Entanglement Inequalities

Hirosi Ooguri

Walter Burke Institute for Theoretical Physics, Caltech
Kavli IPMU, University of Tokyo

Nafplion, Greece, 5 - 11 July 2015

Which CFT's have Gravity Duals?

Which Gravities have CFT Duals?

Tomography from Entanglement

based on arXiv:1412.1879 with Lin, Marcolli, Stoica

Entanglement Inequalities in CFT $\Rightarrow$ Energy Conditions in Gravity

The Structure of Holographic Entropy

based on arXiv:1505.07839 with Bao, Nezami, Stoica, Sully, Walter

Local Geometry in Gravity $\Rightarrow$ New Entanglement Inequalities in CFT

Tomography from Entanglement

based on arXiv:1412.1879,
Phys. Rev. lett. 114 (2015) 221601,
with J. Lin, M. Marcolli, B. Stoica

Tension between **Quantum** and **Gravity**

Non-Local Entanglement ⇔ **Local Geometry**

Holographic Expectations:

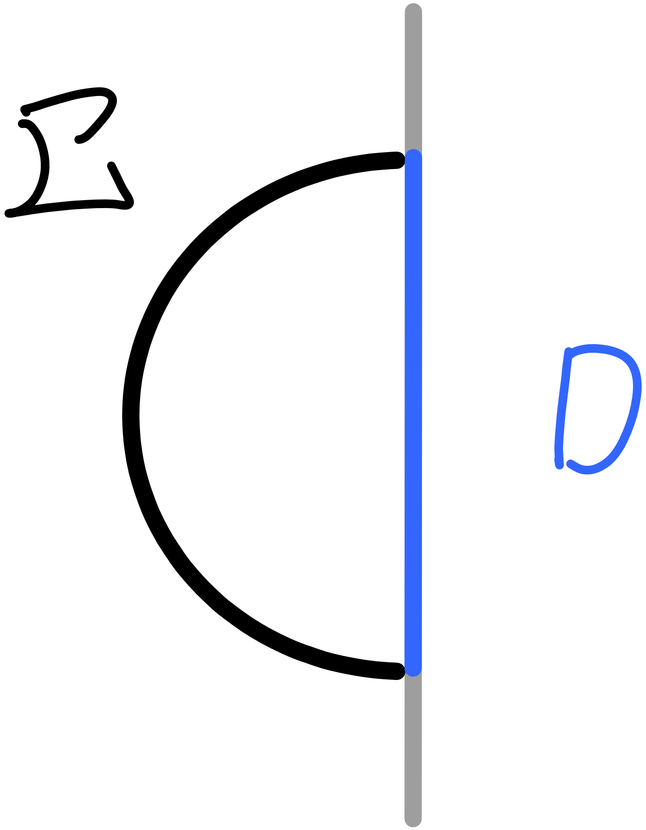
☆ Bulk locality emerges from boundary entanglement.

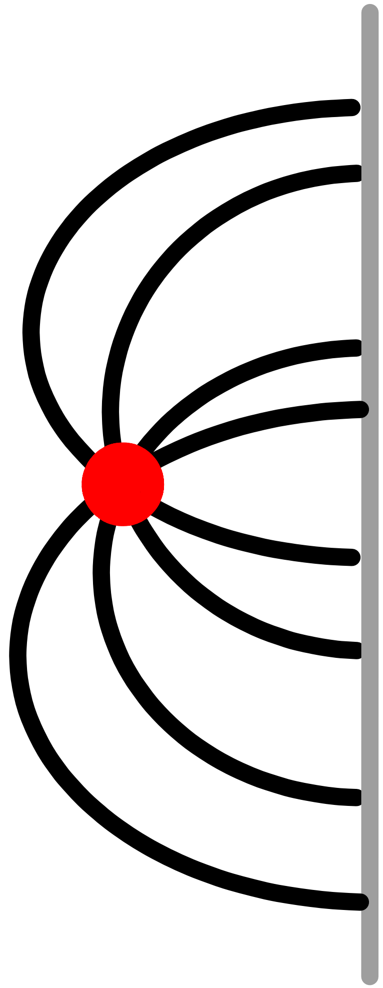
☆ Bulk-boundary relation is non-local.

We will show how these expectations are realized in a specific setup.

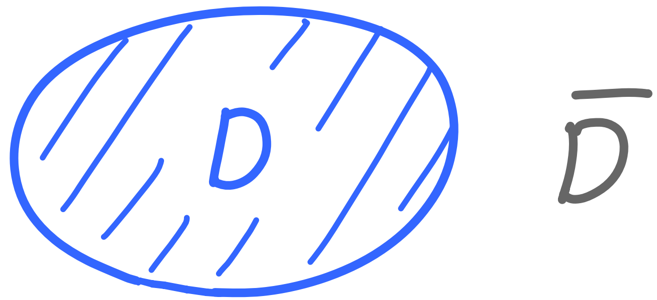
Main Result:

Ryu-Takayanagi formula for **boundary entanglement** can be inverted to diagnose **local data in the bulk**.





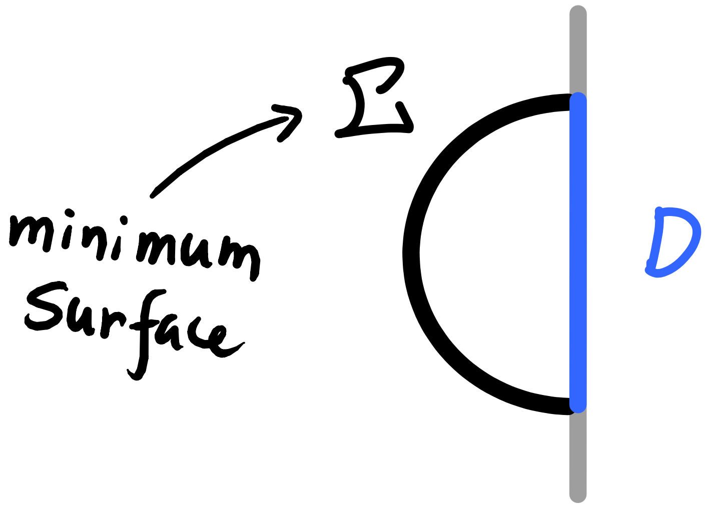
Entanglement Entropy



$|\psi\rangle$: State in CFT

$$\Rightarrow \rho = \mathrm{tr}_{\mathcal{H}_{\bar{D}}} |\psi\rangle\langle\psi|$$

$$S_E = -\,\mathrm{tr}_{\mathcal{H}_D}\, \rho \log \rho$$



Ryu-Takayanagi Formula

$$S_E = \frac{\mathrm{Area}(\Sigma)}{4 G_N}$$

Relative Entropy

$$|\psi_0\rangle,\ |\psi_1\rangle \in \mathcal{H}_{CFT} \Rightarrow \rho_0,\ \rho_1$$

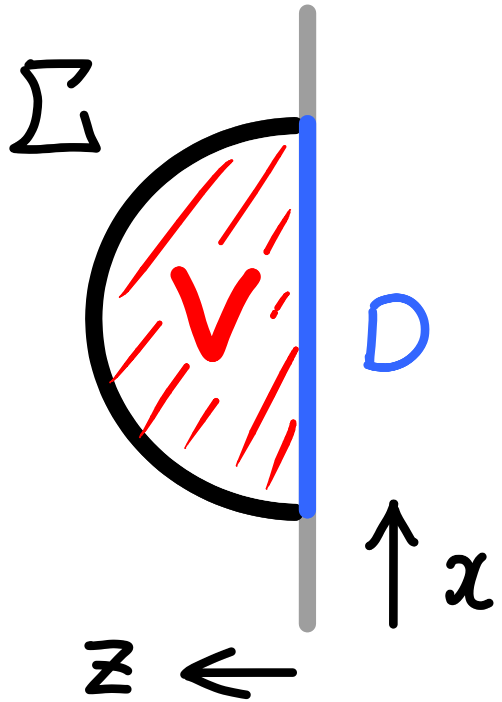
$$S(\rho_1|\rho_0) = \mathrm{tr}[\rho_1 \log(\rho_1/\rho_0)]$$

- measure of distinguishability
- $S(\rho_1|\rho_0) \geqq 0$
- monotonic in |domain|
- not symmetric in ρ_0 and ρ_1

We find: For $|\psi_0\rangle$: vacuum $\Leftrightarrow$ pure AdS_{d+1}

$|\psi_1\rangle$: any state in CFT_d

For small disk D



$$S(\rho_1 | \rho_0) = \frac{8\pi^2 G_N}{R} \times \int_V (R^2 - z^2 - x^2)\, \varepsilon \sqrt{g}\, dz\, d^{d-1}x$$

Bulk Energy Density

$$S(\rho_1 | \rho_0) = \frac{8\pi^2 G_N}{R} \int_V (R^2 - z^2 - x^2)\, \varepsilon \sqrt{g}\, dz\, d^{d-1}x$$

Bulk Energy Density

Entropy Inequalities $\Leftrightarrow$ Positive Energy Conditions

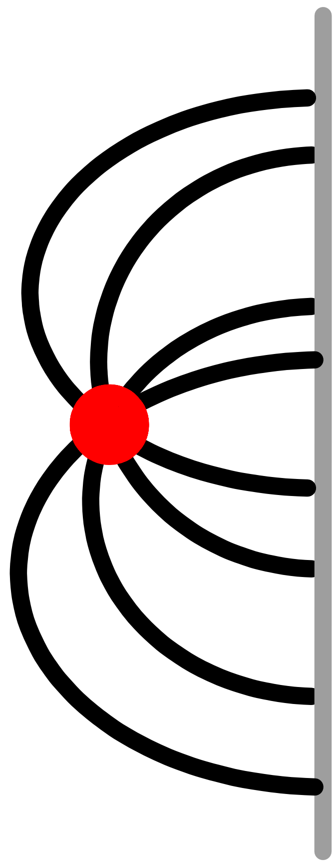
$$S \geq 0 , \quad \frac{\partial}{\partial R} S \geq 0$$

$$\Leftrightarrow \int_V \left[R^2 \pm (z^2 + x^2) \right] \varepsilon \sqrt{g}\, dz\, d^{d-1}x \geq 0$$

$$S(\rho_1 | \rho_0) = \frac{8\pi^2 G_N}{R} \int_V (R^2 - z^2 - x^2)\, \varepsilon \sqrt{g}\, dz\, d^{d-1}x$$

Bulk Energy Density

The relation can be inverted by the Radon transform to express the **bulk energy density** by the **entanglement data on the boundary**.



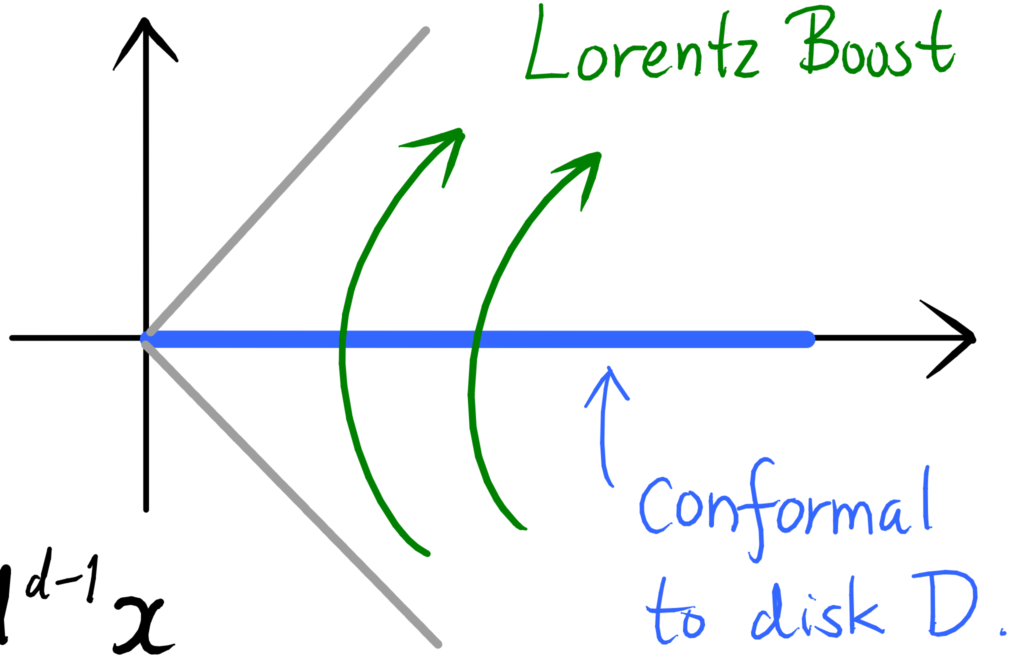
Holographic Expression for Relative Entropy

$$S(\rho_1|\rho_0) = \underbrace{\mathrm{tr}\,\rho_1 \log \rho_1}_{\text{Ryu-Takayanagi}} - \underbrace{\mathrm{tr}\,\rho_1 \log \rho_0}_{\text{modular Hamiltonian}}$$

modular Hamiltonian

$$\rho_0 \propto e^{-H_{mod}}$$

$$H_{mod} = \frac{\pi}{R} \int_D (R^2 - |\vec{x}|^2)\, T^t{}_t \, d^{d-1}x$$



Relative Entropy in terms of Bulk Metric

$$ds^2 = \frac{\ell_{AdS}^2}{z^2}\left[dz^2 + (\eta_{\mu\nu} + h_{\mu\nu})\,dx^\mu dx^\nu\right]$$

$$\Delta\langle H_{mod}\rangle \equiv \mathrm{tr}\,\rho_1 H_{mod} - \mathrm{tr}\,\rho_0 H_{mod}$$

$$= \frac{\ell_{AdS}^{d-1}\cdot d}{16 G_N R}\int_D (R^2 - |\vec{x}|^2)\, h_{ij}\,\eta^{ij} z^{-d}\, d^{d-1}x$$

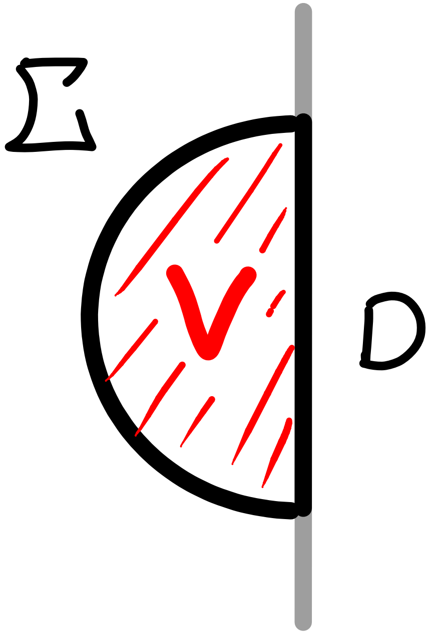
$$\Delta S_E \equiv -\mathrm{tr}\,\rho_1 \log\rho_1 + \mathrm{tr}\,\rho_0\log\rho_0$$

$$= \frac{\ell_{AdS}^{d-1}}{8 G_N R}\int_\Sigma (R^2\eta^{ij} - x^i x^j)\, h_{ij}\, z^{-d}\, d^{d-1}x$$

Using Wald's formalism, one can find a $(d-1)$-form χ such that

$$\Delta \langle H_{mod} \rangle = \int_D \chi$$

$$\Delta S_E = \int_\Sigma \chi$$



$$S(\rho_1 | \rho_0) = \int_D \chi - \int_\Sigma \chi$$

$$= \int_V d\chi$$

$$d\chi \propto R_{tt} - \frac{1}{2} g_{tt} + \Lambda g_{tt}$$

(works w/ higher derivatives, too)

$$S(\rho_1 | \rho_0) = \Delta \langle H_{mod} \rangle - \Delta S_E$$
$$= \int_V d\chi$$

$$d\chi \propto R_{tt} - \frac{1}{2} g_{tt} R + \Lambda g_{tt}$$

To leading order, $\Delta \langle H_{mod} \rangle = \Delta S_E$

First Law of Entanglement

$\Leftrightarrow$ Linearized Einstein Equations

Blanco, et al., arXiv:1305.3182; Lashkari, et al., arXiv:1308.3716; Faulkner, et al., arXiv:1312.7856

Consider backreaction from bulk matter $t_{\mu\nu}$.

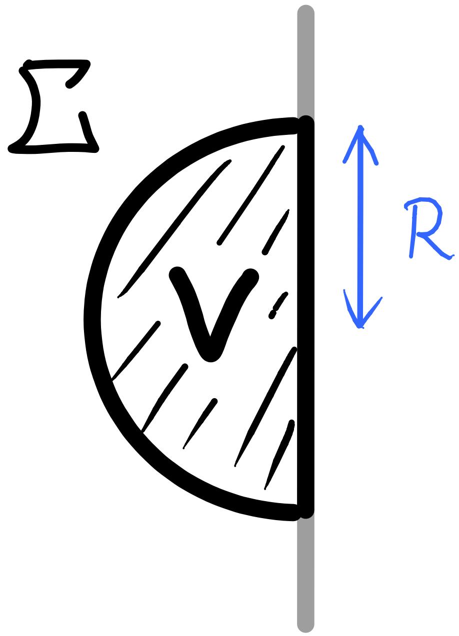
$$
\begin{aligned}
S(\rho_1|\rho_0) &= \Delta\langle H_{mod}\rangle - \Delta S_E \\
&= \int_V dx \\
&= \frac{8\pi^2 G_N}{R}\int_V (R^2-z^2-x^2)\, t^t{}_t \sqrt{g}\, dz\, d^{d-1}x \\
&\geqq 0
\end{aligned}
$$

ε : bulk energy density

$\varepsilon = t^t{}_t$

Boundary Entanglement Inequalities

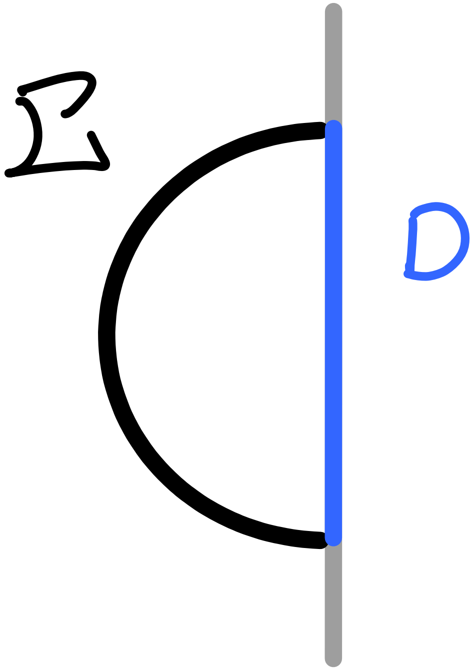
$\Leftrightarrow$ Bulk Energy Conditions



$$\left(\frac{\partial}{\partial R}+\frac{1}{R}\right) S(\rho_1|\rho_0) = 16\pi^2 G_N \int_V \varepsilon \sqrt{g_V}$$

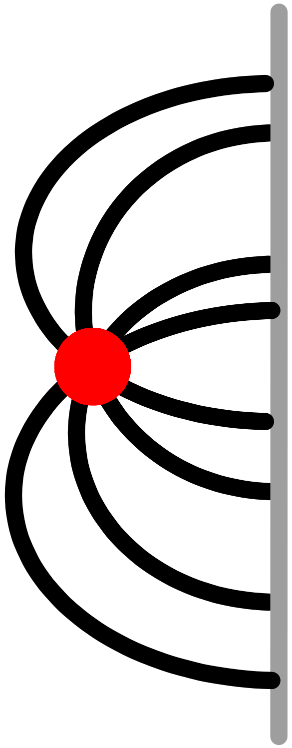
$$\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\right) S(\rho_1|\rho_0) = 16\pi^2 G_N \int_\Sigma \varepsilon \sqrt{g_\Sigma}$$

This relation can be inverted to express the bulk energy density by the relative entropy.



Radon Transform:

$$\int_{\Sigma} \varepsilon \sqrt{g_{\Sigma}} \Rightarrow S(\rho_1|\rho_0)$$

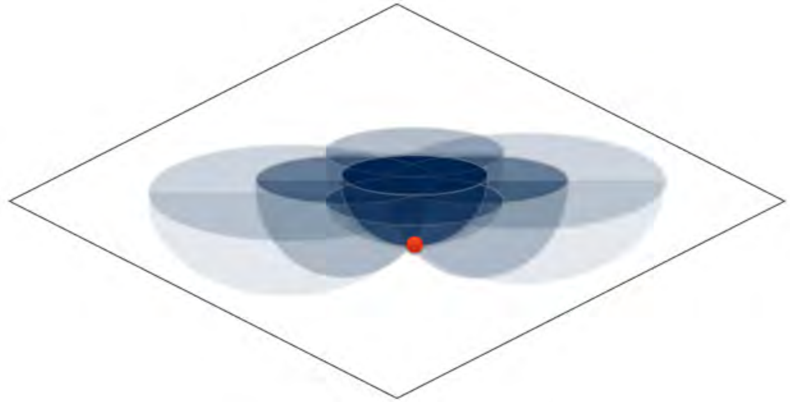


Inverse Radon transform:

$$\int \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2} \right) S(\rho_1|\rho_0)$$

$$\Rightarrow \quad \varepsilon(z, x)$$

Comments



☆ Relation to the Integral Geometry

d=2 : Czech et al., arXiv:1505.05515

Generalization to d > 2:

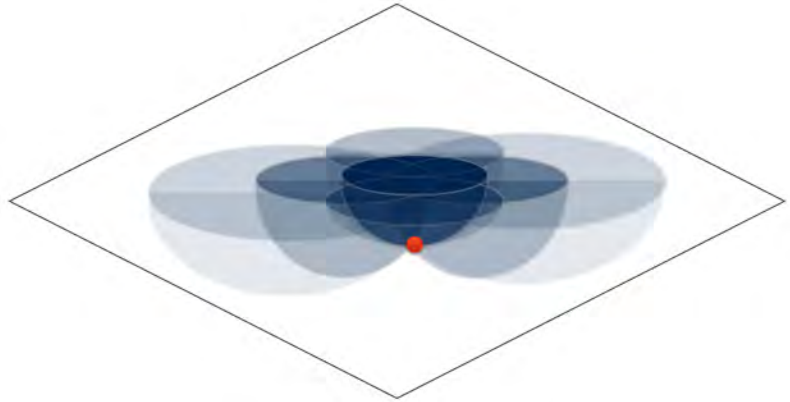
moduli space of points in the spatial section:

$$SO(1, d)/SO(d) = H\_d$$

moduli space of spatial circles on the boundary:

$$SO(1, d)/SO(1, d\text{-}1) = \text{de Sitter}\_d$$

Comments



☆ Relation to other approaches to bulk locality

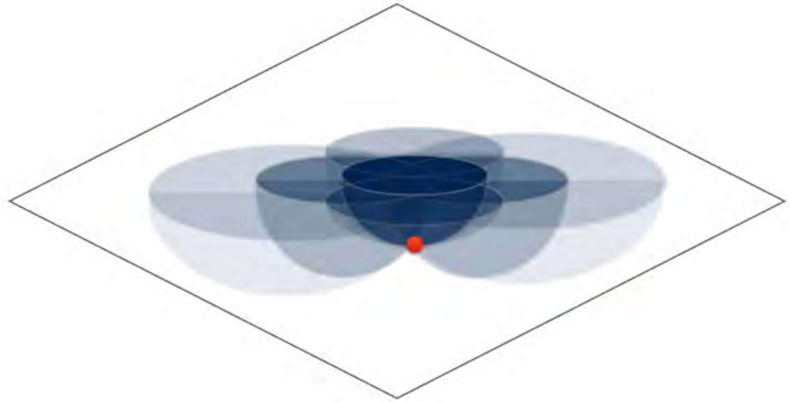
○ Use of Bulk-Boundary Propagators:

Kabat, et al., arXiv:1102.2910, 1311,3020, 1505.03755,
Heemskerk, et al., arXiv:1201.3664.

○ Use of Boundary States:

H. Verlinde, arXiv:1505.05069,
Miyaji, et al., arXiv:1506.01353

Summary



☆ Bulk stress tensor near boundary can be diagnosed by boundary entanglement entropy.

☆ Entanglement inequalities on the boundary are (integrated) positive energy conditions in the bulk.

☆ To do: Go deeper in the bulk interior.

The Structure of Holographic Entropy

based on arXiv:1505.07839
with N. Bao, S. Nezami, B. Stoica, J. Sully, M. Walter

Which CFT's have Gravity Duals?

Which CFT's have Gravity Duals?

Given that the bulk geometry reflect boundary entanglement, the criteria for holography may be stated in terms of entanglement properties.

holography = **hydrodynamic description of entanglement**

What are hydrodynamic criteria?

Entropy Inequalities

(Classical) Shannon Entropy:

There are infinite number of independent entropy inequalities for 4 or more random variables.

$\Longrightarrow$ applications to network coding theory

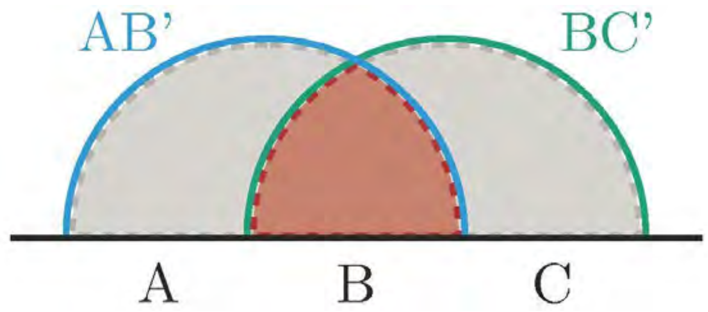
Zhang, Yeung (1998), Matus (2007)

(Quantum) von Neumann Entropy:

General properties are ***not known***.

Entropy inequalities for sub-classes of quantum systems are also of interest for information theorists (e.g., stabilizer states for quantum error correction)

Ryu-Takayanagi formula is known to satisfy 2 types of inequalities:



Strong Subadditivity

$$S_{AB} + S_{BC} \geqq S_B + S_{ABC}$$

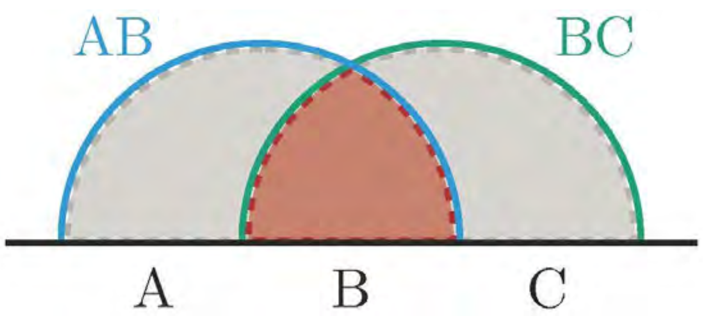
Headrick, Takayanagi, arXiv:0704.3719

Monogamy of Mutual Information

$$S_{AB} + S_{BC} + S_{AC} \geqq S_A + S_B + S_C + S_{ABC}$$

Hayden, Headrick, Maloney, arXiv:1107.2940

Ryu-Takayanagi formula is known to satisfy 2 types of inequalities:



Strong Subadditivity

$$S_{AB} + S_{BC} \geqq S_B + S_{ABC}$$

TRUE for any quantum systems

Monogamy of Mutual Information

$$S_{AB} + S_{BC} + S_{AC} \geqq S_A + S_B + S_C + S_{ABC}$$

NOT TRUE for general quantum systems

Monogamy of Mutual Information

$$S_{AB} + S_{BC} + S_{AC} \geqq S_A + S_B + S_C + S_{ABC}$$

☆ excludes Greenberger-Horne-Zeilinger states.

$$\frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes m} + |1\rangle^{\otimes m} \right)$$

☆ holographic Markov chain is trivial.

Are there other constraints on holographic entanglement entropy?

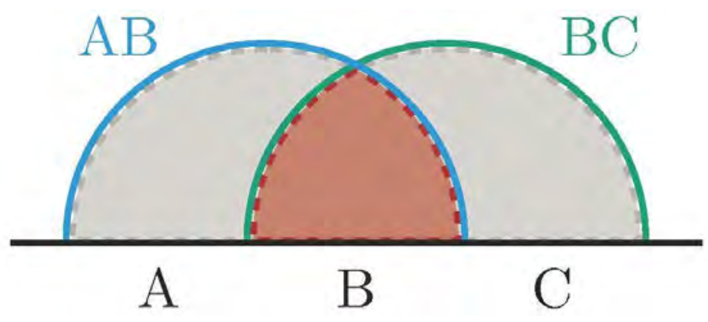
We prove the following properties of holographic states:

☆ The strong subadditivity and the monogamy of mutual information give the **complete set of inequalities** for 4 or less regions. This is in contrast to the situation for general quantum systems, where complete set is not known for 4 or more regions.

☆ New inequalities for 5 or more regions

☆ **For a fixed number of regions**, there are only **finite number of inequalities**. This is *known not to be the case* for the Shannon entropy and is *conjectured not to be the case* for the von Neumann entropy.

Entanglement entropy requires **UV cutoff**
= IR cutoff of the Ryu-Takayanagi formula



Strong Subadditivity

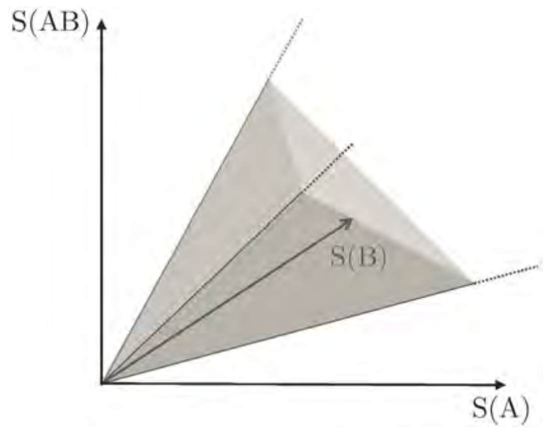
$$S_{AB} + S_{BC} \geqq S_B + S_{ABC}$$

Monogamy of Mutual Information

$$S_{AB} + S_{BC} + S_{AC} \geqq S_A + S_B + S_C + S_{ABC}$$

These inequalities are balanced,
and so are other inequalities we will derive.

Holographic Entropy Cone



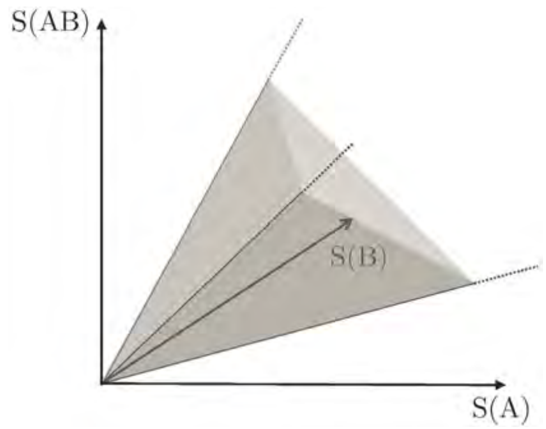
Entanglement entropies for n regions make a vector in (2^n - 1) dimensions.

☆ Entropy vectors make a **convex** cone.
(rescaling of the bulk metric rescale all entropies;
disjoint union of bulk manifolds gives a sum of entropies)

☆ Inside of the cone is populated by holographic constructions.

☆ The cone is **rational polyhedral**.
(with finite number of generators)

Holographic Entropy Cone



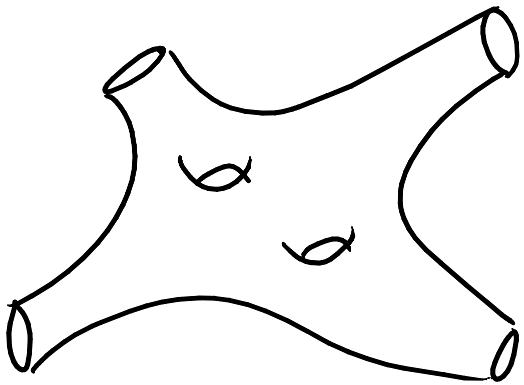
Entanglement entropies for n regions make a vector in (2^n - 1) dimensions.

☆ We have developed a combinatorial method to prove inequalities for holographic entanglement entropies.

☆ These inequalities make a complete set if all extremal rays are realized by holographic construction.

Holographic State

In semi-classical gravity, there is one quantum state for each phase space volume measured in the Planck constant.



For the vacuum Einstein equation with cosmological constant,

☆ constant scalar curvature

☆ vanishing extrinsic curvature on the spacelike section

$\Downarrow$

Time reflection symmetric solution allowing:

☆ analytic contiuation to Euclidean signature

☆ use of the Ryu-Takayanagi formula

Holographic State

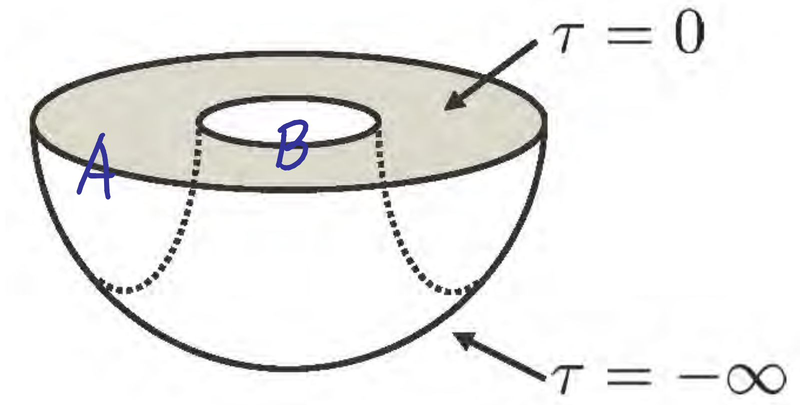
AdS_3 / CFT_2

$$ds^2 = -dt^2 + \cos^2 t \, d\Sigma^2$$

Riemann surface with constant curvature

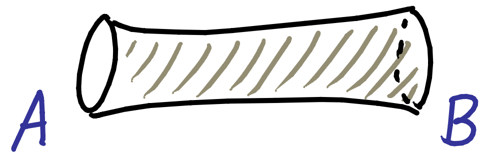
$$\Downarrow$$

$$ds^2 = d\tau^2 + \cosh^2 \tau \, d\Sigma^2$$

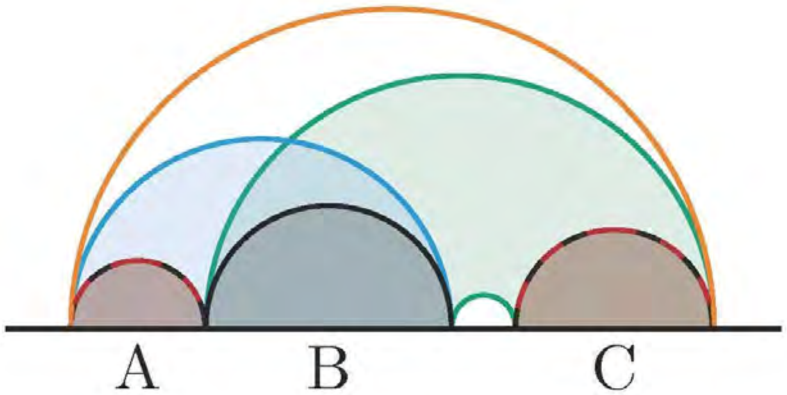


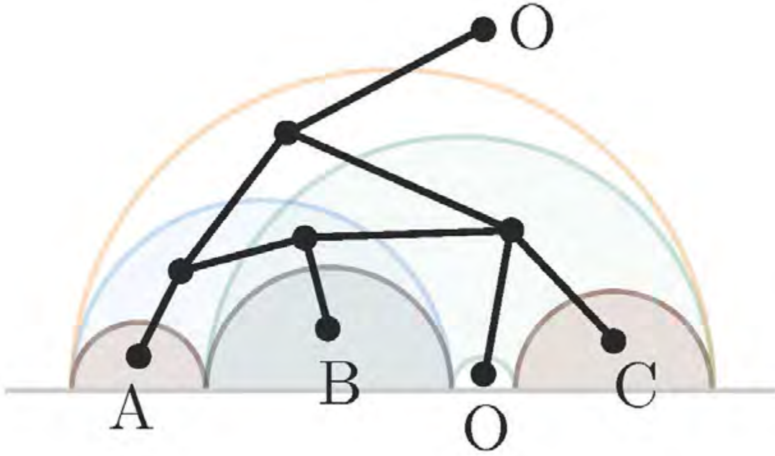
e.g. 2 asymptotic regions

Σ :



From Geometry To Combinatorics

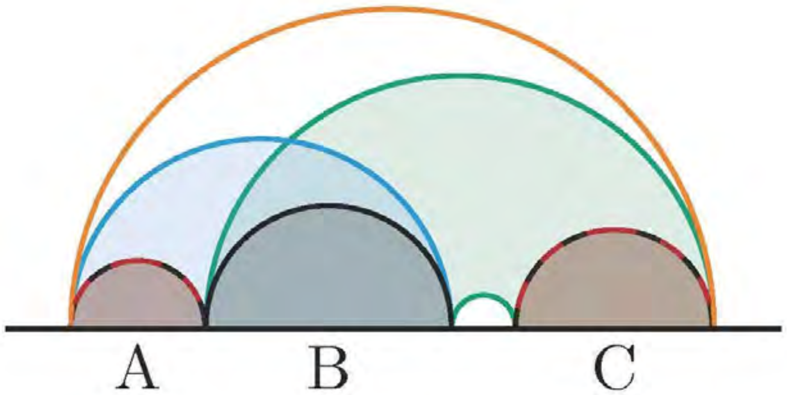


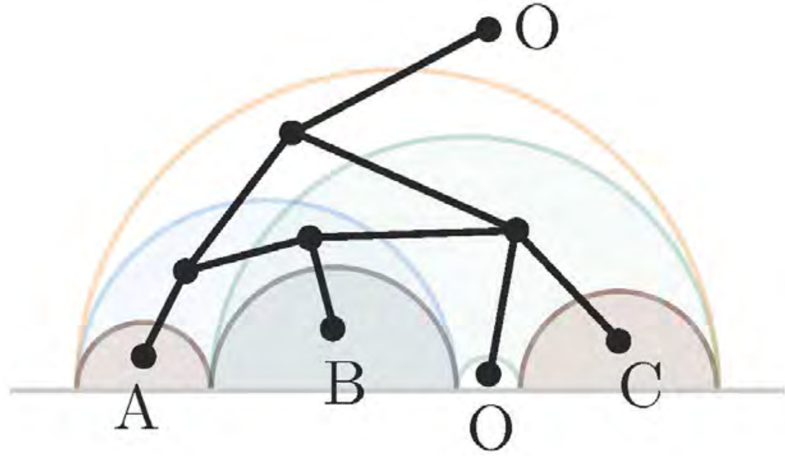


Left: Ryu-Tayayanagi surfaces cut the bulk geometry into pieces.

Right: ☆ Add one vertex for each connected bulk piece.
☆ Add an edge with weight = surface area.
☆ External vertices correspond to boundary regions and the extra vertex O is for the purifying region.

From Geometry To Combinatorics





Discrete entropy:

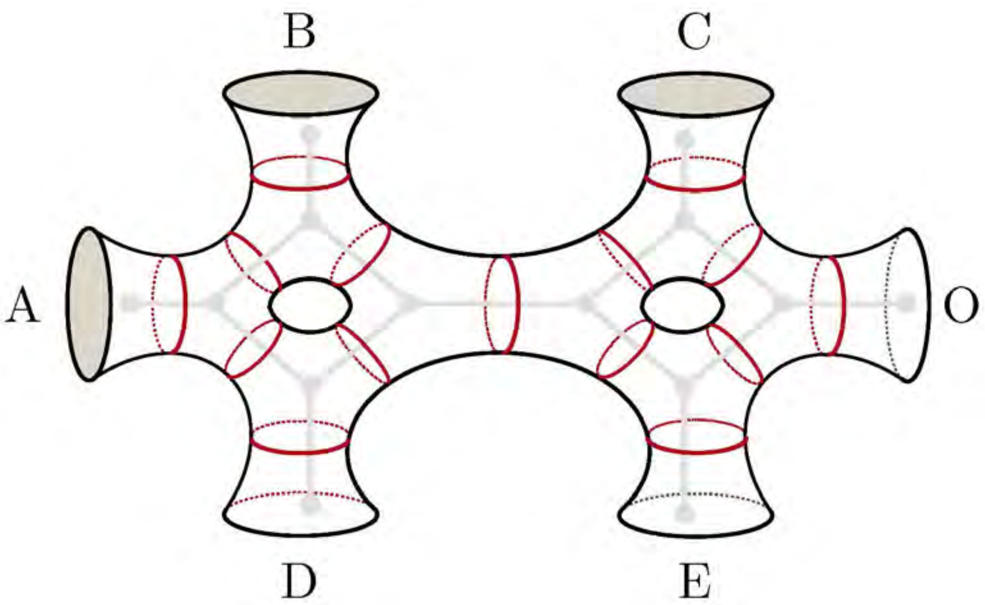
☆ Choose a set of external vertices.

☆ Consider all possible cuts that separate it from its complement.

☆ Minimize the weight.

Discrete entropy reproduces the Ryu-Takayanagi formula.

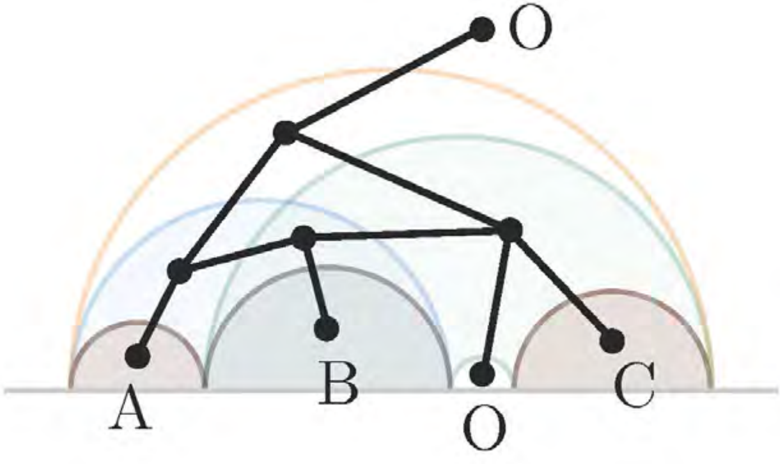
From Combinatorics To Geometry

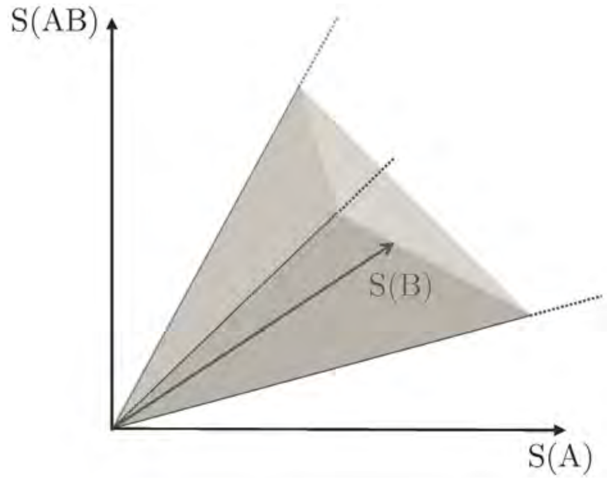


For each graph model,
one can construct a Riemann
surface with constant scalar
curvature such that:

☆ Boundaries correspond to external vertices of the graph.
☆ Discrete entropies are equal to holographic entropies.

Higher dimensional generalization is straightforward.



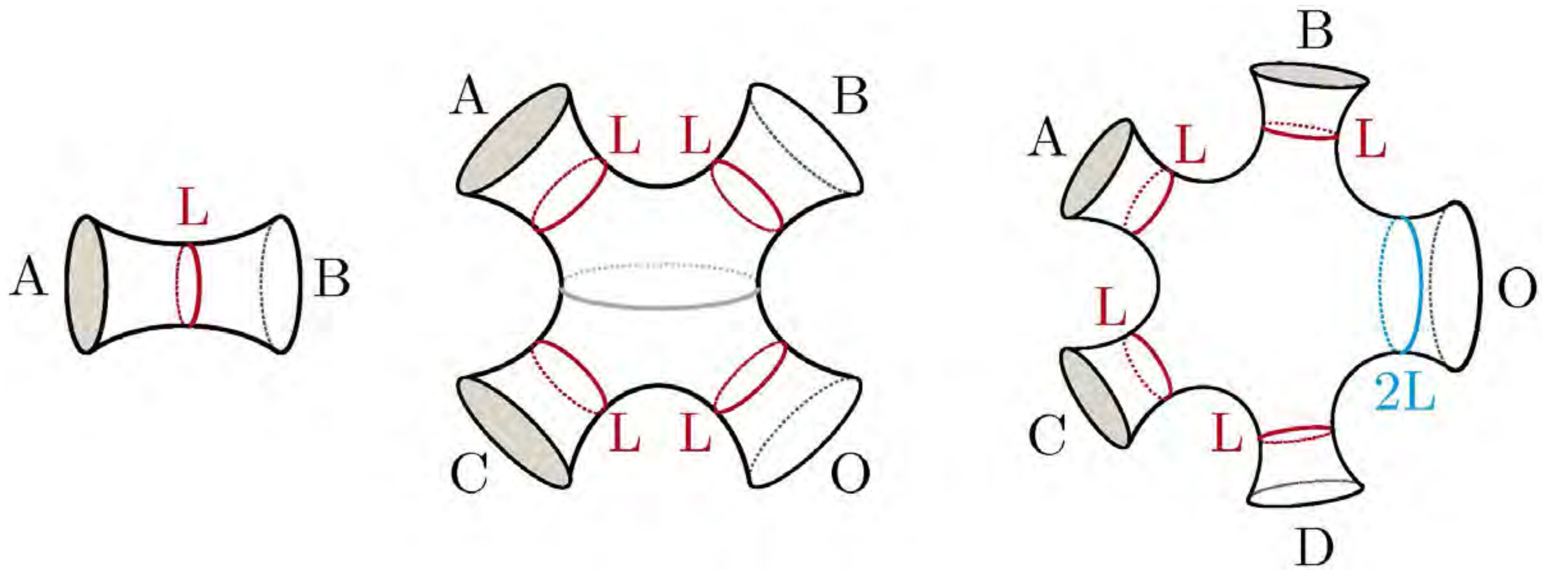


Any holographic entropy of n regions can be described by a graph on 2^{2^n-1} vertices.

The only variable data are edge weights.

Therefore, the holographic entropy cone is **rational polyhedral**.

For 2, 3, 4 regions, we found that all extremal rays generated by the strong subadditivity and the monogamy of mutual information are realizable holographically.



Therefore, they make the complete set of inequalities.

(Note: no genuine 3 party extremal rays)

The combinatorial model enables us to prove the following:

For n regions, $A_1, \cdots, A_n$,

$$\sum_{i=1}^{L} \alpha_i S(I_i) \geqq \sum_{j=1}^{R} \beta_i S(J_j) \qquad I_i, I_j \subset \{A_1, \cdots, A_n\} \quad \alpha_i, \beta_j > 0$$

is a true holographic inequality if there is a map f

$$f: \{0,1\}^L \to \{0,1\}^R$$

such that $\| x - x' \|_\alpha \geqq \| f(x) - f(x') \|_\beta$,

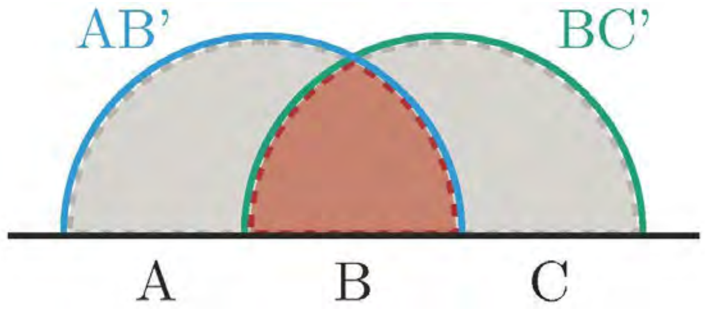
where

$$\| x \|_\alpha = \sum_{i=1}^{L} \alpha_i |x_i|$$

$$\| y \|_\beta = \sum_{j=1}^{R} \beta_j |y_j|$$

weighted Hamming norms

This formalizes and generalizes
the proof of the strong subadditivity.



For n regions, $A_1, \cdots, A_n$,

$$\sum_{i=1}^{L} \alpha_i S(I_i) \geqq \sum_{j=1}^{R} \beta_j S(J_j)$$

$I_i, I_j \subset \{A_1, \cdots, A_n\}$

$\alpha_i, \beta_j > 0$

is a true holographic inequality if there is a map f

$$f: \{0, 1\}^L \to \{0, 1\}^R$$

such that $\| x - x' \|_\alpha \geqq \| f(x) - f(x') \|_\beta$,

where

$$\| x \|_\alpha = \sum_{i=1}^{L} \alpha_i |x_i|$$

$$\| y \|_\beta = \sum_{j=1}^{R} \beta_j |y_j|$$

weighted Hamming norms

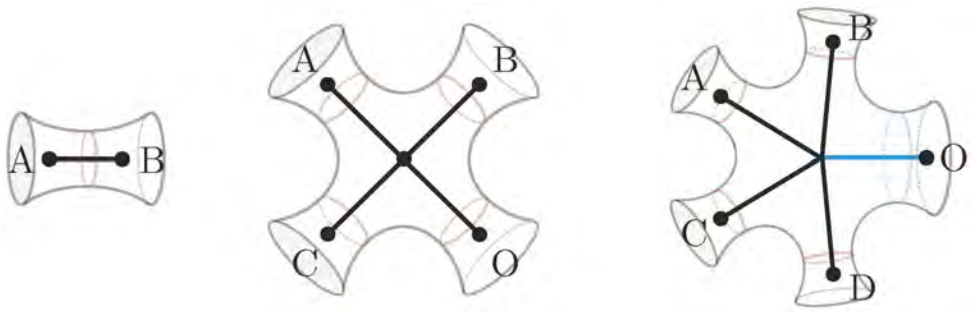
New holographic entanglement inequalities:

For any k and ℓ such that $n \geqq 2k+\ell$

$$\sum_{i=1}^{n} S(A_i \cdots A_{i+k+\ell-1}) \geqq \sum_{i=1}^{n} S(A_{i+\ell} \cdots A_{i+k+\ell-1}) + S(A_1 \cdots A_n)$$

For n=2, this gives the subadditivity.
For n=3, this gives the monogamy of mutual information.
For n > 3, this gives an infinite family of new inequalities.

Summary



☆ We found a **complete set** of holographic entanglement inequalities for 4 or less regions.

☆ We found a new infinite family of inequalities.

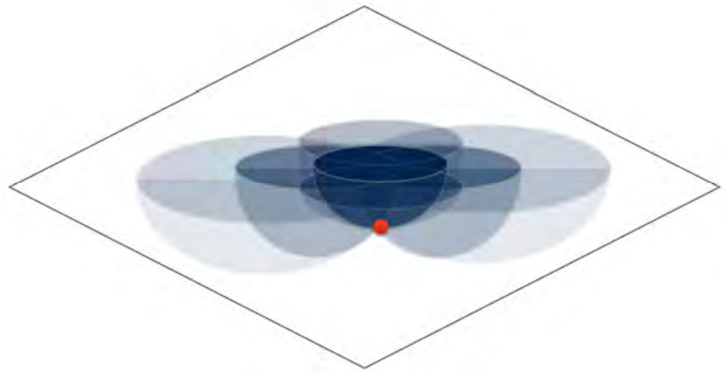
☆ For a fixed number of regions, there are only **finite number of inequalities**.

To do: Find a complete set of inequality for any number of regions.

Tomography from Entanglement

Entropy Inequalities in CFT

⇒ Energy Conditions in Gravity



The Structure of Holographic Entropy

Smooth Holographic Dual

⇒ Entropy Inequalities in CFT