Superconformal Killing Tensors

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Killing Vectors

Killing vector K^m:

$$\mathcal{L}_K g_{mn} = 0 \quad \Leftrightarrow \quad \nabla_{(m}K_{n)} = 0$$

$$g_{mn} = e_m^{\ a} e_{na} \quad \Rightarrow \quad \nabla_{(a}K_{b)} = 0$$

$$\Rightarrow \quad \nabla_a K_b = L_{ab} = -L_{ba}$$

Conformal Killing vector:

$$\nabla_{(m}K_{n)} = S g_{mn} \quad \Rightarrow \quad \nabla_{(a}K_{b)} = S\eta_{ab}$$

$$\Rightarrow \quad \nabla_a K_b = L_{ab} + S\eta_{ab} =: \tilde{L}_{ab}$$

Killing Tensors

This generalizes to n th rank Killing tensors (KTs):

$$\nabla_a K^{b_1 \ldots b_n} = L_a{}^{(b_1 \ldots b_n)}$$
$$\nabla_a K^{b_1 \ldots b_n} = \tilde{L}_a{}^{\{b_1 \ldots b_n\}}$$

where, in the conformal case, a trace is included on the first index pair and curly brackets denote traceless symmetrization. L is skew on the first index pair (plus trace for $\tilde{L}$) as above

Massless pointparticle $x^m(t)$:

$$\mathcal{L} = \tfrac{1}{2}\lambda \dot{x}^m e_m^{\ a} e_{na} \dot{x}^n$$
$$\Rightarrow p_a = \lambda \dot{x}^m e_{ma} \ , \quad \nabla_t p_a = 0 \ , \quad p^a p_a = 0 \ .$$

KTs provide integrals of the motion:

$$K = K^{a_1 \dots a_n} p_{a_1} \dots p_{a_n} \Rightarrow$$

$$\frac{dK}{dt} = \nabla_t K^{b_1 \dots b_n} p_{b_1} \dots p_{b_n} = \dot{x}^m e_m^{\ a} \nabla_a K^{b_1 \dots b_n} p_{b_1} \dots p_{b_n}$$
$$\propto L^{\{a\, b_1 \dots b_n\}} p_{b_1} \dots p_{b_n} p_a = 0 \ .$$

A *trivial* (*reducible*) second rank KT is formed from the metric and Killing vectors:

$$K^{red}_{mn} = k_0 g_{mn} + \sum k_{ij} K^{(i)}_m K^{(j)}_n$$

Aside: Nontrivial second rank KTs are useful in separating variables in Hamilton-Jacobi systems:

$$g^{mn}\partial_m S \partial_n S + a^2 = 0$$

Ex:

Kerr, `(Dolan and Swaminarayan '84)`,

⋮

Myers-Perry black hole, `(Chervonyi and Lunin '15)`

Schouten-Nijenhuis

The phase space of the massless particle carries a symplectic form:

$$\sigma = e^a(dp_a - \omega_a{}^b p_b)$$

The Poisson bracket of two functions on phase space is:

$$(f,g) = -\imath_{X_g}\imath_{X_f}\sigma$$

A CKT is a function K on phase space weakly (Poisson) commuting with the Hamiltonian.

The Poisson bracket of two such functions

$$K = K^{a_1 \dots a_n} p_{a_1} \dots . p_{a_n} \ , \quad L = L^{a_1 \dots a_m} p_{a_1} \dots . p_{a_m} \ ,$$

yields a new CKT,

$$(K, L) = [K, L]^{a_1 \dots a_{n+m-1}} p_{a_1} \dots p_{a_{n+m-1}}$$

where the Schouten-Nijenhuis bracket $[K, L]$ is defined to be

$$\begin{aligned} [K, L]^{a_1 \dots a_q} := \quad & m L^{\{a_1 \dots a_{m-1}|b|} \nabla_b K^{a_m \dots a_q\}} \\ & - n K^{\{a_1 \dots a_{n-1}|b|} \nabla_b L^{a_n \dots a_q\}} \end{aligned}$$

This equips the space of CKTs with a Lie algebra structure.

Superspace

$$Z^M = (x^m, \theta^\mu)$$

$$(E^a, E^\alpha) = E^A = dz^M E_M{}^A$$

$$T^A = DE^A := dE^A + E^B \Omega_B{}^A$$

$$R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B$$

$$T_{\alpha\beta}{}^a = -i(\Gamma^a)_{\alpha\beta} \; ,$$

$$T_{\alpha[bc]} = 0 \; , \quad (\Gamma_a)^{\alpha\beta} T_{\alpha b}{}^a = 0$$

$$T_{ab}{}^c = 0$$

(The last two lines are conventional constraints.)

SuperKilling

SKV:

$$\nabla_A K^B + K^C T_{CA}{}^B = L_A{}^B$$

where $L_A{}^B$ is an element of the structure group Lie algebra.

$$L \to \tilde{L} \,, \quad (....) \to \{....\} \,, \quad \text{for some suitably defined } \{....\},$$

.

and where the scale comes with a non-zero $\tilde{L}_a{}^\beta$

SCKT:

$$K^{a_1 \dots a_n}$$

With our torsion constraints, the component relation

$$\nabla_\alpha K^{b_1 \dots b_n} - inK^{\{b_1 \dots b_{n-1}\gamma}(\Gamma^{b_n\}})_{\gamma\alpha} = 0$$

generates the full SCKT, (all the superfield components) using Ricci and Bianchi identities.

This relation is a direct generalisation of an alternative definition of a SCKV K as preserving the odd tangent bundle

$$< [E_\alpha, K], E^b >= 0$$

using the canonical pairing.

Superparticle:

$$z^M(t) = (x^m(t), \theta^\mu(t))$$

$$\mathcal{L} = \tfrac{1}{2}\lambda \dot{z}^a \dot{z}_a := \tfrac{1}{2}\lambda \dot{z}^M E_M{}^a \, \dot{z}^N E_{Na}$$

$$\Rightarrow p_a = \lambda \dot{z}_a \, , \quad p^2 = 0$$

$$\nabla_t p_a = 0 \, , \quad \dot{z}^\beta (\Gamma \cdot p)_{\beta\alpha} = 0$$

The action is invariant under the fermionic Siegel symmetry (κ symmetry).

The SCKTs provide integrals of the superparticle motion.

$K = K^{a_1 \dots a_n} p_{a_1} \dots p_{a_n} \Rightarrow$

$$\frac{dK}{dt} = \left(\dot{z}^a \nabla_a K^{b_1 \dots b_n} + \dot{z}^\alpha \nabla_\alpha K^{b_1 \dots b_n} \right) p_{b_1} \dots p_{b_n}$$

If $K^{a_1 \dots a_n}$ is symmetric and traceless, the right hand side will vanish (on shell) precisely when

$$\nabla_\alpha K^{b_1 \dots b_n} - inK^{\{b_1 \dots b_{n-1}\gamma}(\Gamma^{b_n\}})_{\gamma\alpha} = 0 \ ,$$

i.e., when $K^{a_1 \dots a_n}$ is a SCKT.

On shell K is also κ-invariant.

Super Schouten-Nijenhuis

The phase space of the massless superparticle carries a "symplectic" form (`Howe and Townsend '91`).

$$\Sigma = E^a Dp_a + T^a p_a$$

The corresponding Poisson bracket reads

$$(f,g) = \left(\tilde{E}_a f \partial^a g - f \leftrightarrow g\right) + i\eta \tilde{E}_\alpha f \frac{(\Gamma \cdot p)^{\alpha\beta}}{p^2} \tilde{E}_\beta g$$

with

$$\tilde{E}_A := E_A{}^M \left(\partial_M + \Omega_{Mb}{}^c p_c (\partial/\partial p_b)\right)$$

Not invertible on shell, but nonsingular for bracket between SCKTs saturated with momenta.

A function K weakly commuting with the Hamiltonian is not automatically a SCKT in this case. We have to impose the condition

$$\nabla_\alpha K^{b_1\dots b_n} - inK^{\{b_1\dots b_{n-1}\gamma}(\Gamma^{b_n\}})_{\gamma\alpha} = 0 \ ,$$

as an additional constraint. The Poisson bracket of two such functions will then yield a new SCKT, whose components are given by a super Schouten-Nijenhuis bracket:

$$[K,L]^{a_1\dots a_q} := mL^{\{a_1\dots a_{m-1}|b|}\nabla_b K^{a_m\dots a_q\}} - nK^{\{a_1\dots a_{n-1}|b|}\nabla_b L^{a_n\dots a_q\}} \\ -imnK^{\{a_1\dots a_{n-1}\gamma}(\Gamma^{a_n})_{\gamma\delta}L^{a_{n+1}\dots a_q\}\delta}$$

This equips the space of SCKTs with a Lie algebra structure.

Components in Flat Superspace

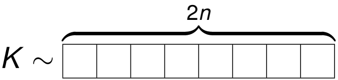
3*D* Killing vector:

$$D_\alpha K^a - iK^\beta (\gamma^a)_{\beta\alpha} = 0$$
$$\Rightarrow K^\alpha = \frac{i}{3}(\gamma_a)^{\alpha\beta} D_\beta K^a$$
$$(\quad K_{ij} := D_{\alpha i} D_{\beta j} K^{\alpha\beta} \qquad N > 1).$$

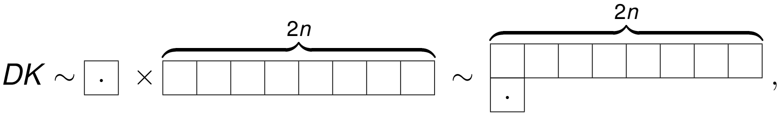
corresponding to the various transformation parameters. Here $K^a \sim K^{\alpha\beta}$.
Conformal Killing equations:

$$\partial_{\{a} K_{b\}} = 0 \, ,$$
$$D^{(\alpha} K^{\beta\gamma)} = 0 \, ,$$
$$(\gamma_{\{a} \partial_{b\}} K)^\alpha = 0 \, .$$

For higher rank SCKTs it is advantageous to use Young tableaux in analyzing the component content.



The constraint on DK is



After m steps, $m \leqslant 2n$ and $m \leqslant N$;

$$D^m K \sim \overbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & \; & \; & \; \\ \hline \cdot & \cdot & \cdot & \cdot & m \\ \cline{1-5} \end{array}}^{2n} ,$$

or, in indices,

$$D^m K \sim (D^m K)^{\alpha_1 \ldots \alpha_{2n-m}}_{i_1 \ldots i_m} .$$

Killing equations:

$$\partial^{(\alpha_1 \alpha_2} K^{\alpha_3 \ldots \alpha_{2n+2})} = 0$$

$$\partial^{(\alpha_1 \alpha_2} K^{\alpha_3 \ldots \alpha_{2n-m+2})}_{i_1 \ldots i_m} = 0$$

- A similar analysis of the components is done in $D = 4, 5, 6$ and $D = 10$ ordinary flat superspace, for arbitrary N.
- For even N and $D = 3, 4$ and $D = 6$ we also introduce SCKTs in Analytic Superspace, which resembles the two component spinor description of $4D$ Minkowski space.

Algebra

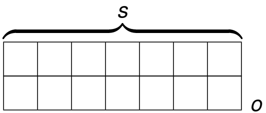
CKTs are related to higher spin algebras, e.g., in the AdS/CFT correspondence. This is explicit in the Eastwood algebras describing the symmetries of the Laplacian: (Eastwood '05)

$$\Delta\mathcal{D} = \mathcal{O}\Delta$$

$$\Rightarrow \mathcal{D} = K^{b_1\dots b_n}\nabla_{b_1}\dots\nabla_{b_n} + \textit{lower}$$

$$\nabla^{\{a}K^{b_1\dots b_n\}} = 0$$

Using an explicit representation of the conformal Lie algebra on $\mathbb{R}^{n+2}$, one finds the $\mathcal{D}$s as differential operators. This results in the rank s CKTs forming an irrep of $\mathfrak{so}(n+1,1)$:



where o denotes the tracefree part.
The algebra $\mathcal{A}_n$ is then shown to be isomorphic to $\mathfrak{U}(\mathfrak{so}(n+1,1))$, modulo certain relations, or to $\bigotimes \mathfrak{so}(n+1,1)/I$, where I is the Joseph ideal.

The generalization to SCKTs is not immediate, since there is no similar conformal embedding available.

However, we can give the components of SCKTs using Young Tableaux, as shown for $3D$ previously. There the starting point was

$$K \sim \overbrace{\square\square\square\square\square\square\square\square}^{2n}$$

a $2n$ symmetric tensor of $\mathfrak{sl}(2)$ satisfying a constraint, and the superfield components were derived by repeatedly applying spinorial derivatives. We may reinterpret *the same* Young Tableau as a Super Young Tableau encoding representation of the *super-conformal* algebra $\mathfrak{spo}(2|N)$. Similarily the SCKT components in $D = 4, 6$ may be represented as Super Young Tableaux. This will be the starting point for the construction of Super Eastwood Algebras.

Further work

- SCKTs components in curved superspace.
- Explicit Higher Spin super Eastwood algebras.
- Generalizing the work of Mikhailov to the supercase, again using Analytic Superspace.
- .....