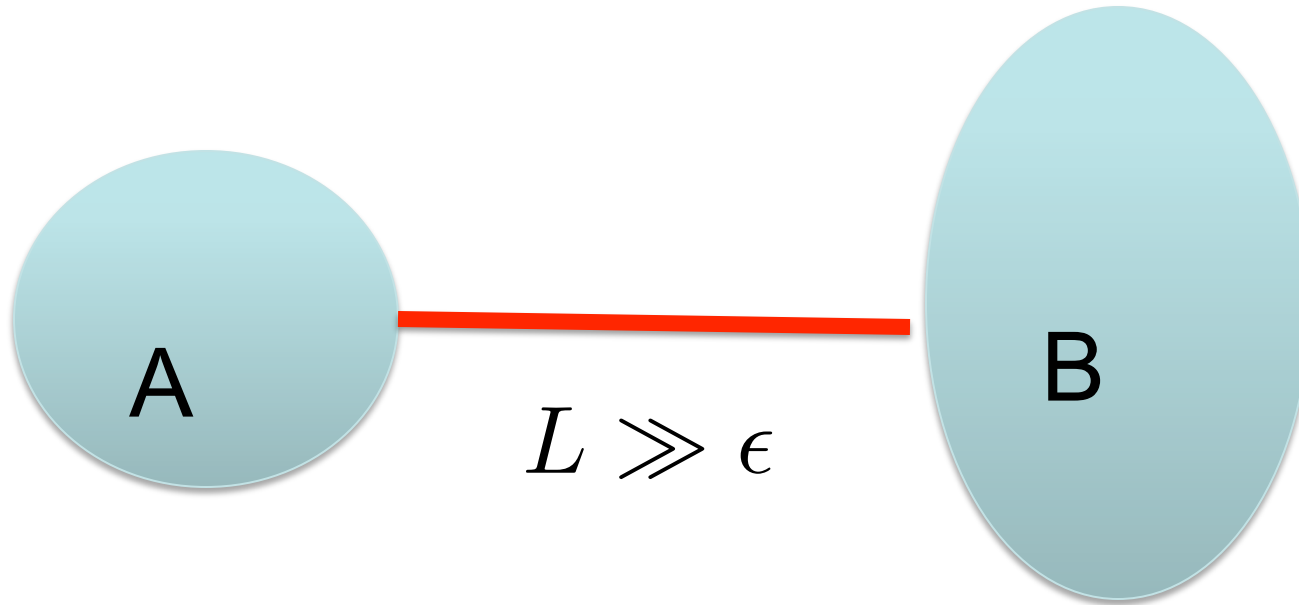
$$\lambda \gg \epsilon$$

It therefore does not contribute to the relative entropy for two such states

U(1) Theory: Classical Term

The Classical term also drops out for this reason from the Mutual Information between two regions, separated by a macroscopic distance, in the continuum limit.

These conclusions are true in 2+1 dimensions also.

Mutual Information

$$I(A, B) = S_A + S_B - S_{A \cup B}$$

Classical Terms Non-Abelian Theories

- Similar arguments also apply for Non-Abelian theories. Classical terms drop out from Relative Entropy and Mutual Information in continuum limit.

- As long as theories are asymptotically free.

- Could be different if behaviour in UV is strongly coupled or for a non-trivial CFT.

Mutual Information and Relative Entropy

As a result, in the continuum limit the Mutual Information only gets contributions from the extractable term.

Similarly for Relative Entropy.

Classical Term

The fact that the classical term drops out was anticipated by Casini, Huerta and Rosabal (CHR).

And verified numerically for free U(1) in 2+1 dim (Casini and Huerta, arXiv:1406.299)

They also argued that Mutual Information and Relative Entropy are independent of choice of centre, in the continuum limit.

Conclusions

Bottom line:

A definition for the entanglement in gauge theories can be given, based on an extended Hilbert space.

It gives rise to a classical and quantum term.

The quantum term is the extractable entanglement.

The Replica Trick path integral agrees with this definition.

Conclusions

Bottom line:

In the continuum, things especially simplify.

The classical term does not contribute to Mutual information and Relative Entropy.

(Also the centre dependence drops out as argued by Cassini, Huerta and Rosabal.)

Conclusions

- A definition for entanglement entropy was given based on an Extended Hilbert Space construction.

- The definition is applicable to Abelian and Non-Abelian Theories, and also to theories with matter.

Conclusions

- It has many nice properties.
  i) It is gauge invariant.
  ii) Agrees with the Replica Trick, etc.

- The resulting entanglement has a Classical and Quantum term.

- The extractable part is given by the Quantum term.

Conclusions

- The classical term does not contribute to mutual information or relative entropy in the continuum limit (with some caveats).

- Thus, these quantities, in the continuum limit, only get a contribution from the extractable part.

Conclusions

- The classical term does contribute to the entanglement entropy. As a result, the extractable part can be different from the full result.

Conclusions

Bottom line:

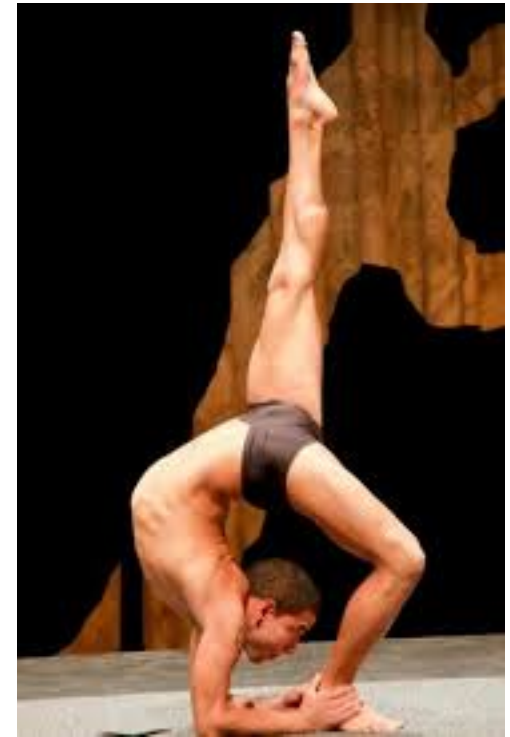A sensible definition for the entanglement in gauge theories can be given.

In the continuum, things especially simplify.

Replica trick path integral in continuum limit gives the correct mutual information for gauge theories.

Conclusions

- Implications for gravity remain to be fully understood.

Thank You !

ಧನ್ಯವಾದಾಲು ಧನ್ಯವಾದ આભાર ধন্যবাদ ಧನ್ಯವಾದಗಳು நன்றி
ಧನ್ಯವಾದಗಳು நன்றி धनयबाद यंनवाद ಧನ್ಯವಾದಾಲು આભારநன்றி ধন্যবাদ
यंनवाद આભાર ಧನ್ಯವಾದಗಳು நன்றி ಧನ್ಯವಾದಾಲು ನನ್ಪಿ
ধন্যবাদ ಧನ್ಯವಾದಾಲು நன்றி આભારধন্যবাদ धनयबाद
ನನ್ಪಿ நன்றி **THANK YOU** ಧನ್ಯವಾದಗಳು
धन्यवाद ﺷﻛﺮﻳﻪ ಧನ್ಯವಾದಗಳು நன்றி धनयबाद यंनवाद ধন্যবাদ ধন্যবাদ
ಧನ್ಯವಾದಾಲು धन्यवाद આભાર ধন্যবাদ ಧನ್ಯವಾದಗಳು நன்றி
ಧನ್ಯವಾದಗಳು நன்றி धनयबाद यंनवाद ಧನ್ಯವಾದಾಲು આભારநன்றி ধন্যবাদ

Banyan Tree In TIFR

Thank you!