Generalized entanglement entropy

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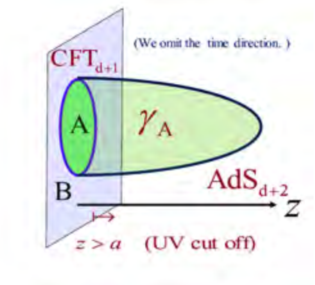
Introduction

- There has been considerable interest recently in entanglement entropy.



(Takayanagi)

- $S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$.
- Holographic Ryu-Takayanagi (RT) prescription: area of co-dimension two minimal surface homologous to A

$$S_A = \frac{\mathcal{A}}{4G_N}$$

- Leading UV divergence: area of separating surface.

Significance of entanglement entropy

A useful computable, particularly in applied holography, but also

- Does entanglement entropy capture global structure in the dual spacetime?
- ER = EPR? (Maldacena and Susskind)

Other measures of entanglement

Many other measures of entanglement: holographic realisations?

- Consider density matrix ρ for theory with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.
- Define entanglement negativity $\mathcal{E} = \log \mathrm{Tr}(\rho^{T_2})$, with T_2 being a partial transpose over $\mathcal{H}_2$.
- Well studied in CFT (Calabrese et al) but replica trick requires a non-integral number of copies of the bulk!

Generalized entanglement entropy

- Generalized holographic entanglement entropy, 1507.xxxxx

and ongoing work with Peter Jones, Da-Wei Pang and William Woodhead.

Outline

- **Entanglement in field theory**
- Generalized holographic entanglement entropy

Field space entanglement entropy

- Consider a quantum field theory with fields $\{\phi_i, \psi_a\}$.
- Given the density matrix of the theory ρ one can define a reduced density matrix ρ_{ϕ_i}

$$\rho_{\phi_i} = \int D\psi_a \, \rho$$

- We will denote the associated von Neumann entropy S_{ϕ_i} as the field space entanglement entropy.
- Operationally we usually compute this using the replica trick.

Examples: free field models

- Massive scalar fields:

$$S = \int d^d x \left((\partial\phi)^2 + (\partial\psi)^2 + m^2(\phi \cos\alpha - \psi \sin\alpha)^2 \right)$$

- Off-diagonal kinetic term:

$$S = \int d^d x \left((\partial\phi)^2 + (\partial\psi)^2 + \mu(\partial\phi)(\partial\psi) \right)$$

Note that these are conformal when $m^2 = 0$ and $\mu = 0$.

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Field space entanglement entropy

- For off-diagonal kinetic term, ground state entropy is (Mollabashi, Shiba and Takayanagi)

$$S_\phi = s(\mu)\left(\frac{V_D}{\epsilon^D} + \cdots + c_0\right)$$

with V_D spatial volume, $d = D + 1$, ϵ UV cutoff and $(s(\mu), c_0)$ computable constants. Note that $s(0) = 0$.

- For the massive model, ground state entropy is

$$S_\phi \sim m^4 \sin^2(2\alpha) V_D \left(\epsilon^{5-d} \log \epsilon\right)$$

i.e. UV finite for $d < 5$. It vanishes for diagonal mass terms.

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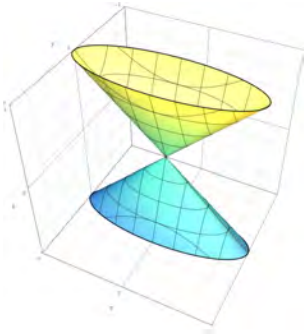
Volume versus area scaling

Entanglement occurs throughout all of the spatial volume, hence the scaling of UV divergences with volume.

Theories with global symmetry

- Next consider a scalar field theory with $SO(n)$ R symmetry acting on n real scalar fields ϕ_i.



- Given ρ we define a reduced density matrix ρ_Ω as

$$\rho_\Omega = \int_\Omega D\phi_i \, \rho,$$

  where Ω is any subregion of the field space R^n.
- Denote the associated entropy as the R symmetry entanglement entropy, S_Ω.

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Global symmetry entanglement entropy

Toy model:

- Two equal mass real scalars in $d = 1$ (quantum mechanics).
- Field space is R^2, R symmetry is $SO(2)$.
- Suppose Ω is a wedge of angle ω in R^2: then in the ground state
$$S_\Omega = -\frac{\omega}{2\pi} \log(\omega(2\pi - \omega)/4\pi^2).$$
- Vanishes for $\omega \to 0$ and gives $\log(2)$ for equal partition of R^2.

General theories

- In interacting theories S_Ω has a volume divergence.
- Bosonic part of SYM: gauge fields and $(10-d)$ adjoint scalars

$$S = \int d^d x \mathrm{Tr} \left(\frac{1}{4g_d^2} F^2 + \frac{1}{2}(D\phi^i)^2 + \frac{g_d^2}{4} \sum [\phi^i, \phi^j]^2 \right)$$

with R symmetry group $SO(10-d)$.

- Extrapolating from toy models,

$$S_\Omega \sim s(\Omega) \frac{V_D}{\epsilon^D} + \cdots$$

in ground state, with Ω defining partition of R^{10-d}.

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Subtleties

- Gauge fields and fermions are problematic.
- Lack of factorisation of Hilbert space.

(*As for usual entanglement entropy.*)

- Global symmetry entanglement entropy can be reformulated in terms of operators, see also (Karch and Uhlemann).

Outline

- Entanglement in field theory
- **Generalized holographic entanglement entropy**

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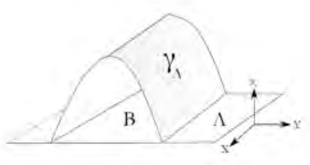
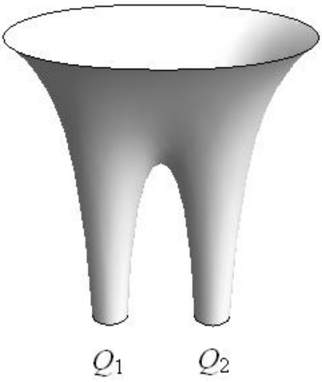
Entanglement and geometry



- In the RT prescription, the bulk geometry is separated into two regions by the minimal surface.
- The degrees of freedom traced out are geometrically separated in the bulk.
- Prescription justified by Lewkowcyz and Maldacena.

Field space entanglement entropy should only be computable holographically if the fields integrated out are geometrically separated in some way.

Consistent with ER=EPR slogan.



- Consider an asymptotically AdS geometry with two internal throats.
- Deep inside each throat, suppose the holographic dual can be described by a given QFT.
- The degrees of freedom of this QFT are entangled with other QFT fields, including those associated with other throat.

Bifurcate throats

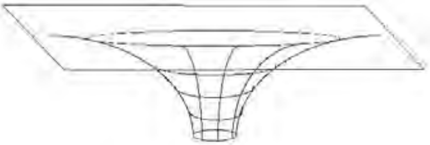
- Integrating out heavy modes, the low energy QFT description has a Lagrangian

$$L = L_{QFT_1} + L_{QFT_2} + L_{\rm int}$$

- Integrating out all degrees of freedom except those in QFT_1, we can compute a field space entanglement entropy.

Geometric formula for the entanglement entropy

Is there a geometric description for the field space entanglement entropy?



- The bulk action contains

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} \, (R + 2\Lambda + L_{\rm irr})$$

- Here

$$L_{\rm irr} = -\frac{1}{2} \left((\partial\Phi)^2 + m^2 \Phi^2 \right)$$

with m^2 corresponding to an irrelevant operator.

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Examples

We can derive such a description explicitly:

- Coulomb branch of $\mathcal{N} = 4$ with separated stacks of branes.
- Coulomb branch of M2-branes, M5-branes and D1-D5 system.
- Decoupling region of near extremal *AdS* Reissner-Nordstrom black holes.

In all cases, the operator is of dimension 2*d*.

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Bifurcate throats

- Coming out of the throat, the bulk geometry is AdS_{d+1} plus irrelevant corrections:

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}\left(1 + \frac{\mu^2}{r^{2(d-\Delta)}} + \cdots\right) dx \cdot dx$$

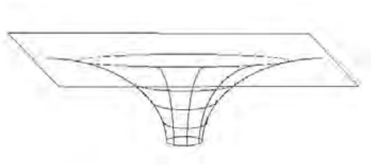
$$\Phi \sim \frac{\mu}{r^{d-\Delta}}$$

- Here μ characterises the irrelevant deformation; $\mu / r^{d-\Delta} < 1$.

Geometric description of entanglement entropy

- Geometric dual of the field space entanglement entropy must have the following properties:
  1. Leading UV divergence behaves as V_D/ϵ^D;
  2. It should vanish as $\mu \to 0$ (i.e. for a non-interacting CFT).
- There are two obvious candidates....

Geometric description of entanglement entropy



- Simplest possibility is the **renormalised spatial volume** of the throat:

$$S = \frac{\mathcal{V}_{\mathcal{R}}}{4G_N} = \frac{1}{4G_N}\int_{\Sigma} d^d x \sqrt{\gamma} - \frac{1}{4(d-1)G_N}\int_{\partial\Sigma} d^{d-1}\sqrt{h} + \cdots$$

- The counterterms ensure that the answer is zero for AdS_{d+1}.

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Renormalized volume

- For our deformed throat geometry:

$$S \sim \mu^2 \frac{V_D}{\epsilon^D} + \cdots$$

  where ϵ is the UV cutoff and $D = d - 1$.
- This is indeed of the same form as the field space entanglement entropy.
- However, S does not vanish for all asymptotically AdS_{d+1} throats i.e. for all backgrounds with $\mu = 0$.

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Why renormalized volume?

Heuristic argument:

(Takayanagi)

- In RT, degrees of freedom are only entangled at the boundary between A and B; extension of this surface into the bulk gives a codimension two surface.
- Here degrees of freedom are entangled through spatial volume; extension into bulk gives a codimension one surface.

Differential entropy

- The other possibility is the **area of the separating surface**:

$$S = \frac{\mathcal{A}}{4G_{d+1}} \sim \frac{V_D}{\epsilon^D}\left(1 - \mu^2\right)$$

- This doesn't vanish for $\mu = 0$, i.e. AdS_{d+1}, but again it has the same structure as the field space entanglement entropy.

Differential entropy

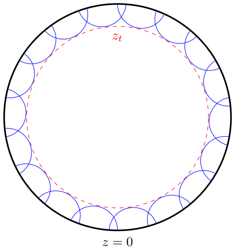
This area is in fact the differential entropy:

- Differential entropy $\mathcal{E}$ is defined as

$$\mathcal{E} = L_x \frac{\partial S_{RT}}{\partial x}$$

  with S_{RT} the entanglement entropy of a strip of width x, L_x the length of the x direction.

- $\mathcal{E}$ computes the area of a surface whose radius is the turning point of the RT surface.



Geometric decoupling is incomplete

- Both geometric candidates for the field space entanglement entropy give non-zero answers for asymptotically AdS throats, i.e. for states/deformations of conformal field theories.

The geometric cutoff removes high energy modes in the low energy conformal field theory, hence associated entanglement entropy.

- *Should differential entropy be interpreted as field space entanglement entropy?*

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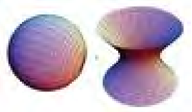
Global symmetry entanglement entropy

- In holography, R symmetry is realised as isometries of the compact space (usually a sphere).
- Global symmetry entanglement entropy was defined by partitioning the field space on which the symmetry acts.
- Natural to propose that the holographic dual quantity partitions the compact space.

Generalized holographic entanglement entropy

- Mollabashi, Shiba and Takyanagi: static codimension two minimal hypersurface in ten dimensions, filling the spatial section of (asymptotically) AdS but divides the compact space.



- Define the **generalised holographic entanglement entropy** as

$$S_G = \frac{\mathcal{A}}{4G_{10}}$$

- Non-zero for $AdS \times S$; divergence scales as spatial volume $\rightarrow$ matches global symmetry entanglement entropy.

Summary

We have discussed two measures of entanglement in QFTs:

1. Integrating out fields - field space entanglement entropy
2. Integrating out part of the field space in an R symmetric theory - global symmetry entanglement entropy

The latter can be described holographically by a codimension two hypersurface partitioning the compact space.

Summary

- The holographic description of a system with non-zero field space entanglement entropy is a throat, deformed by massive scalar fields.
- The renormalized volume of the throat, or the area of its cutoff, capture features of field space entanglement entropy.
- Any geometric cutoff also removes high energy modes of the CFT, so geometric entropies are non-zero even for asymptotically AdS throats.

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