Gravity waves from Kerr/CFT

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based on...

1401.3746 with: A. Strominger
1403.2797 with: S. Hadar, A. Strominger
1504.07650 with: S. Hadar, A. Strominger
1506.08496 with: S. Gralla, N. Warburton
The extreme Kerr throat, NHEK, near-NHEK, etc

- The Kerr metric in Boyer-Lindquist coordinates ($G = c = \hbar = 1$):

\[
\begin{align*}
\text{ds}^2 &= -\frac{\Delta}{\hat{\rho}^2} \left( d\hat{t} - a \sin^2 \theta d\hat{\phi} \right)^2 + \frac{\sin^2 \theta}{\hat{\rho}^2} \left( (\hat{r}^2 + a^2) d\hat{\phi} - ad\hat{t} \right)^2 \\
&\quad + \frac{\hat{\rho}^2}{\Delta} d\hat{r}^2 + \hat{\rho}^2 d\theta^2 ,
\end{align*}
\]

\[
\Delta = \hat{r}^2 - 2M\hat{r} + a^2 , \quad \hat{\rho}^2 = \hat{r}^2 + a^2 \cos^2 \theta .
\]

- Mass $M$, angular momentum $J = aM \leq M^2$.
- Near extremality there is a very long throat

\[\rightarrow\] decoupling metrics: NHEK, near-NHEK
The extreme Kerr throat, NHEK, near-NHEK, etc

\[ \kappa = \sqrt{1 - \frac{a^2}{M^2}} \]

\[ r = \frac{\hat{r} - r_+}{r_+} \]

\[ r = 0 \]
The extreme Kerr throat, NHEK, near-NHEK, etc

- **NHEK:**

\[
ds^2 = 2M^2 \Gamma(\theta) \left[ -R^2 dT^2 + \frac{dR^2}{R^2} + d\theta^2 + \Lambda(\theta)^2 (d\Phi + RdT)^2 \right]
\]

- **near-NHEK:**

\[
ds^2 = 2M^2 \Gamma(\theta) \left[ -r(r + 2\kappa) dt^2 + \frac{dr^2}{r(r + 2\kappa)} + d\theta^2 + \Lambda(\theta)^2 (d\phi + (r + \kappa) dt)^2 \right]
\]

- **Isometry group:** \(SL(2, \mathbb{R})_R \times U(1)_L\). \(\partial_t\) in \(SL(2, \mathbb{R})_R\), \(\partial_\phi\) is \(U(1)\)

- **Asymptotic symmetry group:** Virasoro (L or R), Virasoro-Kac-Moody, …
The extreme Kerr throat, NHEK, near-NHEK, etc
The Kerr/CFT correspondence

‘strong’ Kerr/CFT
the conjecture that quantum gravity in the near horizon region of a near-extreme Kerr is dual to a (warped) 2D CFT
▶ Relevant for quantum black hole puzzles (e.g. $S_{BH} = S_{Cardy}$)
▶ Bottom-up: only some dictionary entries known (e.g. $c, h$)

‘weak’ Kerr/CFT
the fact that gravitational dynamics in the near horizon region of a near-extreme Kerr are constrained by an infinite-dimensional conformal symmetry
▶ Powerful CFT techniques for near-horizon gravitational physics.
▶ Suffices for interesting questions in observational astronomy.
Extreme-Mass-Ratio-Inspirals (EMRIs)

- A primary gravity waves source for eLISA mission
- Slow vs fast plunges
- So far people do PN approximation or numerics
The plan

- Gravity analysis: Solve linearized Einstein equation with source and compute particle number flux at the horizon
  \[ \mathcal{F}^{\text{gravity}} = \frac{dN}{dt} \]

- CFT analysis: Identify the source deformation of the CFT and compute the transition rate out of the vacuum state
  \[ \mathcal{R}^{\text{CFT}} = \frac{dP}{dt} \]

- Glue asymptotically flat region back and find the flux at future null infinity.
NHEK + circle gravity analysis
NHEK + circle: gravity analysis

In NHEK, circular orbits at any radius $R_0$ are marginally stable:

$$ R(T) = R_0 $$
$$ \Phi(T) = -\frac{3}{4}R_0 T + \Phi_0 $$

Consider coupling to a scalar field:

$$ \Box \Psi = g R_0 \delta(R - R_0) \delta(\theta - \pi/2) \delta(\Phi + \frac{3}{4}R_0 T) $$

Killing flow along $\chi = \partial_T - \frac{3}{4}R_0 \partial_\Phi$, so expand accordingly:

$$ \Psi = \sum_{\ell, m} e^{im(\Phi + \frac{3}{4}R_0 T)} S_\ell(\theta) R_{\ell m}(R) $$

where the $S_\ell$'s are spheroidal harmonics obeying $\mathcal{L}^{(2)}_\theta S^m_\ell = -K^m_\ell S^m_\ell$
NHEK + circle: gravity analysis

- Radial equation:

\[
\partial_R \left( R^2 \partial_R R_{\ell m} \right) + \left( 2m^2 - K_\ell + \frac{2\Omega m}{R} + \frac{\Omega^2}{R^2} \right) R_{\ell m} = \frac{M^2}{2\pi} g S_\ell (\pi/2) R_0 \delta(R - R_0)
\]

where \( \Omega = -\frac{3}{4} mR_0 \).

- Homogeneous solutions are confluent hypergeometrics, e.g. Whittakers

\[
W_{im, h-1/2}(-2i\Omega/R), \quad M_{im, h-1/2}(-2i\Omega/R)
\]

where

\[
h \equiv \frac{1}{2} + \sqrt{\frac{1}{4} + K_\ell - 2m^2}
\]

- Boundary conditions:

  @ \( R = 0 \): \( R^{-im} e^{i\Omega/R} \) (ingoing), \( R^{im} e^{-i\Omega/R} \) (outgoing)

  @ \( R = \infty \): \( R^{h-1} \) (Dirichlet), \( R^{-h} \) (Neumann), \( P R^{h-1} + Q R^{-h} \) (“leaky”).
NHEK + circle: gravity analysis

With ingoing boundary conditions at the horizon and Neumann or “leaky” at the boundary, we find that, for real $h$, to leading order, the Klein-Gordon particle number flux down the horizon is given by:

$$\mathcal{F}_{\ell m}^{\text{gravity}} = - \int \sqrt{-g} J_{KG}^R \, d\theta \, d\Phi$$

$$= \frac{g^2 M^6}{12 \pi^2} S_{\ell} (\pi/2) \, R_0 \, m^{-1} \, e^{-\pi m} \frac{|\Gamma(h + im)|^2}{|\Gamma(2h)|^2} \, |M_{im, h-1/2(3im/2)}|^2$$

Summary

- In NHEK there are marginally stable circular orbits at every radius $R_0$.
- The wave equation may be solved analytically in terms of Whittakers.
- With ingoing b.c. at the horizon, we calculate the particle number flux.
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**Summary**

- In NHEK there are marginally stable circular orbits at every radius $R_0$.
- The wave equation may be solved analytically in terms of Whittakers.
- With ingoing b.c. at the horizon, we calculate the particle number flux.
Q: What is the dual of this gravity situation?

A: Holographically driven CFT by external source at frequency $\Omega$:

$$S = S_{CFT} + \sum_{\ell} \int d\Phi dT \, J_\ell(\Phi, T) O_\ell(\Phi, T)$$

with

$$J_\ell(\Phi, T) = \sum_m J_{\ell m} e^{im(\Phi + \frac{3}{4}R_0 T)}$$

Kerr/CFT dictionary: operator $O$ dual to bulk field $\Psi$, with weight $h$

What is $J_{\ell m}$? Bulk solution

$$R_{\ell m} = X \Theta(R_0 - R) R^{in}(R) + Z \Theta(R - R_0) R^{out}(R)$$

Extend inner solution:

$$R^{ext}_{\ell m} = X R^{in}(R)$$

Read off $J_{\ell m}$ from leading term:

$$R^{ext}_{\ell m} \rightarrow J_{\ell m} R^{h-1} + \ldots \quad R \rightarrow \infty$$
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What is $J_{\ell m}$? Bulk solution

$$R_{\ell m} = \chi \Theta(R_0 - R) R^{in}(R) + Z \Theta(R - R_0) R^{out}(R)$$

Extend inner solution:

$$R_{\ell m}^{ext} = \chi R^{in}(R)$$

Read off $J_{\ell m}$ from leading term:

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$$R_{\ell m}^{ext} \rightarrow J_{\ell m} R^{h-1} + \ldots \quad R \rightarrow \infty$$
NHEK + circle: CFT analysis

Applying Fermi’s Golden Rule, the transition rate out of the initial state $|i\rangle$ to any final state $|f\rangle$ is given by the Fourier transform of the two-point function:

$$
R_{\ell m}^{CFT} = 2\pi |J_{\ell m}|^2 \int d\Phi dT \, e^{-im(\Phi + \frac{3}{4}R_0 T)} \langle \mathcal{O}^\dagger(\Phi, T) \mathcal{O}(0, 0) \rangle \\
= \mathcal{C}_\mathcal{O}^2 \left(\frac{2\pi}{2h-1}\right)^2 \frac{(3R_0/4)^{2h-1}}{\Gamma(2h)^2} |J_{\ell m}|^2 m^{2h-1} e^{-\pi m} |\Gamma(h + im)|^2
$$

Plugging in $J_{\ell m}$ and normalizing the operators with $\mathcal{C}_\mathcal{O} = 2^{2h-1}(2h - 1)M/2\pi$ we find:

$$
R_{\ell m}^{CFT} = \frac{g^2 M^6}{12\pi^2} S_{\ell}(\pi/2) R_0 m^{-1} e^{-\pi m} \frac{|\Gamma(h + im)|^2}{|\Gamma(2h)|^2} |M_{im,h-1/2}(3im/2)|^2 = \mathcal{F}_{\ell m}^{\text{gravity}}
$$

Summary

- Dual calculation: holographically driven CFT with source read off from extension of inner solution.
- CFT transition rate matches perfectly gravity particle number flux.
Applying Fermi’s Golden Rule, the transition rate out of the initial state $|i\rangle$ to any final state $|f\rangle$ is given by the Fourier transform of the two-point function:

$$R_{CFT}^{\ell m} = 2\pi |J_{\ell m}|^2 \int d\Phi dT \ e^{-im(\Phi + \frac{3}{4}R_0 T)} \langle O^\dagger(\Phi, T)O(0, 0) \rangle$$

$$= C_O^2 \frac{(2\pi)^2 (3R_0/4)^{2h-1}}{\Gamma(2h)^2} |J_{\ell m}|^2 m^{2h-1} e^{-\pi m} |\Gamma(h + im)|^2$$

Plugging in $J_{\ell m}$ and normalizing the operators with $C_O = 2^{2h-1}(2h-1)M/2\pi$ we find:

$$R_{CFT}^{\ell m} = \frac{g^2 M^6}{12\pi^2} S_\ell(\pi/2) R_0 m^{-1} e^{-\pi m} \frac{|\Gamma(h + im)|^2}{|\Gamma(2h)|^2} |M_{im,h-1/2}(3im/2)|^2 = F_{gravity}^{\ell m}$$

Summary

- Dual calculation: holographically driven CFT with source read off from extension of inner solution.
- CFT transition rate matches perfectly gravity particle number flux.
Gluing back asymptotically flat region

e.g.
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Matched Asymptotic Expansions

How it’s done: for a suitable parameter $k \ll 1$, near solution for $x \ll 1$, far solution for $x \gg k$, match solutions in $k \ll x \ll 1$. 
Gluing back asymptotically flat region

For NHEK + circle problem:

▶ $k = 4M\left(\hat{\omega} - \frac{m}{2M}\right) = -\frac{3}{2} mR_0$

▶ near solution is the NHEK solution

▶ leaky boundary conditions

In this way we get a full solution everywhere which may be used to calculate explicitly desired observables, e.g. the outgoing radiation flux at future null infinity:

\[
\dot{E}_\infty = \frac{g^2 M^4}{144 \pi^2} S_\ell^2 \left(\frac{\pi}{2}\right) \left| W_{im,h-\frac{1}{2}} \left(\frac{3im}{2}\right)\right|^2 (3m^2 R_0/2)^{2\Re[h]} m^{-2} e^{\pi|m|} \times
\]

\[
\left|2h - 1\right|^2 \left|\Gamma(h - im)\right|^4 / \left|\Gamma(2h)\right|^4 \times
\]

\[
\left|1 - (3m^2 R_0/2)^{2h-1} \frac{\Gamma(1-2h)^2}{\Gamma(2h-1)^2} \frac{\Gamma(h-im)^2}{\Gamma(1-h-im)^2}\right|^2
\]
NOW THE MAGIC STARTS!
Magic #1: Near extremal ISCO

Near extremal ISCO is at:

\[ \frac{\hat{r} - r_+}{r_+} = 2^{1/3} \kappa^{2/3} + \mathcal{O}(\kappa) \]

Q: How to get the leading order fluxes in this case?

A: Just put \( R_0 = 2^{1/3} \kappa^{2/3} \) into the extremal formulae!

E.g. plug in:

\[
\dot{E}_\infty = \frac{g^2 M^4}{72 \pi^2} S_\ell^2 (\pi/2) \left| W_{im, h - \frac{1}{2}} (3im/2) \right|^2 \left( 3m^2 R_0 / 2 \right)^{2\text{Re}[h]} m^{-2} e^{\pi|m|} \times \\
\times \left| 2h - 1 \right|^2 |\Gamma(h - im)|^4 / |\Gamma(2h)|^4 \\
\times \left| 1 - (3m^2 R_0 / 2)^{2h - 1} \frac{\Gamma(1-2h)^2}{\Gamma(2h-1)^2} \frac{\Gamma(h-im)^2}{\Gamma(1-h-im)^2} \right|^2
\]
Near extremal ISCO is at:

\[ \frac{\hat{r} - r_+}{r_+} = 2^{1/3} \kappa^{2/3} + O(\kappa) \]

Q: How to get the leading order fluxes in this case?
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E.g. plug in:

\[ \dot{E}_\infty = \frac{g^2 M^4}{72 \pi^2} S_\ell^2 \left( \frac{\pi}{2} \right) \left| W_{im, h - \frac{1}{2}} \left( 3im/2 \right) \right|^2 \left( 3m^2 R_0 / 2 \right)^{2 \Re[h]} m^{-2} e^{\pi |m|} \times \]

\[ \times \left| 2h - 1 \right|^2 \left| \Gamma (h - im) \right|^4 / \left| \Gamma (2h) \right|^4 \]

\[ \times \frac{\left| 1 - \left( 3m^2 R_0 / 2 \right)^{2h - 1} \Gamma \left( 1 - 2h \right)^2 \Gamma (h - im)^2 \right|^2}{\Gamma (2h - 1)^2 \Gamma (1 - h - im)^2} \]
Magic #1: Near extremal ISCO

E.g. this uncovers that what used to look like numerical noise is actually oscillatory behavior in the $\kappa \to 0$ limit
Magic #2: Plunges

Slow plunge in near-NHEK

- Slow plunge orbit that spirals off ISCO

\[ t(r) = \frac{1}{2\kappa} \ln \frac{1}{r(r + 2\kappa)} + t_0 \]

\[ \phi(r) = \frac{3r}{4\kappa} + \frac{1}{2} \ln \frac{r}{r + 2\kappa} + \phi_0 \]

falls into the future horizon in the near-NHEK metric

\[ ds^2 = 2M^2 \Gamma(\theta) \left[ -r(r + 2\kappa)dt^2 + \frac{dr^2}{r(r + 2\kappa)} + d\theta^2 + \Lambda(\theta)^2 (d\phi + (r + \kappa)dt)^2 \right] \]

Q: How to get the solutions in this case?

A: Do a conformal transformation to map to the NHEK + circle problem!
Magic #2: Plunges

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- Slow plunge orbit that spirals off ISCO

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\begin{align*}
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\]

- Q: How to get the solutions in this case?
- A: Do a conformal transformation to map to the NHEK + circle problem!
Slow plunge in near-NHEK

- **Bulk diffeomorphism:**

\[
\begin{align*}
T &= -e^{-\kappa t} \frac{r + \kappa}{\sqrt{r(r + 2\kappa)}} \\
R &= \frac{1}{\kappa} e^{\kappa t} \sqrt{r(r + 2\kappa)} \\
\Phi &= \phi - \frac{1}{2} \ln \frac{r}{r + 2\kappa}
\end{align*}
\]

- **Boundary conformal transformation:**

\[
\begin{align*}
T &= -e^{-\kappa t} \\
\Phi &= \phi
\end{align*}
\]
Magic #2: Plunges

Slow plunge in near-NHEK
Magic #2: Plunges

Fast plunge that spirals off an eccentric last stable orbit