Holographic Two-Point Functions of Conformal Gravity

A.Naseh

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Based on:

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- Conformal gravity action is given by a Weyl squared term in the form
$$S_{CG} = \alpha_{CG} \int d^4x \sqrt{|g|} C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \, ,$$
which is invariant under the Weyl transformation $g_{\mu\nu} \to e^{2\Omega(x)} g_{\mu\nu}$.

- α_{CG} is a dimensionless coupling constant.

- A positive definite Euclidean action and an acceptable Newtonian limit is possible only for $\alpha_{CG} > 0$. ($\alpha_{CG} = 1$) (B. Hasslacher and E. Mottola, 1981).

- The equation of motion extracted from this action is known as the Bach equation (Bach, 1921).
$$\left(\nabla^\delta \nabla_\gamma + \frac{1}{2} R^\delta_\gamma \right) C^\gamma_{\alpha\delta\beta} = 0.$$

- It is well-known that the solutions of Einstein gravity, and as a specific example the AdS space-time, are also solutions of Bach equation.
- Moreover because of the the higher derivative nature of CG, the Bach equation admit also solutions which are not Einstein spaces.
- The most general spherically symmetric solution of CG is given by the line-element (R. J. Riegert,1984)

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2,$$

where $d\Omega^2$ is the line element of the round 2-sphere and

$$A(r) = \sqrt{1 - 12aM} - \frac{2M}{r} - \Lambda r^2 + 2ar.$$

- In contrast with Einstein gravity which is not renormalizable (M. H. Goroff and A. Sagnotti, 1985),
- CG is power-counting renormalizable (K. S. Stelle, 1977)
  - in fact, asymptotically free (J. Julve and M. Tonin, 1978, E. S. Fradkin and A. A. Tseytlin, 1982)
  - and therefore is considered as a possible UV completion of gravity (S. L. Adler, 1983).
- CG is emerged in the Gauge/Gravity duality as a counter term (M. Henningson and K. Skenderis – H. Liu and A. A. Tseytlin, 1998).
- CG also arises in twistor-string theory (N. Berkovits and E. Witten, 2004).

- In recent years CG has been in the center of attention. It is used to explain galactic rotation curves without need for dark matter (P. D. Mannheim, 2010)
- CG is equal (at the linearized level) to Einstein gravity by imposing a special boundary condition (J. Maldacena , 2011).

- More recently, it is shown that the (D. Grumiller, M. Irakleidou, I. Lovrekovic and R. McNees,2014, PRL)
  - On-shell action for the four dimensional conformal gravity is renormalized without need to counterterm.
    - It is observed that the free energy derived from the on-shell action is consistent with the Arnowitt-Deser-Misner mass and Walds definition of the entropy (H. Lu, Y. Pang, C. Pope, and J. F. Vazquez-Poritz,2012).
  - Doing the near boundary analysis, it has been argued that the first two coefficients in the Fefferman-Graham (FG) expansion of the boundary metric can consistently be interpreted as two independent sources for two operators in the boundary theory.
  - The one-point functions of that operators which are the Energy-Momentum (EM) tensor and the Partial Massless Response (PMR) have been worked out.
- Our work is finding the two-point functions of that operators.

- In the FG-gauge

$$ds^2 = \frac{l^2}{\rho^2}\left(-\sigma d\rho^2 + \gamma_{ij} dx^i dx^j\right),$$

it can be proven (A.N and K.Skenderis, unpublished) that close to $\rho = 0$,

$$\gamma_{ij} = \gamma_{ij}^{(0)} + \frac{\rho}{L}\gamma_{ij}^{(1)} + \frac{\rho^2}{L^2}\gamma_{ij}^{(2)} + \frac{\rho^3}{L^3}\gamma_{ij}^{(3)} + \ldots$$

- Specification of boundary conditions (D. Grumiller, M. Irakleidou, I. Lovrekovic and R. McNees,2014)

$$\delta\gamma_{ij}^{(0)}|_{\partial M} = 2\lambda\gamma_{ij}^{(0)}, \qquad \delta\gamma_{ij}^{(1)}|_{\partial M} = \lambda\gamma_{ij}^{(1)},$$

- Consistency of the boundary conditions checked by
  - Considering on-shell action and the variational principle.
  - On-shell action for that metric remains finite.
  - Free-energy from the on-shell action leads to consistent thermodynamics.
  - The entropy derived in this way is the same as the entropy derived using Wald's Noether charge technique.

- The first variation of CG action is (D. Grumiller, M. Irakleidou, I. Lovrekovic and R. McNees,2014)

$$\delta S_{CG} = e.o.m + \int_{\partial M} d^3x \sqrt{-\gamma}\,(T^{ij}\delta\gamma_{ij} + P^{ij}\delta K_{ij}).$$

- T^{ij} and P^{ij} are holographic response functions conjugate to the sources γ^0_{ij} and γ^1_{ij} respectively
  - $T^{ij} \quad \rightarrow \quad$ Brown-York stress tensor.
  - $P^{ij} \quad \rightarrow \quad$ Partially massles response.
  - $\gamma^{(1)}_{ij}$ plugged the linearized CG-e.o.m around (A)dS background exhibits partial masslessness behaviour.

- In order to find two-point functions, one should calculate the second variation of the on-shell action with respect to the corresponding sources. Alternatively one can obtain it by varying the one-point function with respect to the source

$$\langle P_{ij}(x)P_{kl}(0)\rangle = \frac{i}{\sqrt{-\gamma_{(0)}}}\frac{\delta\langle P_{ij}(x)\rangle}{\delta\gamma^{kl}_{(1)}(0)},$$

$$\langle T_{ij}(x)T_{kl}(0)\rangle = \frac{i}{\sqrt{-\gamma_{(0)}}}\frac{\delta\langle T_{ij}(x)\rangle}{\delta\gamma^{kl}_{(0)}(0)},$$

$$\langle P_{ij}(x)T_{kl}(0)\rangle = \frac{i}{\sqrt{-\gamma_{(0)}}}\frac{\delta\langle P_{ij}(x)\rangle}{\delta\gamma^{kl}_{(0)}(0)} = \frac{i}{\sqrt{-\gamma_{(0)}}}\frac{\delta\langle T_{kl}(0)\rangle}{\delta\gamma^{ij}_{(1)}(x)},$$

- This is where the information from inside the bulk comes into play. That is, we should solve exactly the equations of motion by specifying convenient boundary conditions.

- To find the linearized equations of motion for metric fluctuations around the AdS_4 space-time we substitute $\gamma_{ij} = \eta_{ij} + f_{ij}$ in Bach equation. In this way

$$r\left(\frac{1}{3}\Box f'' - \partial^i\partial^j f''_{ij}\right) - \frac{1}{2}\partial^a\partial^b f'_{ab} + \frac{1}{6}\Box f' + \frac{1}{12}\Box^2 f - \frac{1}{12}\partial^a\partial^b\Box f_{ab} = 0\,,$$

$$\begin{aligned}
&r^3\left(\frac{16}{3}f''''\eta_{ij} - 16f''''_{ij}\right) + r^2\Big(\Big(16f''' + \frac{8}{3}\Box f'' - \frac{4}{3}\partial^a\partial^b f''_{ab}\Big)\eta_{ij} - 48f'''_{ij} \\
&-8\Box f''_{ij} - \frac{4}{3}\partial_i\partial_j f'' + 4\partial_i\partial^a f''_{aj} + 4\partial_j\partial^a f''_{ai}\Big) + r\Big(\Big(4f'' + \frac{4}{3}\Box f' - \frac{2}{3}\partial^a\partial^b f'_{ab} \\
&-\frac{1}{3}\partial^a\partial^b\Box f_{ab} + \frac{1}{3}\Box^2 f\Big)\eta_{ij} - 12f''_{ij} - 4\Box f'_{ij} + 2\partial_i\partial^a f'_{aj} + 2\partial_j\partial^a f'_{ai} - \frac{2}{3}\partial_i\partial_j f' \\
&+\partial_i\partial^a\Box f_{aj} + \partial_j\partial^a\Box f_{ai} - \frac{1}{3}\partial_i\partial_j\Box f - \Box^2 f_{ij} - \frac{2}{3}\partial_i\partial_j\partial^a\partial^b f_{ab}\Big) = 0.
\end{aligned}$$

$$2r^2\left(\partial^j f'''_{ij} - \frac{1}{3}\partial_i f'''\right) + r\left(-\partial_i f'' + \frac{1}{2}\partial^a\Box f'_{ia} - \frac{1}{6}\partial_i\Box f' - \frac{1}{3}\partial_i\partial^a\partial^b f'_{ab}\right) = 0\,,$$

- The equations of motion are looking complicated to solve.
- However by using the reparametrization invariance and the Weyl symmetry, one can write a simple form for the linearized equations

$$\frac{3}{2}(\Box - \frac{4}{3}\Lambda)(\Box - \frac{2}{3}\Lambda)\tilde{h}_{\mu\nu} = 0\,,$$

- In this way, solutions are classified into two different modes with the following differential equation

$$(\Box + a^2)\tilde{h}_{\mu\nu} = 0\,, \qquad a^2 = 2\,,4\,.$$

- To construct a time-like mode it is possible to choose $p_i = E\delta_i^t$ as a time-like three-momentum.
- The e.o.m becomes

$$
\begin{aligned}
&4r^2\tilde{h}''_{rr} + 6r\tilde{h}'_{rr} + (rE^2 + a^2 - 4)\tilde{h}_{rr} = 0\,,\\
&4r^2\tilde{h}''_{ir} + 6r\tilde{h}'_{ir} + (rE^2 + a^2 - 4)\tilde{h}_{ir} - 4r\partial_i\tilde{h}_{rr} = 0\,,\\
&4r^2\tilde{h}''_{ij} + 6r\tilde{h}'_{ij} + (rE^2 + a^2 - 4)\tilde{h}_{ij} - 4r\partial_i\tilde{h}_{rj} - 4\partial_j\tilde{h}_{ri} + 8r\eta_{ij}\tilde{h}_{rr} = 0.
\end{aligned}
$$

- Solving the above equations with appropriate boundary conditions and after that using below transformation to convert the solution to FG gauge:

$$\xi^\mu(r, x^i) = e^{-iEt}\xi^\mu(r),$$

$$r^{-1}f_{\mu\nu} = \tilde{h}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu.$$

- The Ghost modes are solutions for $a^2 = 4$.
- Imposing proper boundary conditions
  - We consider the " infalling boundary condition in the Bulk". That means that we require the behavior of solution around the Poincare horizon of the bulk, i.e. at $r \to \infty$, to be of the form $e^{\pm iE(t-\sqrt{r})}$, so that the fluctuation modes go toward the horizon as time passes and do not come out of it.
  - The expansion of these modes near the boundary at $r = 0$ contains no r_0 terms. We impose this condition in order to make sure that while computing the two-point functions by varying one-point functions with respect to the source $\tilde{h}_{ij}^{(1)}$, the source $\tilde{h}_{ij}^{(0)}$ is automatically turned off in the last step of calculation.

- After imposing proper boundary condition, the covariant time-like solution in terms of three-momentum p_i as follows

$$
\begin{aligned}
f_{ij}^{G(1,2)} &= i(1-e^{i|p|\sqrt{r}})\frac{1}{|p|}(p_i\epsilon_j^{1,2}+p_j\epsilon_i^{1,2}),\\
f_{ij}^{G(3,4)} &= \sqrt{r}e^{i|p|\sqrt{r}}\, M_{ij}^{1,2},\\
f_{ij}^{G(5)} &= \frac{2i}{|p|}(1-e^{i|p|\sqrt{r}})(\eta_{ij}-2\frac{p_ip_j}{p^2})-\sqrt{r}e^{i|p|\sqrt{r}}(\eta_{ij}-\frac{p_ip_j}{p^2})\\
&\quad +i|p|r\frac{p_ip_j}{p^2},
\end{aligned}
$$

where $\epsilon_i^{1,2}$ are two transverse polarizations from which, $M_{ij}^{1,2}$ are constructed

$$
M_{ij}^1=\epsilon_i^1\epsilon_j^1-\epsilon_i^2\epsilon_j^2\,,\qquad M_{ij}^2=\epsilon_i^1\epsilon_j^2+\epsilon_i^2\epsilon_j^1\,.
$$

- The Einstein modes are solutions for $a^2 = 2$.

- After imposing proper boundary conditions,

$$
\begin{aligned}
f_{ij}^{E(1,2)} &= (p_i \epsilon_j^{1,2} + p_j \epsilon_i^{1,2})\,,\\
f_{ij}^{E(3,4)} &= e^{i|p|\sqrt{r}}(1 - i|p|\sqrt{r})M_{ij}^{1,2}\,,\\
f_{ij}^{E(5)} &= p_i p_j\,, \quad f_{ij}^{E(6)} = -2\eta_{ij} + r p_i p_j.
\end{aligned}
$$

- The precise form of one-point functions are

$$
\begin{aligned}
&P_{ij} = -\tfrac{4\,\sigma}{\ell}\,E^{(2)}_{ij}\,,\\
&T_{ij} = \sigma\big[\tfrac{2}{\ell}\,(E^{(3)}_{ij} + \tfrac{1}{3}E^{(2)}_{ij}f^{(1)}) - \tfrac{4}{\ell}\,E^{(2)}_{ik}\psi^{(1)k}_{j} + \tfrac{1}{\ell}\,f^{(0)}_{ij}E^{(2)}_{kl}\psi^{kl}_{(1)}\\
&+\tfrac{1}{2\ell^3}\,\psi^{(1)}_{ij}\psi^{(1)}_{kl}\psi^{kl}_{(1)} - \tfrac{1}{\ell^3}\,\psi^{(1)}_{kl}\,\big(\psi^{(1)k}_{i}\psi^{(1)l}_{j} - \tfrac{1}{3}\,f^{(0)}_{ij}\psi^{(1)k}_{m}\psi^{lm}_{(1)}\big)\big]\\
&-4\,D^k B^{(1)}_{ijk} + i\leftrightarrow j,
\end{aligned}
$$

- where $\psi^{(n)}_{ij}$ is defined as the traceless part of $f^{(n)}_{ij}$

$$
\psi^{(n)}_{ij} = f^{(n)}_{ij} - \frac{1}{3}f^{(0)}_{ij}f^{(n)}\,, \qquad f^{(n)} = f^{ij}_{(0)}f^{(n)}_{ij}\,.
$$

and

$$
\begin{aligned}
E^{(2)}_{ij} &= -\frac{1}{2l^2}\psi^{(2)}_{ij} + \frac{\sigma}{2}\left(R^{(0)}_{ij} - \frac{1}{3}f^{(0)}_{ij}R^{(0)}\right) + \frac{1}{8l^2}f^{(1)}\psi^{(1)}_{ij}\\
B^{(1)}_{ijk} &= \tfrac{1}{2\ell}\left(D_j\psi^{(1)}_{ik} - \tfrac{1}{2}\,f^{(0)}_{ij}\,D^l\psi^{(1)}_{kl}\right) - j\leftrightarrow k,
\end{aligned}
$$

and $E^{(3)}_{ij}$ has complicated expression!!

- The correlation functions of two Partial Massless and two Stress Tensor operators are respectively

$$
\langle P_{ij} P_{kl} \rangle = i \frac{\delta \langle P_{ij} \rangle}{\delta f^{kl}_{(1)}} = \frac{1}{2|p|^3}(p_i p_k \Theta_{jl} + p_i p_l \Theta_{jk} + p_j p_k \Theta_{il} + p_j p_l \Theta_{ik}) + \frac{1}{|p|^3}(\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk} - \Theta_{ij}\Theta_{kl}),
$$

$$
\left\langle T_{ij} T_{kl} \right\rangle = i \frac{\delta T_{ij}}{\delta f^{kl}_{(0)}} = \frac{1}{|p|}(\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk} - \Theta_{ij}\Theta_{kl}) \,,
$$

where

$$
\Theta_{ij} = \eta_{ij} p^2 - p_i p_j \,.
$$

- And for mixed correlation functions we obtain

$$\langle T_{ij} P_{kl} \rangle = i \frac{\delta \langle T_{ij} \rangle}{\delta f^{kl}_{(1)}} = i \frac{\delta \langle P_{ij} \rangle}{\delta f^{kl}_{(0)}} = 0\,.$$

- This result is consistent with this fact that the correlation function of two operators with different scaling dimensions in a CFT vacuum is zero.
- Actually in the calculation of correlation functions, a relevant question is that, in which vacuum one is performing the computations.
- To answer this question, remind that we considered the linearisation of metric around the AdS_4 vacuum and observed that the on-shell perturbations respect the asymptotically AdS_4 form. According to asymptotically AdS_4 form of solutions, the dual field theory is a CFT.

- To further check the result one can use the Ward identites which come from the constraints on one-point functions in presence of sources

$$f^{(0)}_{ij}T^{ij}+\frac{1}{2}\psi^{(1)}_{ij}P^{ij}=0,\qquad f^{(0)}_{ij}P^{ij}=0.$$

$$2D_iT^{ij}+2D_iP^i{}_k\,h^{kj}_{(1)}+2P^i{}_k\,D_ih^{kj}_{(1)}=P^{ik}D^jh^{(1)}_{ik}\,.$$

- Our correlation functions satisfy for example

$$\frac{\delta f^{(0)}_{ij}}{\delta f^{kl}_{(0)}}P^{ij}+f^{(0)}_{ij}\frac{\delta P^{ij}}{\delta f^{kl}_{(0)}}=0\,,$$

and other constraints.

- Note that P_{ij} is traceless. According to the York decomposition of a traceless tensor A_{ij} we have

$$A_{ij} = \nabla_i V_j + \nabla_j V_i + P_{ij}^{TT} + (\nabla_i \nabla_j - \frac{1}{3}\eta_{ij}\nabla^2)S\,,$$

with V_i being a transverse vector ($\nabla^i V_i = 0$), P_{ij}^{TT} a transverse-traceless tensor and S being a scalar.

- Moreover, PMR modes are actually massive gravitons with $M^2 = \frac{2\Lambda}{3}$.

- Massive gravitons with this special amount of mass contain only spin two and spin one parts (S.Deser and A.Waldron, 2012). Thus

$$P_{ij} = \nabla_i V_j + \nabla_j V_i + P_{ij}^{TT}.$$

- Our boundary correlation functions also in full agreement with above fact in the bulk side.

- These are our results for two-point correlators which all match the expectations for a CFT

$$\langle T_{ij} T_{kl} \rangle = A_T \, \hat{\Theta}_{ij,kl} \frac{1}{|x|^2} \,, \qquad \langle T_{ij} P_{kl} \rangle = 0$$
$$\langle P^{TT}_{ij} | P^{TT}_{kl} \rangle = A_P \, \hat{\Theta}_{ij,kl} log|x|, \qquad \langle V_i(x) | V_j(0) \rangle = A_V \frac{x_i x_j}{|x|^4} \,,$$

where

$$\hat{\Theta}_{ij,kl} \; = \; \hat{\Theta}_{ik} \hat{\Theta}_{jl} + \hat{\Theta}_{il} \hat{\Theta}_{jk} - \hat{\Theta}_{ij} \hat{\Theta}_{kl},$$

and

$$A_T = 4\pi, \qquad A_V = -4\pi, \qquad A_P = -4\pi.$$

- The amplitudes A_V and A_P are negative.
- This observation leads to the conclusion that the dual field theory is non-unitary.

- We calculate the two-point functions for EM and PMR operators that have been identified as two response functions for two independent sources in the dual CFT.
- The correlation function of EM with PMR tensors turns out to be zero which is expected according to the conformal symmetry.
- The two-point function of EM is that of a transverse and traceless tensor, and the two-point function of PMR which is a traceless operator contains two distinct parts, one for a transverse-traceless tensor operator and another one for a vector field, both of which fulfill criteria of a CFT.
- We also observe the absence of the scalar part in PMR.
- It was claimed that the CG is unitary (C.M.bender and P.Mannheim, 2007,PRL). Our result shows that CG is not unitary at least in the linearized level

Thank You