Entanglement Entropy and Duality in AdS_4

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Section 1

Motivation

- Black Hole Physics suggest Einstein equations could be effective thermodynamic relations for some underlying degrees of freedom.

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- AdS/CFT correspondence could suggest that these underlying degrees of freedom are the boundary conformal field theory degrees of freedom.
- More recently, it has been suggested that the relation between gravity and thermodynamics should not be attributed to thermal statistics, but rather to quantum statistics related to quantum entanglement physics.
- Enforcement of entanglement thermodynamics first law at linear order is equivalent to the linearized Einstein's equations (for Dirichlet boundary conditions).

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- We study the duality in the bulk from the point of view of entanglement physics, as it may provide new light to its understanding
- Such study can act as a consistency benchmark for the RT conjecture

Section 2

Entanglement Entropy and Holography

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The most popular definition of an entanglement measure is the entanglement entropy, which is Shannon entropy defined on the spectrum of the reduced density matrix ρ_A,

$$S := -\mathrm{Tr} \rho_A \ln \rho_A .$$

The density matrix ρ_A is hermitian and positive semidefinite, thus, one may define the modular Hamiltonian as

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$$\delta S_A = \delta \langle H_A \rangle .$$

This is the direct analog of the first law of thermodynamics for entanglement physics.

Ryu-Takayanagi conjecture connects the entanglement entropy for a region A defined by the entangling surface ∂A in the boundary field theory to the area of an extremal co-dimension two open surface in the bulk gravitational dual theory with the **same boundary** ∂A.

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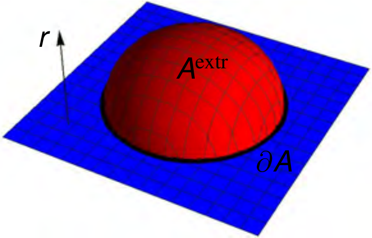

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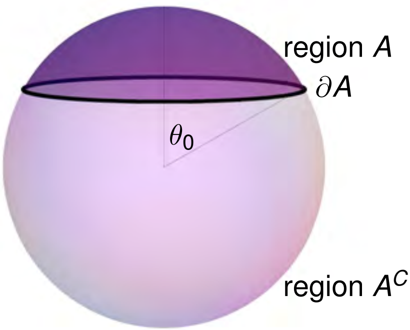

then the extremal surface is given by (unperturbed AdS_4)

$$t = t_0,\ \theta\left(r\right) = \arccos\left(\cos\theta_0\sqrt{1+\frac{1}{r^2}}\right),$$

$$\rho \in \left[\cot\theta_0, \infty\right),\ \varphi \in \left[0, 2\pi\right),$$

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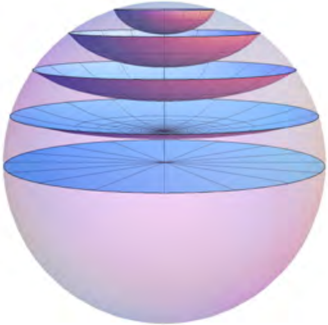

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- The first term is the co called *Area Law* and it is divergent,
- The second term is *universal* and finite.

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Since variation of the boundary metric is allowed, it looks reasonable that **the entangling loop may get disturbed**, thus, this has to be taken into account in the variational problem determining the extremal surface.

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Varying the metric one finds

$$\delta A = \left.\frac{\delta A\left(g, X\right)}{\delta g}\right|_{g=g_0, X=X_0} \delta g + \left.\frac{\partial A\left(g, b\right)}{\partial b}\right|_{g=g_0, b=b_0} \delta b := \delta A_g + \delta A_b.$$

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The term occuring from the variation of the surface equations is vanishing as a consequence of the extremality of the unperturbed surface.

The first term can be calculated as

$$\delta A_g = \frac{1}{2} \int d^2\sigma \sqrt{\gamma_0} (\gamma_0)^{ab} \delta\gamma_{ab},$$

where γ_{ab} is the induced metric on the unperturbed extremal surface.

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The second term can be calculated by varying the unperturbed result with respect to θ_0.

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In some cases, the modular Hamiltonian generates a geometric flow and then it can admit a local expression.

A known example is that of the region A being a disk of radius R in a Minkowski boundary.

In such cases, the modular Hamiltonian can be expressed in terms of the holographic energy-momentum tensor and a conformal Killing vector that leaves invariant the entangling surface ∂A and its causal development,

$$\delta E = \int_C d\Sigma^\mu T_{\mu\nu} \zeta^\nu ,$$

where C is any spacelike surface with boundary ∂A.

Our case can be connected with that of a disk in Minkowski space through a coordinate transformation. Then one yields

$$\zeta = \frac{2}{\sin\theta_0}\left[\left(\cos\left(t - t_0\right)\cos\theta - \cos\theta_0\right)\partial_t - \sin\left(t - t_0\right)\sin\theta\partial_\theta\right].$$

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Selecting C to coincide with A,

$$\delta E = \frac{\pi}{\sin\theta_0}\int_0^{\theta_0} d\theta \sin\theta \left(\cos\theta - \cos\theta_0\right) T_{tt}.$$

Section 3

Linearized Gravity in AdS_4

We consider linear metric perturbations around AdS_4 background

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$f(r) = 1 - \frac{\Lambda}{3}r^2.$$

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There are two classes of perturbations

- Axial perturbations

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + 2e^{-i\omega t}\sin\theta\frac{dP_l(\cos\theta)}{d\theta}\left(h_0(r)\,dt + h_1(r)\,dr\right)d\varphi.$$

- Axial perturbations

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$
$$+\,2e^{-i\omega t}\sin\theta\frac{dP_l(\cos\theta)}{d\theta}\left(h_0(r)\,dt + h_1(r)\,dr\right)d\varphi.$$

- Polar perturbations

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$
$$+\,e^{-i\omega t}P_l(\cos\theta)\Bigg[H_0(r)\left(f(r)\,dt^2 + \frac{dr^2}{f(r)}\right)$$
$$+2H_1(r)\,dtdr + K(r)\,r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\Bigg].$$

For **both** classes Einstein's equations become equivalent to the **same** effective Schrödinger problem with respect to the tortoise coordinate x ($dx = \frac{dr}{f(r)}$),

$$-\frac{d^2\Psi(x)}{dx^2} + \frac{l(l+1)}{\sin^2 x}\Psi(x) = \omega^2\Psi(x).$$

All functions of r in the metric can be expressed in terms of the solutions of the effective Schrödinger problem.

An asymptotic expansion of Ψ yields

$$\Psi = l_0 + \frac{l_1}{r} + \frac{l_2}{r^2} + \frac{l_3}{r^3} + \dots$$

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Only two parameters are independent.
$l_0 = 0$ corresponds to Dirichlet boundary conditions.
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- Axial perturbations

Dirichlet conditions for Ψ correspond to **Dirichlet** conditions for the metric

- Polar perturbations

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Section 4

Entanglement Entropy for Metric Perturbations in AdS_4

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$$\delta A_g = -2\pi e^{-i\omega t_0}\left[\lim_{r\to\infty}\left[rJ_1 + J_0\left(l\left(l+1\right)-\omega^2\right)\right]\sin\theta_0 P_l\left(\cos\theta_0\right) + \cot\theta_0 \frac{l\left(l+1\right)}{2l+1} J_0\left(P_{l+1}\left(\cos\theta_0\right) - P_{l-1}\left(\cos\theta_0\right)\right)\right].$$

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$$\left.+\cot\theta_0\frac{l\left(l+1\right)}{2l+1}J_0\left(P_{l+1}\left(\cos\theta_0\right)-P_{l-1}\left(\cos\theta_0\right)\right)\right].$$

Notice that the result is divergent for non-Dirichlet boundary conditions for the metric.

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For both kinds of perturbations the result is

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Notice that the result is in general divergent.

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$$\begin{aligned}\delta S = -\frac{\pi}{4} e^{-i\omega t_0} \Bigg[\lim_{r\to\infty} \left(r J_1 + J_0 \left(l \left(l+1\right) - \omega^2 \right) \right) \sin\theta_0 \, P_l \left(\cos\theta_0\right) \\ + \cot\theta_0 \frac{l\left(l+1\right)}{2l+1} J_0 \left(P_{l+1}\left(\cos\theta_0\right) - P_{l-1}\left(\cos\theta_0\right)\right)\Bigg] \\ + \frac{\pi}{2} \lim_{r\to\infty} r \cos\theta_0 \, \delta\theta \left(t_0\right).\end{aligned}$$

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$$\begin{aligned}\delta E = & -2\pi \frac{l(l+1)}{2l+1} J_0 e^{-i\omega t_0} \cot\theta_0 \left(P_{l+1}(\cos\theta_0) - P_{l-1}(\cos\theta_0)\right) \\ & - \pi l(l+1) J_0 e^{-i\omega t_0} \sin\theta_0 P_l(\cos\theta_0) .\end{aligned}$$

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$$\delta E = -2\pi \frac{l\,(l+1)}{2l+1} J_0 e^{-i\omega t_0} \cot\theta_0 \left(P_{l+1}\left(\cos\theta_0\right) - P_{l-1}\left(\cos\theta_0\right)\right) \\ - \pi l\,(l+1)\, J_0 e^{-i\omega t_0} \sin\theta_0 P_l\left(\cos\theta_0\right).$$

Notice that both results are finite.

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Demanding $\delta S = \delta E$ yields

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$$\delta\theta\left(t_0\right) = \frac{1}{2} e^{-i\omega t_0}\left[J_1 + \frac{J_0}{r}\left(\frac{1}{2} l\left(l+1\right) - \omega^2\right)\right] \tan\theta_0 \, P_l\left(\cos\theta_0\right).$$

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These expressions take care of divergence issues.
Is there any geometric picture of these results?

For polar perturbations, the induced metric on the spherical slices is

$$ds^2 = r^2 \left(1 + K\left(r\right) e^{-i\omega t} P_l\left(\cos\theta\right)\right) \left(d\theta^2 + \sin^2\theta d\varphi^2\right) .$$

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The asymptotic expansion for $K(r)$ is

$$K(r) = -J_1 - \frac{J_0}{r}\left(\frac{1}{2} l(l+1) - \omega^2\right) + \mathcal{O}\left(\frac{1}{r^2}\right),$$

which implies that the deformation of the entangling curve can be written as

$$\delta\theta(t_0) = -\frac{1}{2} e^{-i\omega t_0} K(r) \tan\theta_0\, P_l(\cos\theta_0)$$

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but

$$\left(1 + \tfrac{1}{2} K(r)\, e^{-i\omega t_0} P_l(\cos(\theta_0 + \delta\theta(t_0)))\right) r \sin(\theta_0 + \delta\theta(t_0)) =$$
$$\left(1 + \tfrac{1}{2} K(r)\, e^{-i\omega t_0} P_l(\cos\theta_0)\right) r \sin(\theta_0 + \delta\theta(t_0)) = r \sin\theta_0 + \mathcal{O}\left(\tfrac{1}{r}\right).$$

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which means that although the polar cap region and its boundary undergo perturbations, **the line element of the entangling curve remains constant!**

The same trivially holds for axial perturbations, where the induced metric on the spherical slices of the boundary is not perturbed and the entangling curve is not deformed.

Summarizing, we see that for all kind of gravitational perturbations of AdS_4 space-time (axial or polar) and for all kind of boundary conditions, the holographic realization of the first law of thermodynamics requires that the entangling curve undergoes **an isoperimetric deformation.**

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Summarizing, we see that for all kind of gravitational perturbations of AdS_4 space-time (axial or polar) and for all kind of boundary conditions, the holographic realization of the first law of thermodynamics requires that the entangling curve undergoes **an isoperimetric deformation.**

As the first law of thermodynamics is a triviality from the CFT point of view, the above statement should **not** be perceived as a condition for classification of the entangling curves according to the validity of the first law.

On the contrary it should be understood as an additional prescription to the RT conjecture. Namely the region A, for whom the entanglement entropy has a holographic manifestation through formula $S_A = \frac{\mathrm{Area}(A^{\mathrm{extr}})}{4G_N}$ is **the one** whose entangling surface deforms with time in such a way that its line element remains constant.

Section 5

Electric-Magnetic Duality in AdS_4 and Entanglement

One can calculate the holographic energy-momentum and Cotton tensors for both kinds of perturbations to discover that

$$T_{ab}^{\mathrm{polar}} = C_{ab}^{\mathrm{axial}},$$
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or in other words, if the axial and polar perturbations are characterized by **same frequency**, **same boundary conditions for** Ψ, meaning opposing boundary conditions for the metric.

The above duality can be better understood in terms of the Weyl tensor. If one defines the dual Weyl tensor like

$$\tilde{C}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\kappa\lambda} C_{\kappa\lambda\rho\sigma} ,$$

then a dual metric can be defined as,

$$\tilde{C}_{\mu\nu\rho\sigma}\left(g\right) = C_{\mu\nu\rho\sigma}\left(\tilde{g}\right) .$$

At linear level, the classification of the perturbations to axial and polar modes resolves the highly non-trivial, non-local duality relations,

$$\tilde{g}^{\mathrm{polar}} = g^{\mathrm{axial}}, \qquad \tilde{\Psi}_{\mathrm{polar}} = -\frac{2i}{\omega}\Psi_{\mathrm{axial}},$$
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$$\tilde{g}^{\text{polar}} = g^{\text{axial}}, \qquad \tilde{\Psi}_{\text{polar}} = -\frac{2i}{\omega}\Psi_{\text{axial}},$$
$$\tilde{g}^{\text{axial}} = g^{\text{polar}}, \qquad \tilde{\Psi}_{\text{axial}} = \frac{i\omega}{2}\Psi_{\text{polar}}.$$

In both cases the relation between $\tilde{\Psi}$ and Ψ enforces that dual perturbations are corresponding to the same solution of the identical effective Schrödinger problems and thus, they are characterised by identical frequency and boundary conditions for $\Psi(x)$.

In four dimensions, the energy-momentum and Cotton tensors are given by appropriate elements of the Weyl tensor like

$$T_{ab} = -\lim_{r\to\infty} r^3 C_{arbr},$$
$$C_{ab} = \lim_{r\to\infty} r^3 \tilde{C}_{arbr},$$

giving the interpretation of the relation between the holographic energy-momentum and Cotton tensor as an electric-magnetic duality with respect to the radial ADM decomposition of the Weyl tensor.

We define the dual entanglement entropy as

$$\tilde{S}_A = \frac{\mathrm{Area}\left(\tilde{A}^{\mathrm{extr}}\right)}{4G_N},$$

where $\tilde{A}^{\mathrm{extr}}$ is the extremal surface with respect to the dual metric $\tilde{g}$ ($\tilde{C}_{\mu\nu\rho\sigma}(g) = C_{\mu\nu\rho\sigma}(\tilde{g})$)

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Since the unperturbed AdS_4 space is self dual $\tilde{g}^{AdS} = g^{AdS}$, the unperturbed dual extremal surface is identical to the unperturbed extremal surface.
Thus, the variation of the dual entanglement entropy for axial and polar perturbations can be calculated in the same way as the variation of entanglement entropy.

A direct consequence of

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Similarly one can define the variation of the dual modular Hamiltonian as

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is

$$\delta \tilde{E}^{\mathrm{axial}} = \delta E^{\mathrm{polar}},$$
$$\delta \tilde{E}^{\mathrm{polar}} = \delta E^{\mathrm{axial}}.$$

Since it is true that $\delta S = \delta E$ for each kind of perturbations, it is also true that

$$\delta \tilde{S} = \delta \tilde{E}$$

for each kind of perturbations.

Section 6

Discussion

- We calculated the variations of the entanglement entropy and the expectation value of the modular Hamiltonian for linear metric perturbations in AdS_4 background and **general** boundary conditions.

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- We show that validity of the entanglement thermodynamics first law demands an **isoperimetric** time evolution for the entangling curve.

- We calculated the variations of the entanglement entropy and the expectation value of the modular Hamiltonian for linear metric perturbations in AdS_4 background and **general** boundary conditions.
- We show that validity of the entanglement thermodynamics first law demands an **isoperimetric** time evolution for the entangling curve.
- It would be interesting to study this kind of isoperimetric deformations of the entangling curve in the presence of metric perturbations which do not preserve the entangling curve symmetry, the axial symmetry in our case.

- The electric-magnetic duality can be implemented introducing a dual entanglement entropy and a dual modular Hamiltonian, such that a dual Ruy-Takayanagi conjuncture and a dual entanglement thermodynamics first law holds.

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- This is a positive consistency check for the validity of RT conjecture.
- It would be interesting to check whether such constructions can be achieved in other backgrounds, for example in asymptotically AdS_4 black holes.

- As a final comment, it would be very interesting to better understand the connection between the dual entanglement entropy and modular Hamiltonian and the reduced density matrix of the boundary conformal field theory.

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- Notice that the electric magnetic duality interchanges boundary conditions for the metric, thus, the time evolution of the entangling curve and thus the region A is in general **different** for the initial and dual CFTs. This is expected because of the **non-local** realization of electric-magnetic-duality.

Thank you very much