Scale vs Conformal from Entanglement Entropy

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9th CRETE REGIONAL MEETING IN STRING THEORY

- In quantum field theory, scale invariance has an interpretation in terms of particle physics. In scale-invariant theory, the strength of particle interactions does not depend on the energy of the particles involved.
- In statistical mechanics, scale invariance is a feature of phase transitions. The key observation is that near a phase transition or critical point, fluctuations occur at all length scales.
- Universality is the observation that widely different microscopic systems can display the same behavior at a phase transition. Thus phase transitions in many different systems may be described by the same underlying scale-invariant theory.

- The modern way to think about that scale invariant theories comes from the concept of Renormalization Group (RG) flow. The scale invariant theories live at the fixed point of RG flows.
- Therefore, one of the fundamental questions in theoretical physics is to understand the structure of fixed points of the RG flow(s).
- Further importance of this question stems from the relation between fixed points of the RG flow and quantum gravity which could be provided by using the AdS/CFT correspondence.

- Remarkably, with a few known exceptions, unitary scale-invariant relativistic field theories always exhibit full conformal symmetry.
- The mechanism behind symmetry enhancement remains poorly understood.

What is the necessary and sufficient condition for this enhancement?

- A general scale invariant theory (SFT) has a local conserved scale current S^μ [Wess (1960)]

$$S^\mu = x^\nu T^\mu_\nu + V^\mu,$$

where V^μ is the so-called 'virial current'.

- The conservation of scale current gives

$$0 = \partial_\mu S^\mu = T^\mu_\mu + \partial_\mu V^\mu,$$

which means that for scale invariant theories

$$T^\mu_\mu = -\partial_\mu V^\mu$$

- A conformal current must be of the form [Wess(1960)]

$$K^{\mu} = v^{\nu} T^{\mu}_{\nu} + (\partial.v)\tilde{V}^{\mu} + \partial_{\nu}\partial.v\, L^{\nu\mu},$$

where $\partial_{(\mu}v_{\nu)} = \frac{2}{D} g_{\mu\nu}\partial.v$ and $\tilde{V}^{\mu}$ is the same as V^{μ} up to the possible addition of a conserved current and $L_{\mu\nu}$ is some local operator.

- Conservation of the conformal current ($\partial_{\mu}K^{\mu} = 0$) gives

$$T^{\mu}_{\mu} = -\partial_{\mu}\tilde{V}^{\mu} \quad \textit{plus} \quad \tilde{V}^{\mu} = -\partial_{\nu}L^{\nu\mu}.$$

- Obviously if the 'virial current' in a SFT is conserved, that SFT actually is a CFT.

- The Less obvious case where a unitary SFT can be a CFT is when the virial current is a total derivative, i.e, if

$$V_\mu = -\partial_\nu L^{\nu\mu}.$$

In that case one can define an improved stress-energy tensor

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{3}(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box)L,$$

which is conserved and traceless [Polchinski(1987)].

- In $D = 2$, scale invariance plus unitarity implies conformal invariance but with some implicit assumptions [Zamolodchikov(1986),Polchinski(1987)]
- The stress-energy tensor has canonical scaling. This assumption would be correct for the theories which have a discrete spectrum of scaling dimensions.
  - Scale invariant but not conformal invariant theory of [Riva and Cardy(2005)] does not satisfy this assumption (apart from the violation of unitarity).
- The stress-energy tensor 2-point function exists.
  - Scale invariant but not conformally invariant unitary 2D sigma-models violate this assumption because of their non-compact target space [Hull and Townsend (1986)].

- For $D \geq 3$, the situation was unclear up to 2011.
- Polchinski in 1987 undertook a detailed review of the pre-existing literature and found no counterexamples.
- No candidate for Virial current, and thus are automatically conformally invariant
  - the Belavin-Migdal-Banks-Zaks fixed points for non-abelian gauge theories coupled to fermions in $D = 4$ [Belavin and Migdal (1974),Banks and Zaks(1982)].
  - The Wilson-Fisher $\lambda\phi^4$ fixed point in $D = 4 - \epsilon$ [Wilson,1973].

- Systematic searches among theories having candidates for a nonconserved virial current have not turned up any counterexamples either.
- Nontrivial candidates for a non-conserved V_μ. However, this never happens for the one-loop fixed points in $4 - \epsilon$ dimensions
  - Multi-field generalizations of $\lambda\phi^4$

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{4!}\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l, \qquad V_\mu = M^{[ij]}\phi_i\partial_\mu\phi_j.$$

setting the one-loop β-functions to zero, have $T_\mu^\mu = 0$ and are conformally invariant [Polchinski(1988)].

  - Weyl fermions with arbitrary quartic and Yukawa interactions [Dorigoni,Rychkov(2009)].

$$\tilde{\mathcal{L}} = \bar{\psi}_a\bar{\sigma}^\mu\partial_\mu\psi_a + \frac{1}{2}(y^{iab}s_i\psi_a\psi_b + y^{*iab}s_i\bar{\psi}_a\bar{\psi}_b),$$

$$V_\mu^{\mathcal{L}+\tilde{\mathcal{L}}} = M^{[ij]}\phi_i\partial_\mu\phi_j + N^{ab}\bar{\psi}_a\bar{\sigma}_\mu\psi_b.$$

Consequently, a general conjecture was that the Zamolodchikov-Polchinski theorem is true also in $D \geq 3$.

- This conjecture is false, at least in $D = 3$ and in $D \geq 5$.
- The counterexample is astonishingly simple: It is the free Maxwell theory [El-Showk,Nakayama,Rychkov (2011)].
- The main idea is that to show the impossibility of improving the stress-energy tensor.
- Another way to see this is that the field strength operator $F_{\mu\nu}$ is neither a primary nor a descendant.

But what happen in $D = 4$?

No non-trivial counterexample up to yet!

- It is shown that in perturbation theory about a conformal fixed point the only possible asymptotics is conformally invariant [Luty,Polchinski,Rattazi(2012),Baume,Keren-Zur,Rattazzi,Vitale (2014)].
  - The method in the first one is based on approach of Komargodski and Schwimmer to proof of a-theorem and the second one is based on using the local Callan-Symanzik equation idea.
- It is shown that limit cycles associated with scale but not conformally invariant unitary theories do not exist in perturbation theory [Fortin,Grinstein,Stergiou(2015)].

- At non-perturbative level, this problem can be studied by using the non-perturbative effects in field theory such as scaling anomalies.
- Under the global scale transformations we have [Farnsworth,Luty,Prelipina(2013)]

$$\int d^4x\sqrt{-g}\langle T^\mu_\mu + \nabla_\mu V^\mu\rangle = \int d^4x\sqrt{-g}\left(-aE_4 + cW^2 - eR^2\right).$$

- The coefficients a and c are the standard conformal anomaly coefficients of a CFT while the e term appears only in a SFT.
- It is shown that unitarity imposes that $e \geq 0$ [Farnsworth,Luty,Prelipina(2013),Bzowski and Skenderis(2014)].
- In the presence of a dimension two scalar operator $\mathcal{O}_2$, the term $\xi \int d^4xR\mathcal{O}_2$ can be added to the action, which only shifts the anomaly coefficient e, [Farnsworth,Luty,Prelipina(2013)].

- The two-point function of the trace of stress-energy tensor in a 4D anomalous scale invariant theory is given by

$$\langle T(q)T(-q)\rangle = -e\, q^4 \log\frac{q^2}{\mu^2} + C(\mu)q^4,$$

where μ is an arbitrary renormalization scale and $C(\mu)$ is a scheme dependent constant.

- Note that the Fourier transformation of just the q^4 term is a derivative of delta function, so if $e = 0$ we have

$$\langle T(x)T(0)\rangle = 0, \qquad x \neq 0$$

which means that in a unitary theory, T must be equal to zero as an operator identity and the scale invariant theory becomes fully conformal.

- It is shown that unitary scale-invariant field theories must be either conformal field theories, or the trace of the stress-energy tensor behaves like a generalized free field.
[Dymarsky,Komargodski,Schwimmer,Theisen(2013)].

- Moreover it is shown that if no scalar operator of dimension precisely 2 appears in the spectrum of a theory which it's stress-energy tensor is generalized free field, that theory would be conformal.
[Dymarsky,Farnsworth,Komargodski,Luty,Prilepina(2014)].

- In the presence of an operator of dimension 2 that can mix with T, one can show that there is at least one improvement such that T is not a generalized free field.
[Dymarsky,Farnsworth,Komargodski,Luty,Prilepina(2014)].

- Thus the only loophole which is remained in that proof is the case where the stress-energy tensor is generalized field and the scalar operator with dimension precisely 2 exists in the spectrum[1].
- It would be nice to fill this gap and prove that none of the possible improved stress-energy tensors in this situation is a generalized free field.
- In this talk we discuss how the EE can inform us about this subject.

[1] We thank Z. Komargodski for many helpful clarifications.

- Consider a relativistic SFT on the 4 dimensional manifold $\mathcal{M}$.

- The entanglement entropy is defined by first taking a pure state and then tracing the modes which reside inside an entangling surface Υ. The result of this tracing, is a mixed state with a certain density matrix ρ_Υ. This entangling surface,Υ, is a submanifold of $\mathcal{M}$ at a fixed time.

- The strategy to calculate the entanglement entropy is that to first obtain the reduced trace $Tr_\Upsilon({\rho_\Upsilon}^n)$ to find the R´enyi entropy

$$S_n(\rho_\Upsilon) = \frac{1}{1-n} \log Tr_\Upsilon({\rho_\Upsilon}^n).$$

- After that the entanglement entropy is given by

$$S_{EE} = \lim_{n \to 1} S_n = -\partial_n \log Tr_\Upsilon({\rho_\Upsilon}^n)\Big|_{n=1}.$$

- For a closed connected surface Υ, we can define a length scale s. Therefore

$$s\frac{d}{ds}S_{EE} = -\partial_n \int_{\mathcal{M}_n} d^4x\sqrt{-g}\left(-aE_4 + cW^2 - eR^2\right)\Big|_{n=1} + \int_{\mathcal{M}_1} d^4x\sqrt{-g}\left(-aE_4 + cW^2 - eR^2\right).$$

- A n-sheeted 4 dimensional manifold $\mathcal{M}_n$, in general contains conical singularities. Calculating the integral of metric curvatures on manifolds with conical singularities [Fursaev,Solodukhin(2013)]

$$\int_{\mathcal{M}_n} d^4x\sqrt{-g}E_4 = n\int_{\mathcal{M}_1} d^4x\sqrt{-g}E_4 + 8\pi(1-n)\int_{\partial\Upsilon} d^2\chi\sqrt{-\gamma}R[\gamma],$$

$$\int_{\mathcal{M}_n} d^4x\sqrt{-g}W^2 = n\int_{\mathcal{M}_1} d^4x\sqrt{-g}W^2 + 8\pi(1-n)\int_{\partial\Upsilon} d^2\chi\sqrt{-\gamma}K[g;t,s;\mathcal{K}^\alpha_{ij}],$$

$$\int_{\mathcal{M}_n} d^4x\sqrt{-g}R^2 = n\int_{\mathcal{M}_1} d^4x\sqrt{-g}R^2 + 8\pi(1-n)\int_{\partial\Upsilon} d^2\chi\sqrt{-\gamma}R[g],$$

where

$$K[g;t,s;\mathcal{K}^\alpha_{ij}] = 2W_{\mu\nu\alpha\beta}t^\mu s^\nu t^\alpha s^\beta - [\mathcal{K}^\alpha_{ij}\mathcal{K}^{\alpha ij} - \frac{1}{2}(\mathcal{K}^{\alpha i}_i)^2],$$

γ_{ij} and $\mathcal{K}^\alpha_{ij}$ are the intrinsic metric and the extrinsic curvature of $\partial\Upsilon$, $\alpha = \{t,s\}$ indexing the two normal directions (one timelike t^μ and one spacelike s^μ) and the first term on the right hand side of K is nothing but the pullback of the Weyl tensor onto $\partial\Upsilon$.

- At the end, one arrives at

$$s\frac{d}{ds}S_{EE} = -8\pi\int_{\partial\Upsilon} d^2\chi\sqrt{-\gamma}\Big(aR[\gamma] - cK[g;t,s;\mathcal{K}^{\alpha}_{ij}] + eR[g]\Big).$$

- The point which should be stressed is that $s\frac{d}{ds}S_{EE}$ is equal to the minus of C_{univ} which captures the important physical information [Ryu, Takayanagi(2006)] .
- For $\partial\Upsilon = S^2$, the $C_{univ}\big|_{S^2}$ is a measure of degrees of freedom [Ryu, Takayanagi(2006)].
- We like specially to know the effect of e-anomaly in the universal quantity $C_{univ}\big|_{S^2}$.

- For simplicity, we take a conformally flat metric, $g_{\mu\nu} = e^{-2\tau}\eta_{\mu\nu}$ as a background metric.

- Therefore

$$C_{univ}\Big|_{S^2} = 16\pi\left(a + 3\;e\int_{S^2} d^2\chi\left[\Box\tau - (\partial\tau)^2\right]\right).$$

- For a generic 4D CFT, the scale anomaly dictates that on conformally flat backgrounds, the universal part of entanglement entropy across a sphere is positive.

- Based on this fact, let explore the consequences of assuming a positive sign for $C_{\text{univ}}\Big|_{S^2}$ on such backgrounds in a 4D SFT.

- In a unitary SFT, $e \geq 0$.
- If we assume that $e > 0$, one can check that for any positive value of a, there exists a τ for which the $C_{univ}\big|_{S^2}$ becomes negative.
- Thus, in the absence of a dimension two scalar operator $\mathcal{O}_2$ in the spectrum of a SFT, we have shown that the positivity of $C_{univ}\big|_{S^2}$ suggests that a SFT is a CFT.

- In the presence of $\mathcal{O}_2$, one can add the term $\xi \int d^4 x R \mathcal{O}_2$ to the action in order to change the trace of an energy-momentum tensor. Therefore,

$$C_{univ}\Big|_{S^2} = 16\pi \left(a + 3\left(e - \alpha\xi\right) \int_{S^2} d^2\chi \left[\Box\tau - (\partial\tau)^2\right] \right).$$

where α is a positive number.

- For example, for a free scalar theory, the universal part of EE is calculated by using a heat Kernel method, [Fursaev, Patrushev, and Solodukhin (2013)], which leads to $e = 1/72$ and $\alpha = 1/12$.
- Interestingly, the positivity of $C_{univ}\big|_{S^2}$ fixes the coefficient of the nonlinear coupling term to $\xi = e/\alpha$, where for the free scalar theory it becomes $\xi = 1/6$. This value for ξ is exactly the one to have a conformal scalar theory.

- The only loophole in the previous proofs was that the trace of stress-energy tensor could be generalized free field and a scalar operator with dimension precisely 2 exists in the spectrum.
- Interestingly, positivity of $C_{univ}(S^2)$ in the absence of a dimension two scalar operator $\mathcal{O}_2$ in the spectrum of a SFT, suggests that SFT is a CFT.
- In the presence of $\mathcal{O}_2$, we show that this assumption can fix the coefficient of the nonlinear coupling term $\int d^4xR\mathcal{O}_2$ to a conformal value.

Thank YOU