Turbulence and Random Geometry

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Outline

Turbulence

Turba is a Latin word for crowd. Turbulence originally refers to the disorderly motion of a crowd. Scientifically it refers to a complex and unpredictable motions of a fluid.

Turbulence

- Fluid turbulence is a major unsolved problem of physics.

Turbulence

- Most fluid motions in nature at all scales are turbulent. Aircraft motions, river flows, atmospheric phenomena, astrophysical flows and even blood flows are some examples of set-ups where turbulent flows occur.
- Despite centuries of research, we still lack an analytical description and understanding of fluid flows in the non-linear regime.
- Insights to turbulence hold a key to understanding the principles and dynamics of non-linear systems with a large number of strongly interacting degrees of freedom far from equilibrium.

Anomalous Scaling

- We propose an analytical formula for the anomalous scaling exponents of inertial range structure functions in incompressible fluid turbulence.
- The formula is a Knizhnik-Polyakov-Zamolodchikov (KPZ)-type relation and is valid in any number of space dimensions:

$$\xi_n - \frac{n}{3} = \gamma^2(d)\xi_n(1 - \xi_n) \ , \qquad (1)$$

where $\gamma(d)$ is a numerical real parameter that depends on the number of space dimensions d.

Navier-Stokes Equations

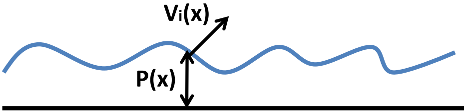
- Consider incompressible fluid flows in $d \geq 2$ space dimensions. They are the relevant flows when the velocities are much smaller than the speed of sound.
- The incompressible Navier-Stokes (NS) equations provide a mathematical formulation of the fluid flow evolution. They read

$$\partial_t v^i + v^j \partial_j v^i = -\partial_i p + \nu \partial_{jj} v^i + f^i, \qquad \partial_i v^i = 0, \quad i = 1, ..., d \, , \tag{2}$$

- v^i is the fluid velocity and p is the fluid pressure. f^i an external random force. ν is the kinematic viscosity.

Geometrization of the Fluid Variables

- The fluid pressure and velocity in the geometrical picture



Fluid pressure and velocity in the geometrical picture. The pressure P(x) measures the deviation of the perturbed event horizon from the equilibrium solution. The velocity vector field Vi(x) is the normal vector.

Reynolds Number

- An important dimensionless parameter in the study of fluid flows is the Reynolds number $\mathcal{R}_e = \frac{lv}{\nu}$, where l is a characteristic length scale, v is the velocity difference at that scale, and ν is the kinematic viscosity.
- The Reynolds number quantifies the relative strength of the non-linear interaction compared to the viscous term.
- When the Reynolds number is of order a thousand or more, one observes numerically and experimentally a turbulent structure of the flow.
- This phenomenological observation is general, and fluid details are of no importance.

Turbulence in Nature

- Most flows in nature are turbulent. This is simple to see by noting that the viscosity of water is $\nu \simeq 10^{-6} \frac{m^2}{sec}$, while that of air is $\nu \simeq 1.5 \times 10^{-5} \frac{m^2}{sec}$. Thus, a medium size river has a Reynolds number $\mathcal{R}_e \sim 10^7$.

Turbulent Flows

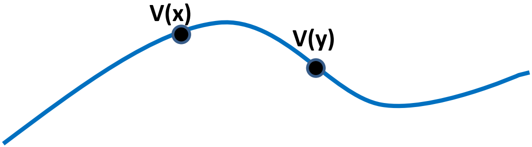
- The turbulent velocity field exhibits highly complex spatial and temporal structures and appears to be a random process. Thus, even though the NS equations are deterministic (in the absence of a random force), a single realization of a solution to the NS equations is unpredictable.
- Instead of studying individual solutions to the NS equations, one is led to consider the statistics of the solutions.
- The statistics can be defined in various ways. One can use an ensemble average by averaging over initial conditions. Turbulence that is reached in this way is a decaying one. Alternatively, one can introduce a random force. This allows reaching a sustained steady state turbulence with an energy source and a viscous sink.

Statistical Properties

- Numerical and experimental data show that the statistical average properties exhibit a universal structure shared by all turbulent flows, independently of the details of the flow excitations.
- One defines the inertial range to be the range of distance scales $L_V \ll r \ll L_F$, where the scales L_V and L_F are determined by the viscosity and forcing, respectively.
- Turbulence at the inertial range of scales reaches a steady state that exhibits statistical homogeneity and isotropy.

The Statistical Approach

- Consider the statistics of velocity difference between points separated by a fixed distance.



Structure Functions

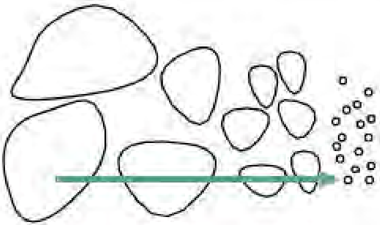
- Define the longitudinal velocity difference between points separated by a fixed distance $r = |\vec{r}|$

$$\delta v(r) = \big(\vec{v}(\vec{r}, t) - \vec{v}(0, t)\big) \cdot \frac{\vec{r}}{r} \,. \tag{3}$$

- The structure functions $S_n(r) = \langle(\delta v(r))^n\rangle$ exhibit in the inertial range a scaling $S_n(r) \sim r^{\xi_n}$.
- The exponents ξ_n are universal, and depend only on the number of space dimensions.

K41 Theory

- In 1941 Kolmogorov argued that in three space dimensions the incompressible non-relativistic fluid dynamics in the inertial range follows a cascade breaking of large eddies to smaller eddies, called a direct cascade, where energy flux is being transferred from large eddies to small eddies without dissipation.



Scale Invariance

- He further assumed scale invariant statistics, that is

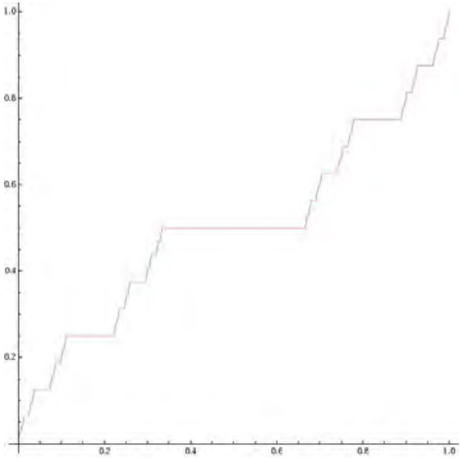
$$P(\delta v(r))\delta v(r) = F\left(\frac{\delta v(r)}{r^h}\right), \tag{4}$$

  where $P(\delta v(r))$ is the probability density function, and h is a real parameter.
- Treating the mean viscous energy dissipation rate ϵ as a constant in the limit of infinite Reynolds number, he deduced a linear scaling of the exponents $\xi_n = n/3$.

Intermittency

- All direct cascades are known numerically and experimentally to break scale invariance and do not simply follow Kolomogorov scaling.
- Note, that in two space dimensions the energy cascade is inverse, that is the energy flux is instead transferred to large scales. Kolmogorov's assumption that the random velocity field is self-similar is incorrect in direct cascades, but it seems to hold in the inverse cascade.
- The self-similarity assumption misses the intermittency of the turbulent flows. Thus, in order to calculate the scaling exponents one has quantify the inertial range intermittency effects.

Intermittency



Anomalous Scaling

- The calculation of the anomalous exponents and their deviation from the Kolmogorov scaling is a major open problem.

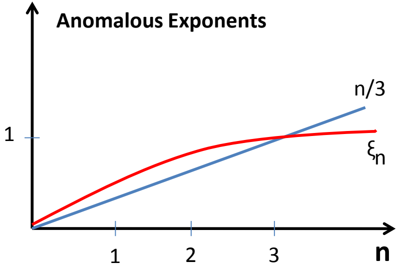


Figure 1: Anomalous exponents versus the linear Kolomogorov scaling in three space dimensions. The plot is a schematic draw of numerical simulations and experimental data.

Multifractal Structure

- Since 1941 many multifractal models of turbulence have been proposed. These express multifractality directly in terms of fluctuations of the velocity increments or of the energy dissipation.
- For example, Kolmogorov and Obukhov proposed to replace the constant global average ϵ with local averages ϵ_r over a volume of dimension r. One then considers ϵ_r as a lognormally distributed random variable with variance $\sigma^2 \sim -ln(r)$.

Gaussian Multiplicative Cascade

- Mandelbrot argued that one should think about the energy cascade as a random multiplicative process. In this case a random measure can be formalised mathematically as a limit of a “Gaussian multiplicative cascade".The lognormal model assumes “refined self-similarity"

$$P(\delta v(r))\delta v(r) = F\left(\frac{\delta v(r)}{(\langle \epsilon_r \rangle r)^{1/3}}\right), \tag{5}$$

which leads to a formula for ξ_n producing physically inconsistent supersonic velocities at large n and a violation of the convexity inequality.

Gravitational Dressing

- We propose that in intermittent turbulence, Kolmogorov linear scaling itself is evaluated with respect to a lognormal random measure.
- This is a “gravitational" dressing of Kolmogorov scaling which is inspired by the relation of fluid dynamics and black hole horizon dynamics in one higher space dimension.
- We propose the dressing of Kolmogorov scaling is via a KPZ (Knizhnik-Polyakov-Zamolodchikov)-type relation. This gives an analytical formula for the scaling exponents of incompressible fluid turbulence in any number of space dimensions $d \geq 2$.
- It reads

$$\xi_n - \frac{n}{3} = \gamma^2(d)\xi_n(1 - \xi_n) \ , \tag{6}$$

where $\gamma(d)$ is a numerical real parameter that depends on the number of space dimensions d.

Coupling to a Random Geometry

- This means changing the Euclidean measure dx on a R^d to $d\mu_\gamma(x) = e^{\gamma\phi(x)-\frac{\gamma^2}{2}}dx$, where the Gaussian random field $\phi(x)$ has covariance $\phi(x)\phi(y) \sim -\log|x-y|$ when $|x-y|$ is small (but still in the inertial range).
- Physically, the notion of distance r is modified in the new measure. Consider a set of scaling exponents ξ_0 with respect to the Euclidean measure. Denote the same set of exponents, but now with respect to the random measure, by ξ.
- Then ξ and ξ_0 are related by the KPZ relation

$$\xi - \xi_0 = \gamma^2(d)\xi(1-\xi). \quad (7)$$

Coupling to a Random Geometry

- Mathematically, this is a known method to obtain a multifractal structure from a fractal one.
- Our proposal is that one can incorporate the effect of intermittency at the inertial of range of scales by coupling to a random geometry in this way and evaluating the Kolmogorov linear scaling exponents $\xi_0 = \frac{n}{3}$ with respect to the random measure.
- Physically, it is highly nontrivial that the steady state statistics of turbulence can be viewed as such a combination of the scale invariant statistics and intermittency.
- Note, that intermittent features appear at short length scales, and this is when the effects of the random field ϕ are prominent.

Coupling to a Random Geometry

- We conjecture that the $e^{\gamma\phi(x)-\frac{\gamma^2}{2}}$ is proportional to local energy flux field $\epsilon(x)$ in the direct cascade of the turbulent fluid.
- This has some similarities to the Kolmogorov-Obukhov lognormal model. In that case, refined self-similarity implies the following simple dressing of Kolmogorov scaling

$$\langle(\delta v(r))^n\rangle \sim r^{\xi_n} \sim \langle(\epsilon_r)^{n/3}\rangle r^{n/3}. \tag{8}$$

- Evaluating the expectation of the lognormal energy dissipation, one finds

$$\xi_n - \frac{n}{3} = \gamma^2 \frac{n}{3}\left(1 - \frac{n}{3}\right). \tag{9}$$

- This formula fails as it implies ξ_n is a decreasing function for large enough n, which violates basic physical inequalities. Here we assume instead the fluctuating dissipation field acts as a random measure.

KPZ and Generalisations

- The KPZ relation was first derived by coupling a two-dimensional conformal field theory (CFT) to gravity and analyzing the effect of quantum gravity on the scaling dimensions of the CFT. This has been dubbed “gravitational dressing".
- The KPZ relation has been generalized in various directions. First, to an arbitrary number of dimensions without reference to a conformal field theory structure.

KPZ and Generalisations

- Second, to a more general random field than the lognormally distributed one (Schramm et. al.)

$$\xi_0 = \xi - \log_2 E[W^{\xi}] \tag{10}$$

  where E is the expectation and W is the random variable associated with the measure (not necessarily a lognormal one).
- We will not use the latter generalization here, but it may be valuable in the study of steady state statistics of other non-linear dynamical systems out of equilibrium.

An Exact Formula

- We propose that the scaling exponents of incompressible fluid turbulence ξ_n in any number of space dimensions d satisfy the KPZ-type relation (6).
- Solving for ξ_n we get

$$\xi_n = \frac{\left((1+\gamma^2)^2 + 4\gamma^2(\frac{n}{3}-1)\right)^{\frac{1}{2}} + \gamma^2 - 1}{2\gamma^2} , \qquad (11)$$

  where in choosing the branch we required finite exponents ξ_n.
- $\gamma(d)$ is a numerical real parameter that depends on the number of space dimensions d. It can be determined from any moment, for instance, from the energy spectrum.

Properties

- There are several immediate properties of the formula (11) that we can see. First, using $n = 3$ in (11) one gets the exponent $\xi_3 = 1$ in any dimension, an exact result derived by Kolmogorov which agrees with numerical simulations and experiments.
- Second, the scaling exponent ξ_2 is a monotonically increasing function of γ, while the exponents $\xi_n, n > 3$ are monotonically decreasing functions of γ.
- Third, in the limit $n \to 0$ we get that $\xi_n \to 0$, as expected.
- Fourth, in the limit $\gamma \to 0$ we have $\xi_n \to \frac{n}{3}$, that is scale invariant statistics with no intermittency.

Properties

- Fifth, in the limit $\gamma \to \infty$, we have $\xi_n \to 1$, as in Burgers turbulence. The scaling exponents take values in the range $\frac{2}{3} \leq \xi_2 \leq 1$, and $1 \leq \xi_n \leq \frac{n}{3}$ for $n \geq 3$. We will propose that the limit $\gamma \to \infty$, is the limit of infinite number of space dimensions d. The subleading correction, relevant for developing a systematic $\frac{1}{d}$ expansion reads

$$\xi_n = 1 + \frac{1}{\gamma^2}\left(\frac{n}{3} - 1\right) + O\left(\frac{1}{\gamma^4}\right) . \qquad (12)$$

- Sixth, in the limit $n \to \infty$ for fixed γ, we have $\xi_n \to \frac{1}{\gamma}\left(\frac{n}{3}\right)^{\frac{1}{2}}$, thus growing as $\sqrt{n}$.
- Seventh, at the "critical" value $\gamma = 1$ we get $\xi_n = \left(\frac{n}{3}\right)^{\frac{1}{2}}$.

Analytical Constraints on the Scaling Exponents

- If there exist two consecutive even numbers $2n$ and $2n+2$ such that $\xi_{2n} > \xi_{2n+2}$, then the velocity of the flow cannot be bounded. Using (11) it is straightforward to show that $\xi_{2n} \leq \xi_{2n+2}$ for any γ, thus (11) satisfies the absence of supersonic velocity requirement.
- The second condition is that of convexity. For any three positive integers $n_1 \leq n_2 \leq n_3$, the scaling exponents satisfy the convexity inequality that follows from Hölder inequality $(n_3 - n_1)\xi_{2n_2} \geq (n_3 - n_2)\xi_{2n_1} + (n_2 - n_1)\xi_{2n_3}$. Using (11) it is straightforward to show that the Hölder inequality holds. Equality is achieved when $\gamma = 0$, when $\gamma \to \infty$ and when $n_i = n_j$ for some $i \neq j$ and arbitrary γ.

The Energy Spectrum

- The structure function $S_2(r) \sim r^{\xi_2}$ gives the energy spectrum of the fluid.
- Using (11) we see that ξ_2 is a monotonic function of γ that takes values in the range $\frac{2}{3} \leq \xi_2 \leq 1$ when γ goes from zero to infinity.
- In momentum space a deviation from the Kolmogorov spectrum for small γ (small d) reads

$$E(k) \sim k^{-\frac{5}{3}-\frac{2\gamma^2}{9}} \ . \tag{13}$$

- For large γ (large d) we have

$$E(k) \sim k^{-2+\frac{1}{3\gamma^2}} \ . \tag{14}$$

Experimental and Numerical Data

- The anomalous scaling exponents (11) depend on the parameter γ, which is a function of d. We do not know the exact expression of γ, but it can be calculated knowing one of the structure functions, such as the energy spectrum $\gamma = \left(\frac{\xi_2 - \frac{2}{3}}{\xi_2(\xi_2 - 1)}\right)^{\frac{1}{2}}$.
- With this knowledge we can then make an infinite number of predictions. In the following we will compare the analytical expression (11) to the available numerical and experimental data in various dimensions.

Two Space Dimensions

- In two space dimensions the energy cascade is an inverse cascade, where the energy flux flows to scales larger than the injection scale.
- In this case, one has the energy spectrum agreeing with the Kolmogorov scaling $\xi_2 = \frac{2}{3}$.
- Using (11), this implies that $\gamma(2) = 0$, and that all the other scaling exponents follow the Kolmogorov scaling $\xi_n = \frac{n}{3}$.

Three Space Dimensions

- In three space dimensions we first use the data for the anomalous scaling exponents (Benzi) from wind tunnel experiments at Reynolds number $\sim 10^4$.
- This experimental data is consistent with numerical data from simulations of the Navier-Stokes equations, see e.g.(Gotoh).
- We fit (6) to this data using a least squares fit.

Three Space Dimensions

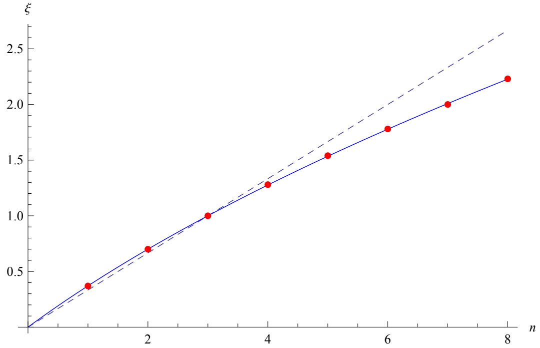


Figure : Fit of (6) (blue) to experimental data. The dashed line represents Kolmogorov scaling. The best fit value of the free parameter γ^2 is about 0.161. The error on the data is about ± 1 percent.

Three Space Dimensions

- Consider the numerical results for low order structure function exponents and non-integer *n*. The numerical data is consistent with experiment at Reynolds number 10^4. For this data, the fitted value of γ^2 is about 0.159.

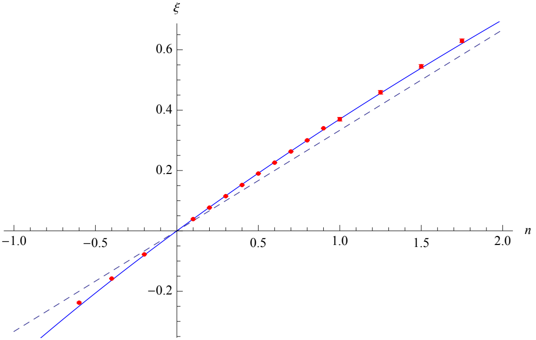


Figure : Fit of (6) (blue) to numerical data of numerical low moments (red). The dashed line represents Kolmogorov scaling. The best fit value of the free parameter γ^2 is about 0.159.

Three Space Dimensions

- Note that if our conjectured relation between the random measure and the local energy dissipation field is correct, one can determine γ^2 independently by measuring the scaling exponent of the two point function, $\langle \epsilon(x)\epsilon(0) \rangle \sim x^{-\gamma^2}$.
- This value has been found to be ≈ 0.2, which in our formula is still consistent with the data.

Four Space Dimensions

- In four space dimensions, numerical simulations of the Navier-Stokes equations were performed in (Gotoh2007). The authors found an increase in intermittency, i.e. $\xi_n^{(4)} > \xi_n^{(3)}$ for $n < 3$, while $\xi_n^{(4)} < \xi_n^{(3)}$ for $n > 3$.
- We took the data for the structure function exponents in 4d given in (Gotoh) and performed a fit to (6). This is shown in Figure 3. Although taken at a relatively low Reynolds number, the results are in agreement with a simple increase in the γ^2 parameter in our formula (6). The value of γ^2 in four space dimensions is fitted to about 0.278.
- Note that their numerical data for same simulation in three space dimensions predicts γ^2 about 0.188, which is higher than the experimental data above. This could be related to the relatively low Reynolds numbers involved.

Four Space Dimensions

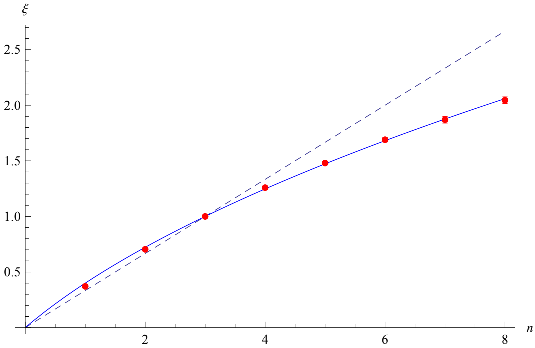


Figure : Fit of (6) to the 4d exponents given in (Gotoh2007). The solid line is the 4d fit with γ^2 about 0.278.

Intermittency

- In order to observe intermittency one has to study the short distance statistical properties of the fluid flow. There are various measures for intermittency, such as $F_n(r) = \frac{S_n(r)}{S_2(r)^{\frac{n}{2}}}, \quad n \geq 3$. $F_n(r)$ are expected to grow as a power-law in the limit $r \to 0$, while staying in the inertial range of scales.
- We can analyze the properties of $F_n(r)$ using (11). They scale as $\sim r^{\alpha}$, where α is a decreasing function of γ.
- In the limit $\gamma \to 0$ one has $\alpha \to 0$ and no intermittency, while as $\gamma \to \infty$ we get the maximal intermittency $\alpha = \frac{2-n}{2}$.
- Numerically, one sees a clear growth of $F_n(r), n \geq 4$ in the limit $r \to 0$, when as we increase the number of space dimensions in the simulation. The data is not accurate enough to observe the growth when $n = 3$.

Large d Limit

- It was conjectured that in the limit of infinite d all the exponents ξ_n approach the same value, one, as in Burgers turbulence.
- With our formula (6) this means that γ goes to infinity in the limit of infinite d, and therefore $\xi_n = 1$ for any n.
- This suggests the interesting possibility of having a systematic $\frac{1}{d}$ expansion (12).

Summary

- It incorporates intermittency in a novel way by dressing the Kolmogorov linear scaling via a coupling to a lognormal random geometry.
- The formula has one real parameter γ that depends on the number of space dimensions.
- The scaling exponents satisfy the convexity inequality, and the supersonic bound constraint.
- They agree with the experimental and numerical data in two and three space dimensions, and with numerical data in four space dimensions.
- Intermittency increases with γ, and in the infinite γ limit the scaling exponents approach the value one, as in Burgers turbulence.
- At large n the nth order exponent scales as $\sqrt{n}$.

Outlook

- Our proposal to incorporate the intermittency at the inertial range of scales by a gravitational dressing using a random geometry was inspired by the mapping between (Navier-Stokes) fluid flows and black hole horizon geometry.
- The main challenge is to determine analytically the function $\gamma(d)$.
- While equilibrium statistics is characterized by the Gibbs measure, there is yet no analog of this for non-equilibrium steady state statistics. We speculate that there is a general principle that allows us to consider the steady state statistics of out of equilibrium systems as a gravitationally dressed scale invariant one.