Holographic boundary conformal field theory

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9'th Crete regional meeting in String theory

Main motivation

We usually send the boundaries to infinity to get rid of the boundary terms, but ...

The real world has boundaries!

Content

- “Holographic calculation of boundary terms in conformal anomaly,”
  A.F.A and S. N. Solodukhin,
  Phys. Lett. B **769**, 25 (2017) [arXiv:1702.00566 [hep-th]].
- “Holographic calculation of entanglement entropy in the presence of boundaries,”
  A.F.A, C. Berthiere, D. Fursaev and S. N. Solodukhin,
  Phys. Rev. D **95**, no. 10, 106013 (2017) [arXiv:1703.04186 [hep-th]].

Conformal invariant boundary conditions

- conformal scalar field in d dimensions:

$$\text{Dirichlet b. c. :} \quad \phi|_{\partial\mathcal{M}_d} = 0\,,$$
$$\text{Robin b. c. :} \quad (\partial_N + \frac{(d-2)}{2(d-1)}k)\phi|_{\partial\mathcal{M}_d} = 0\,,$$

- massless Dirac fermion in dimension $d = 4$:

$$\Pi_-\psi|_{\partial\mathcal{M}} = 0\,, \quad (\nabla_N + K/2)\Pi_+\psi|_{\partial\mathcal{M}} = 0\,,$$

where $\Pi_\pm = \frac{1}{2}(1 \pm +i\gamma_* N^\mu \gamma_\mu)$

- gauge field A_μ:

$$\text{absolute b. c. :} \quad N^\mu F_{\mu\nu} = 0\,,$$
$$\text{relative b. c. :} \quad N^\mu F^*_{\mu\nu} = 0\,,$$

Conformal anomaly in dimension $d = 3$ [Fursaev, Solodukhin]

In this case there are no bulk terms in the anomaly. The whole contribution comes from the boundary.

$$\int_{\mathcal{M}_3} \langle T \rangle = \frac{c_1}{96} \chi[\partial \mathcal{M}_3] + \frac{c_2}{256\pi} \int_{\partial \mathcal{M}_3} \mathrm{Tr}\, \hat{k}^2 \,,$$

where $\chi[\partial \mathcal{M}_3] = \frac{1}{4\pi} \int_{\partial \mathcal{M}_3} \hat{R}$.
If manifold $\mathcal{M}_3$ is flat then using the Gauss-Codazzi equation we arrive at

$$\int_{\mathcal{M}_3} \langle T \rangle = \frac{1}{256\pi} \int_{\partial \mathcal{M}_3} \left((c_2 - \frac{2}{3} c_1) \mathrm{Tr} k^2 + (\frac{2}{3} c_1 + \frac{c_2}{2}) k^2 \right) .$$

If $c_2 = \frac{2}{3} c_1$ then $\mathrm{Tr} k^2$ drops out above. As we will see this is exactly what happens in the holographic calculation.

Conformal anomaly in $d = 4$ [Fursaev, Solodukhin]

In 4 dimensions

$$\int_{\mathcal{M}_4} \langle T \rangle = -\frac{a}{180} \chi[\mathcal{M}_4] + \frac{1}{1920\pi^2} \Big(\int_{\mathcal{M}_4} b \,\mathrm{Tr}\, W^2 \\ -8b_1 \int_{\partial \mathcal{M}_4} W^{\mu\nu\alpha\beta} n_\mu n_\beta \hat{k}_{\nu\alpha} \Big) + \frac{c}{280\pi^2} \int_{\partial \mathcal{M}_4} \mathrm{Tr}\, \hat{k}^3 \,,$$

where $\hat{k}_{ij} = k_{ij} - \frac{1}{3}\gamma_{ij} k$ and

$$\chi[\mathcal{M}_4] = \frac{1}{32\pi^2} \int_{\mathcal{M}_4} (R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) \\ -\frac{1}{4\pi^2} \int_{\partial \mathcal{M}_4} (-k^{\mu\nu} R_{n\mu n\nu} + k^{\mu\nu} R_{\mu\nu} + k R_{nn} - \frac{1}{2} k R - \frac{1}{3} k^3 \\ + k \,\mathrm{Tr}\, k^2 - \frac{2}{3} \mathrm{Tr}\, k^3) \,, .$$

Conformal anomaly in $d = 4$

If $\mathcal{M}_4$ is flat then the

$$\int_{\mathcal{M}_4} \langle T \rangle = \frac{1}{\pi^2} \int_{\partial \mathcal{M}_4} \Big(\frac{a}{720} (-\frac{1}{3} k^3 + k \mathrm{Tr} k^3 - \frac{2}{3} \mathrm{Tr} k^3) + \frac{c}{280} (\mathrm{Tr} k^3 - k \mathrm{Tr} k^2 + \frac{2}{9} k^3) \Big) \, .$$

Charges [Fursaev, Solodukhin]

$$\text{real}\quad \text{scalar}: a = 1\,, \quad b_1 = b = 1\,, \quad c = 1 \;(\text{Dirichlet b. c.})\,,$$
$$\text{real}\quad \text{scalar}: a = 1\,, \; b_1 = b = 1\,, \; c = \frac{7}{9} \;(\text{Robin b. c.})\,,$$
$$\text{Dirac}\quad \text{fermion}: a = 11\,, \; b_1 = b = 6\,, \; c = 5, \;(\text{mixed b. c.})\,,$$
$$\text{gauge boson}: a = 62, b_1 = b = 12, c = 8 \;(\text{absolute or relative b. c.}).$$

Conformal anomaly in $d = 4$: $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills multiplet [A.F.A, Solodukhin]

The free field multiplet consists of: $n_s = 6$ scalars, $n_f = 2$ Dirac fermions and $n_v = 1$ gauge bosons.
Introducing $\Delta n = n_s^D - n_s^R$ we find

$$a = 90(N^2 - 1)\,, \quad b = b_1 = 30(N^2 - 1)\,, \quad c = (\frac{70}{3} + \frac{1}{2}\Delta n)(N^2 - 1)\,.$$

and hence the integral anomaly is (we focus only on the boundary terms)

$$\int_{\mathcal{M}_4} \langle T \rangle_{SYM} = \frac{(N^2-1)}{24\pi^2} \int_{\partial\mathcal{M}_4} [\frac{3}{2}(k^{\mu\nu} + k n^\mu n^\nu - \frac{2}{3} k g^{\mu\nu}) R_{\mu\nu}$$
$$+(k \mathrm{Tr} k^2 - \frac{5}{9} k^3) + \frac{3\Delta n}{70} \mathrm{Tr} k^3]\,.$$

Conformal anomaly in $d = 4$: $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills multiplet [A.F.A, Solodukhin]

if $n_s^D = n_s^R = 3$, the term $\mathrm{Tr}k^3$ drops out from the anomaly. In Ricci flat spacetime we have then

$$\int_{\mathcal{M}_4} \langle T \rangle_{SYM} = \frac{(N^2-1)}{24\pi^2} \int_{\partial\mathcal{M}_4} (k\mathrm{Tr}k^2 - \frac{5}{9}k^3)\,.$$

This is exactly the condition that the preserved supersymmetry in $\mathcal{N} = 4$ superconformal theory is maximal. In this case the boundaries preserve 1/2 of supersymmetry. [Gaiotto and Witten]

Holographic boundary conformal field theory

$$AdS_{d+1} \xrightarrow{\text{boundary}} \mathcal{M}_d \xrightarrow{\text{boundary}} \partial\mathcal{M}_{d-1} \xrightarrow{\text{extension to the bulk}} \mathcal{S}_d.$$

such that $\partial\mathcal{S}_d = \partial\mathcal{M}_d$.

- *Takayanagi's prescription:*[1]

$$W^T_{grav} = -\frac{1}{16\pi G}\int_{AdS_{d+1}}(R+2\Lambda) - \frac{1}{8\pi G}\left[\int_{\mathcal{M}_d} K + \int_{\mathcal{S}_d}(K+T)\right],$$

then

$$W_{gr} = -\int_{\mathcal{M}_d} \langle T\rangle \ln\epsilon\,.$$

Varying the action w.r.t the boundary metric, γ_{ij} one gets

$$K_{ij} - \gamma_{ij}(K+T) = 0\,,$$

<u>Too restrictive!</u>

[1] Series of papers by Takayanagi, Fujita, Tonni, Nozaki, Ugajin

Holographic boundary conformal field theory

- *Minimal surface prescription: [A.F.A, Solodukhin]*

In this proposal we extend the boundary minimally into the bulk.
On has to modify the gravitational action by adding a boundary volume term

$$W_{gr}^{\min} = -\frac{1}{16\pi G}\int_{AdS_{d+1}}(R-2\Lambda) - \frac{1}{8\pi G}[\int_{\mathcal{M}_d} K + \int_{\mathcal{S}_d}\lambda]\,.$$

Profile of $\mathcal{S}_d$ is specified solving

$$K_{\mathcal{S}} = 0\,.$$

Holographic setup

$$ds^2 = \frac{1}{4\rho^2}d\rho^2 + \frac{1}{\rho}g_{AB}(\rho, X)dX^A dX^B \;, \;\; X^A = \{r, x^i\}, i = 1, 2, 3\,.$$

the boundary is located at $r = 0$.

$$g_{AB} = (1 + \rho g_{rr}^{(1,0)} + r\rho g_{rr}^{(1,1)} + \cdots)dr^2$$
$$+(g_{ij}^{(0,0)} + r g_{ij}^{(0,1)} + r^2 g_{ij}^{(0,2)} + \cdots + \rho g_{ij}^{(1,0)} + r\rho g^{(1,1)} + \cdots_{ij})dx^i dx^j\,.$$

where

$$\begin{aligned}
g_{ij}^{(0,0)} &= \gamma_{ij}^{(0)}\,, \\
g_{ij}^{(0,1)} &= -2k_{ij}\,, \\
g_{ij}^{(0,2)} &= k_{ij}^2 - R_{rirj}\,, \\
g_{AB}^{(1,0)} &= -\frac{1}{2}(R_{AB}^{(0)} - \frac{1}{6}R^{(0)}g_{AB}^{(0)})\,, \\
&\vdots
\end{aligned}$$

Holographic boundary terms of conformal anomaly

[A.F.A, Solodukhin]

- $d = 3$

$$\int_{\mathcal{M}_3} \langle T \rangle_{\rm hol} = \frac{\lambda}{64\pi G_N} \int_{\partial\mathcal{M}_3} k^2 \, .$$

- $d = 4$

$$\int_{\mathcal{M}_4} \langle T \rangle_{\rm hol,ms} = \frac{N^2}{24\pi^2} \int_{\partial\mathcal{M}_4} [\frac{3}{2}(k^{\mu\nu} + kn^{\mu}n^{\nu} - \frac{2}{3}kg^{\mu\nu})R_{\mu\nu} \\ +(k{\rm Tr}k^2 - \frac{5}{9}k^3)] \, .$$

This precisely matches (for $N \gg 1$) the anomaly computed for the free super-multiplet, for 4-dim $\mathcal{N} = 4$ SYM with maximal SUSY preservation.

Entanglement Entropy

- Consider a quantum mechanical system in a pure ground state which is described by $|\psi\rangle$ ($\rho = |\psi\rangle\langle\psi|$).

- *Reduced* density operator:

$$\rho_A = \text{Tr}_B\, \rho = \text{Tr}_B |\psi\rangle\langle\psi|\,.$$

Then the EE is

$$S_{EE}(A) = -\,\text{Tr}\, \rho_A \log \rho_A\,.$$

Rényi entropy

In a QFT, we firstly construct the Rényi entropy as

$$S_n(A) = \frac{1}{1-n} \log \operatorname{Tr} \rho_A{}^{n} \,,$$

The EE reads then

$$S_{EE}(A) = \lim_{n \to 1} S_{RE}(A).$$

The main challenge is the computation on a manifold with conical singularity

$$S_n = -\partial_n \log \operatorname{Tr} \rho_A^n = (n\partial_n - 1) W_n \,.$$

but

$$W_n = -\frac{(-1)^{2s}}{2} \int_{\epsilon^2} \frac{d\tau}{\tau} \operatorname{Tr} K_n(\triangle^{(s)}, \tau) \,.$$

$$\operatorname{Tr} K_n(\triangle^{(s)}, \tau) \simeq \sum_{p=0} a_p(\triangle^{(s)}, n)\, \tau^{(p-d)/2} \,, \qquad \tau \to 0 \,.$$

and we know many thing about the heat kernels on the cones!

EE in general d

$$S_{EE}(\Sigma) = \frac{s_{d-2}}{\epsilon^{d-2}} + \frac{s_{d-4}}{\epsilon^{d-4}} + \cdots + s_{log} \log \epsilon + f \,,$$

where

$$s_{d-2} \propto A(\Sigma).$$

$$s_{log} \propto \mathcal{A} \text{ (conformal anomaly)} \,.$$

EE for BCFTs in $d = 4$ [A.F.A, Berthiere, Fursaev, Solodukhin]

In four dimensions, the entanglement entropy has the following asymptotic dependence on the UV cut-off ϵ,

$$S(\Sigma) = \frac{s_2}{\epsilon^2} + \frac{s_1}{\epsilon} + s_{log} \log \epsilon + \cdots \, ,$$

The logarithmic term s_{log} is a combination of conformal invariants constructed on Σ and its boundary $\mathcal{P} = \partial\Sigma$

$$s_{log} \;=\; \frac{a}{720\pi}\left[\int_\Sigma R_\Sigma + 2\int_{\mathcal{P}} k_p\right] - \frac{b}{240\pi}\int_\Sigma [W_{ijij} - \operatorname{Tr} \hat{k}_i^2] \;+\; d\, F_d \;+\; e\, F_e \, ,$$

$$F_d = -\frac{1}{40\pi}\int_{\mathcal{P}} \hat{k}_{\mu\nu} v^\mu v^\nu \, , \qquad F_e = -\frac{1}{\pi}\int_{\mathcal{P}} (N \cdot p_i)(\hat{k}_i)_{\mu\nu} v^\mu v^\nu \, .$$

F_e reflects properties of extrinsic geometry of Σ at the boundary and F_d keeps the information about the extrinsic geometry of the boundary itself.

Charges

Theory	a	b	c	d	boundary condition
real scalar	1	1	1	1	Dirichlet
real scalar	1	1	$\frac{7}{9}$	$-\frac{2}{3}$	Robin
Dirac spinor	11	6	5	1	mixed
gauge boson	62	12	8	7	absolute/relative

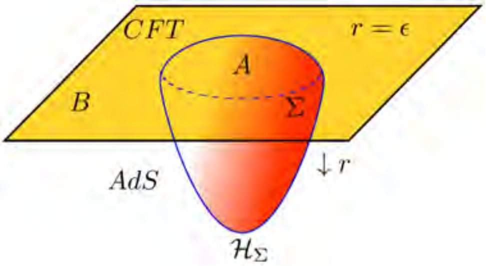
EE for $\mathcal{N} = 4$ supermultiplet

$$
\begin{aligned}
a &= (N^2-1)\Big(3\cdot 1 + 3\cdot 1 + 2\cdot 11 + 1\cdot 62\Big) = 90(N^2-1)\,,\\
b &= (N^2-1)\Big(3\cdot 1 + 3\cdot 1 + 2\cdot 6 + 1\cdot 12\Big) = 30(N^2-1)\,,\\
c &= (N^2-1)\Big(3\cdot 1 + 3\cdot \frac{7}{9} + 2\cdot 5 + 1\cdot 8\Big) = \frac{70}{3}(N^2-1)\,,\\
d &= (N^2-1)\Big(3\cdot 1 - 3\cdot \frac{2}{3} + 2\cdot 1 + 1\cdot 7\Big) = 10(N^2-1)\,.
\end{aligned}
$$

$$
s_{log}^{(SYM)} = \frac{N^2-1}{8\pi}\left(\Big[\int_\Sigma R_\Sigma + 2\int_{\mathcal{P}=\partial\Sigma} k_p\Big] + \int_\Sigma \mathrm{Tr}\,\hat{k}_i^2 - 2\int_{\mathcal{P}} \hat{k}_{\mu\nu} v^\mu v^\nu\right)\,.
$$

Holographic EE

Ryu-Takayanagi's proposal (06)



$$S_{hol}[\Sigma] = \frac{A[\mathcal{H}]}{4G} \, ,$$

Holographic EE for BCFT in $d = 4$ [A.F.A, Berthiere, Fursaev, Solodukhin]

We cast the AdS_5 bulk metric in the form

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(-dt^2 + dr^2 + (\gamma_{ij} - k_{ij} r)^2 dx^i dx^j\right)\ ,\ i, j = 1, 2\,.$$

In four dimensions a suitable profile $x = x(r, \rho)$, subject to boundary condition $x(r, \rho = 0) = 0$, which minimizes the area functional above is

$$x(r, \rho) = \rho^2 \left(c_1 + c_2 r + c_2 (k_{11} + \frac{k_{22}}{2}) r^2 + \mathcal{O}(r^3)\right) + \cdots\ .$$

The RT surface terminates at the hypersurface $\mathcal{S}$ which was the extension of the boundary into the bulk

$$r(\rho) = r_1 \rho + r_2 \rho^2 + \cdots\ ,$$

Holographic EE for BCFT in $d = 4$

Putting things together one finds

$$S_{log}^{(hol)}[\Sigma, \mathcal{P}] = \frac{N^2}{8\pi} \left(\left[\int_\Sigma R_\Sigma + 2 \int_{\mathcal{P}=\partial\Sigma} k_p \right] + \int_\Sigma \mathrm{Tr}\, \hat{k}_i^2 - 2 \int_{\mathcal{P}} \hat{k}_{ij} v^i v^j \right) \log \epsilon \,,$$

A perfect match again!

Summary

- We compute the boundary terms of the conformal anomaly in $d = 3$ and $d = 4$ by proposing the minimal surface prescription in the context of Holography. We observe a perfect agreement with the results in field theory side for the free super-multiplet, for 4-dim $\mathcal{N} = 4$ SYM when SUSY is maximally preserved.
- Following our proposal we suggest how to perform the RT calculation to obtain the EE in presence of the boundaries. We confirm again a perfect match with the results in field theory side.