Nafplion July 2015

Discrete and $U(1)$ symmetries in F-theory

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GREECE

Outline of the Talk

- Introductory remarks
- Rational points on Elliptic curves
- F-theory and Elliptic Fibration
- F-GUTs with discrete symmetries
- Mordell-Weil $U(1)$ and GUTs
- Concluding Remarks

$\mathcal{A}$

Properties of Ordinary $GUTs$

★ interesting features

▲ Gauge coupling unification

▲ Assembling of SM fermions in a few irreps.

▲ Charge Quantisation

★ deficiencies

▲ fermion mass hierarchy and mixing not predicted

▲ Yukawa Lagrangian poorly constrained

▲ Baryon number non-conservation

... Solution requires new insights ...

Discrete and $U(1)$ symmetry extensions

▲ These appear naturally in $\mathcal{F}-\mathcal{THEORY}$ constructions ▲

New Ingredients from F-theory

★ Discrete and $U(1)$ symmetries:

- necessary tools to suppress or eliminate undesired superpotential terms

★ Fluxes :

- ... truncate GUT irreps, eliminate coloured Higgs triplets, induce chirality...

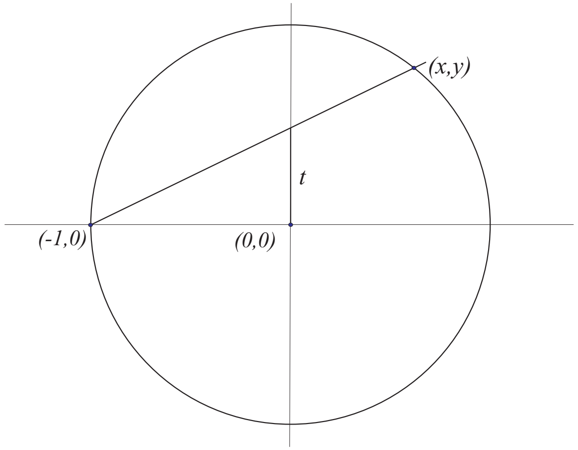
★ “Internal” Geometry :

- ... determines SM arbitrary parameters from a handful of topological properties

$\mathcal{B}$

Rational Points on Elliptic Curves

Rational Points (R.P.) on Conics



- Choose one R.P. on conic - taken here to be $(-1,0)$.
- Project all others on a line (here axis y):

$$x = \frac{1-t^2}{1+t^2} \qquad y = \frac{2t}{1+t^2}$$

$$\Downarrow$$

R.P. on line 1-1 with R.P. on circle

★ Real Rational Elliptic Curves

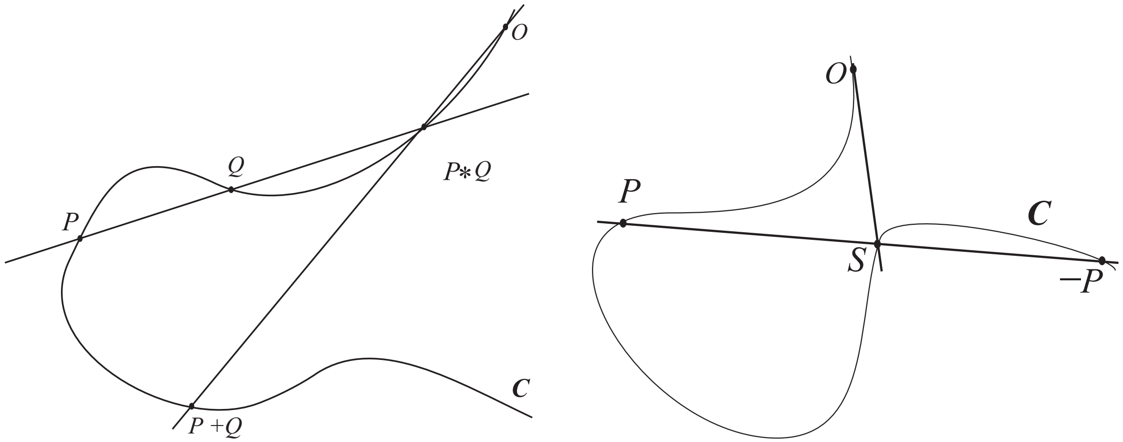
- General cubic equation with rational coefficients $f(x, y) = 0$:

$$f = a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3 + a_5 x^2 + a_6 x y + a_7 y^2 + a_8 x + a_9 y + a_{10}$$

- *rational points on elliptic curve?* Non-trivial to find **but:**

They obey a group law!

The Group Law on Elliptic Curves



The addition law: $P + Q$ (left).

($P, Q =$ *rational* $\rightarrow P + Q$ *rational.*)

The opposite element $P + (-P) = \mathcal{O}$ (right)

Mordell Theorem

$\Downarrow$

The Rational Points on Elliptic Curve constitute a finitely generated Abelian Group

$\Downarrow$

Mordell - Weil Group

Any cubic equation with a rational point can be written in:

★ Weierstrass form:

$$y^2 = x^3 + fx + g$$

▲ Two important quantities characterising elliptic curves:

1. The Discriminant:

$$\Delta = 4f^3 + 27g^2$$

... classifies the curves with respect to its singularities

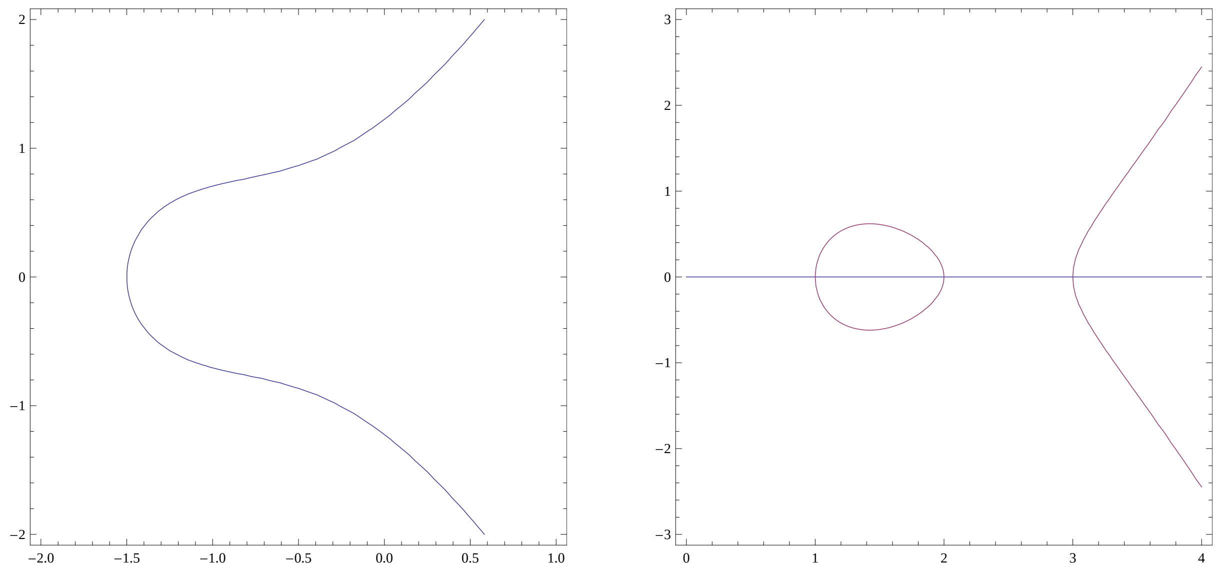
2. The j-invariant function:

$$j = 4\,\frac{(24f)^3}{4f^3 + 27g^2}$$

... takes the same value for equivalent elliptic curves

The role of the Discriminant

▲ $\mathcal{A}$: Non-singular curves: $\Delta \neq 0$.

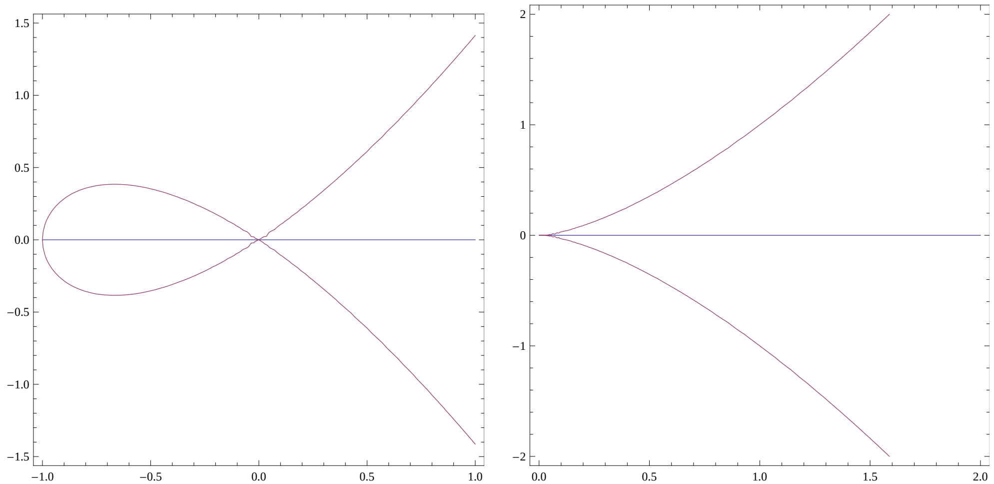


examples of non-singular curves $(\Delta \neq 0)$:

1 real root (left), 3 real roots (right).

▲ $\mathcal{B}$: Singular cases: Discriminant: $\Delta = 0$

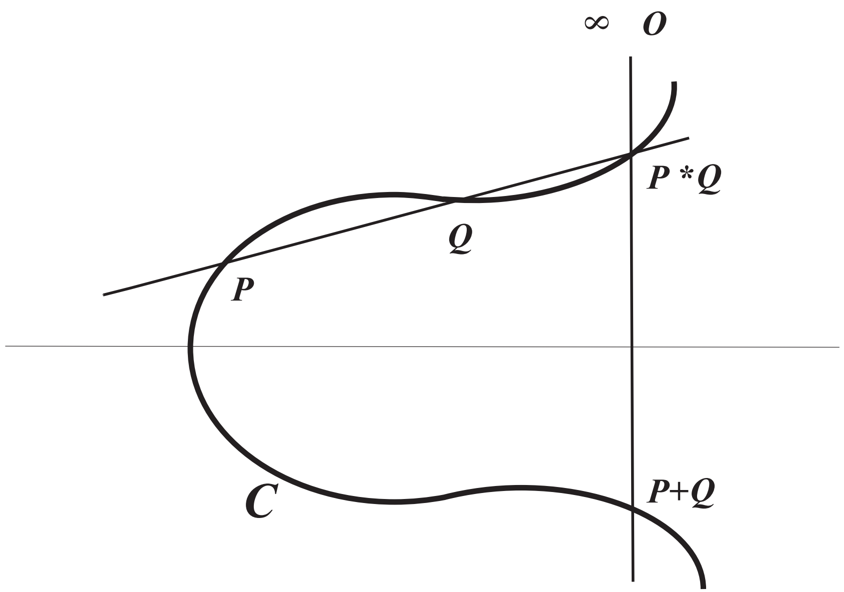
$$
\begin{aligned}
y^2 &= (x-a)^2(x+b) \\
y^2 &= x^3
\end{aligned}
$$



Singular curves ($\Delta = 0$) :

double root (left), cusp (right)

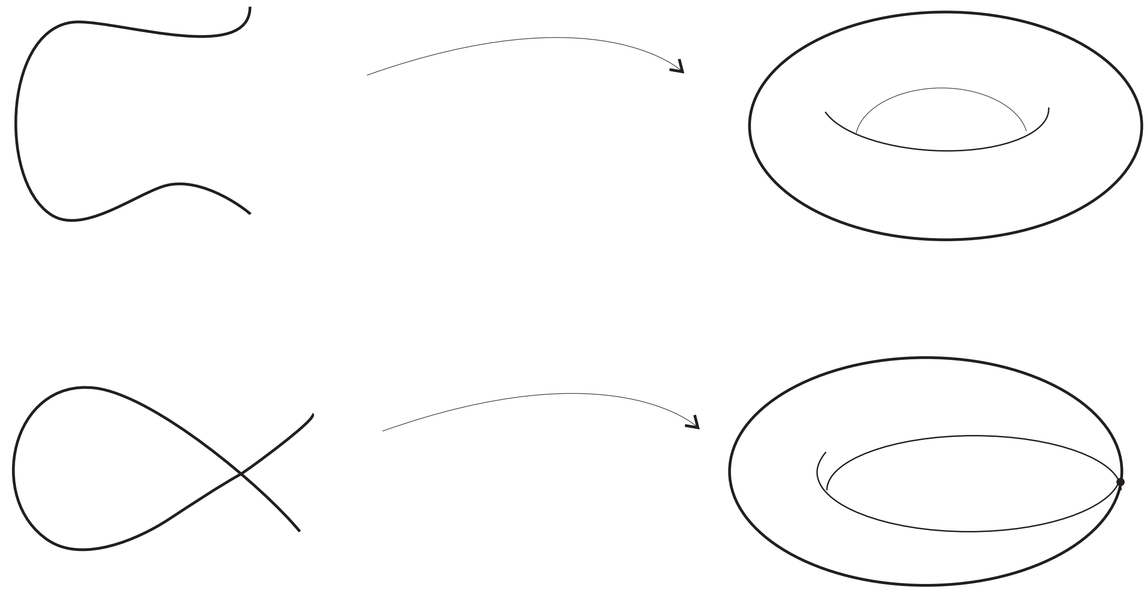
★ **Weierstrass form ... x- symmetric curve**:



Addition on Weierstrass form: The zero element $\mathcal{O}$ is at infinity.

★ **Weierstrass** equation with complex coefficients

Real ***Complex***



Complex coefficients:$\longrightarrow$ topology of torus.

Non-singular curve ($\Delta \neq 0$) "upgrades" to normal torus

Singular curve ($\Delta = 0$) corresponds to torus with a pinched radius.

$\mathcal{C}$

F-theory and Elliptic Fibration

★ F-theory ★

(*Vafa 1996*)

$\Downarrow$

Geometrisation of Type II-B superstring

II-B: *closed string spectrum obtained by combining left and right moving open strings with NS and R-boundary conditions:*

$$(NS_+, NS_+),\ (R_-, R_-),\ (NS_+, R_-),\ (R_-, NS_+)$$

Bosonic spectrum:

(NS_+, NS_+): graviton, dilaton and 2-form KB-field:

$$g_{\mu\nu},\ \phi,\ B_{\mu\nu} \rightarrow B_2$$

(R_-, R_-): scalar, 2- and 4-index fields (p*-form potentials*)

$$C_0, C_{\mu\nu}, C_{\kappa\lambda\mu\nu} \rightarrow C_p,\ p = 0, 2, 4$$

Definitions (*F-theory bosonic part*)

1. *String coupling:* $g_{IIB} = e^{-\phi}$

2. *Combining the two scalars* $C_0,\ \phi$ *to one modulus:*

$$\tau = C_0 + i\, e^{-\phi} \to C_0 + \frac{i}{g_{IIB}}$$

IIB - *action (see e.g. Denef, 0803:1194):*

$$S_{IIB} \quad \propto \quad \int d^{10}x\sqrt{-g}\, R - \frac{1}{2}\int \frac{1}{(\mathrm{Im}\tau)^2} d\tau \wedge *d\bar{\tau}$$

$$+ \quad \frac{1}{\mathrm{Im}\tau} G_3 \wedge *\overline{G}_3 + \frac{1}{2}\tilde{F}_5 \wedge *\tilde{F}_5 + C_4 \wedge H_3 \wedge F_3$$

Property:

Invariant under $SL(2,Z)$ S-duality:

$\tau \to \frac{a\tau+b}{c\tau+d}$

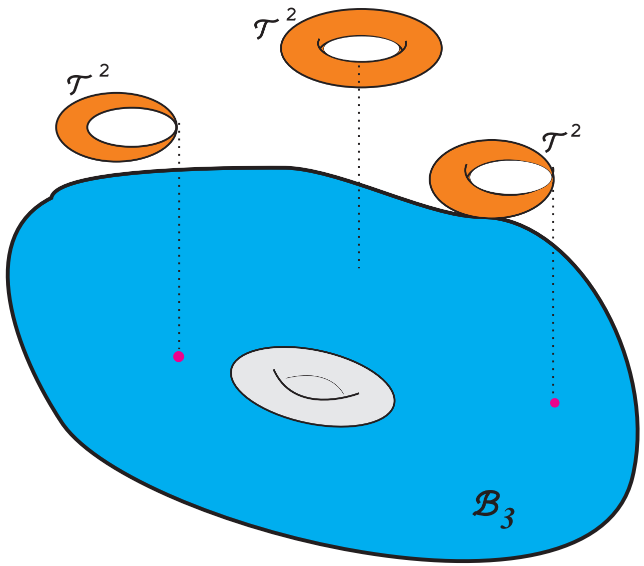
$\mathcal{FIBRATION}$

F-theory $\mathcal{R}^{3,1} \times \mathcal{X}$

$\Rightarrow$ $\mathcal{X}$, elliptically fibered **CY** 4-fold over B_3 $\Leftarrow$

$\Downarrow$

▲ a torus $\tau = C_0 + \imath/g_s$ at each point of B_3 ▲



CY 4-fold: Red points: pinched torus $\Rightarrow$ 7-branes $\perp B_3$

Elliptic Fibration

described by $\mathcal{W}$eierstraß $\mathcal{E}$quation

$$y^2 \quad = \quad x^3 + f(z)\, x w^4 + g(z) w^6$$

For each point of B_3, the above equation describes a torus

1. x, y, z homogeneous coordinates
2. $f(z),\ g(z) \rightarrow$ 8^{th} and 12^{th} degree polynomials.
3. Discriminant

$$\Delta(z) = 4\, f^3 + 27\, g^2$$

Fiber singularities at

$$\Delta(z) = 0 \rightarrow 24 \text{ roots } z_i$$

$$\Downarrow$$

j-invariant function can be written in terms of modulus τ

$$
\begin{aligned}
j(\tau) &= 4\,\frac{(24f)^3}{\Delta} \qquad (1)\\
&\propto e^{-2\pi i\tau} + 744 + \mathcal{O}(e^{2\pi i\tau}) \qquad (2)\\
\Delta &= \prod_{i=1}^{24}(z - z_i) \qquad (3)
\end{aligned}
$$

Solving

$$\tau \approx \frac{1}{2\pi i}\log(z - z_i)$$

Circling around z_i:

$$\tau \to \tau + 1 \;\Rightarrow C_0 \to C_0 + 1$$

$\to \tau$, C_0 undergo Monodromy.

At $z = z_i$ $\exists$ source of RR-flux which is interpreted as a:

$D7$-brane at $z = z_i$

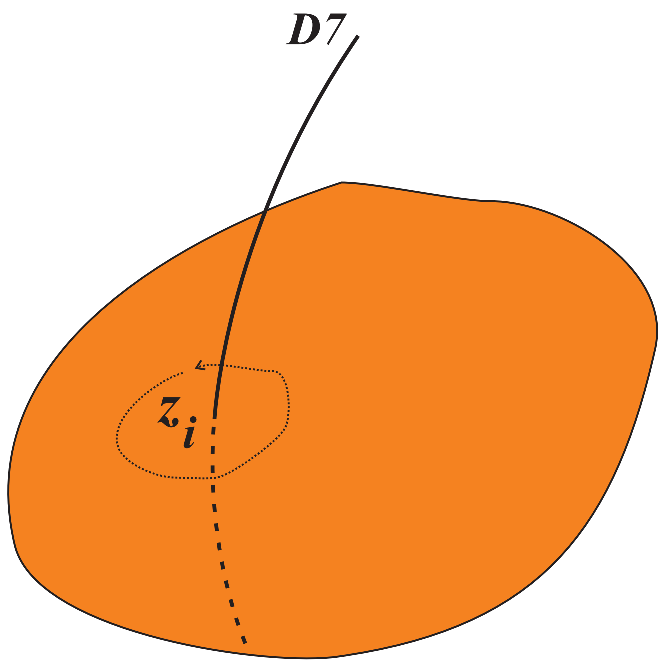


Figure 1: Moving around z_i, $\log(z) \to \log|z| + i(2\pi + \theta)$ and $\tau \to \tau + 1$

Kodaira classification:

- Type of Manifold **singularity** is specified by the vanishing order of $f(w),\ g(w)$ polynomials
- **Singularities** are classified in terms of $\mathcal{A}\,\mathcal{D}\,\mathcal{E}$ Lie groups.

Interpretation of geometric singularities

$$\Downarrow$$

CY_4-**Singularities** $\rightleftarrows$ **gauge symmetries**

$$\textbf{Groups} \quad \to \quad \begin{cases} SU(n) \\ SO(m) \\ \mathcal{E}_n \end{cases}$$

Tate's Algorithm

$$y^2 + a_1 x\, y\, z + a_3 y\, z^3 = x^3 + a_2\, x^2 z^2 + a_4 x\, z^4 + a_6 z^6$$

Table: *Classification of Elliptic Singularities w.r.t. vanishing order of Tate's form coefficients a_i:*

Group	a_1	a_2	a_3	a_4	a_6	Δ
$SU(2n)$	0	1	n	n	$2n$	$2n$
$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
$SU(5)$	0	1	2	3	5	5
$SO(10)$	1	1	2	3	5	7
$\mathcal{E}_6$	1	2	3	3	5	8
$\mathcal{E}_7$	1	2	3	3	5	9
$\mathcal{E}_8$	1	2	3	4	5	10

$\mathcal{D}$

F-theory Model Building

Basic ingredient in F-theory:

$D7$ - brane

GUTs are associated to 7-branes wrapping certain classes of *'internal'* **2-complex dim.** surface $\mathbf{S} \subset B_3$

▲ Gauge symmetry:

$$\mathcal{E}_8 \to \mathbf{G_{GUT}} \times \mathcal{C}$$

▲ $G_{GUT} = SU(5),\ SO(10),\ \ldots$

convenient description in the context of spectral cover

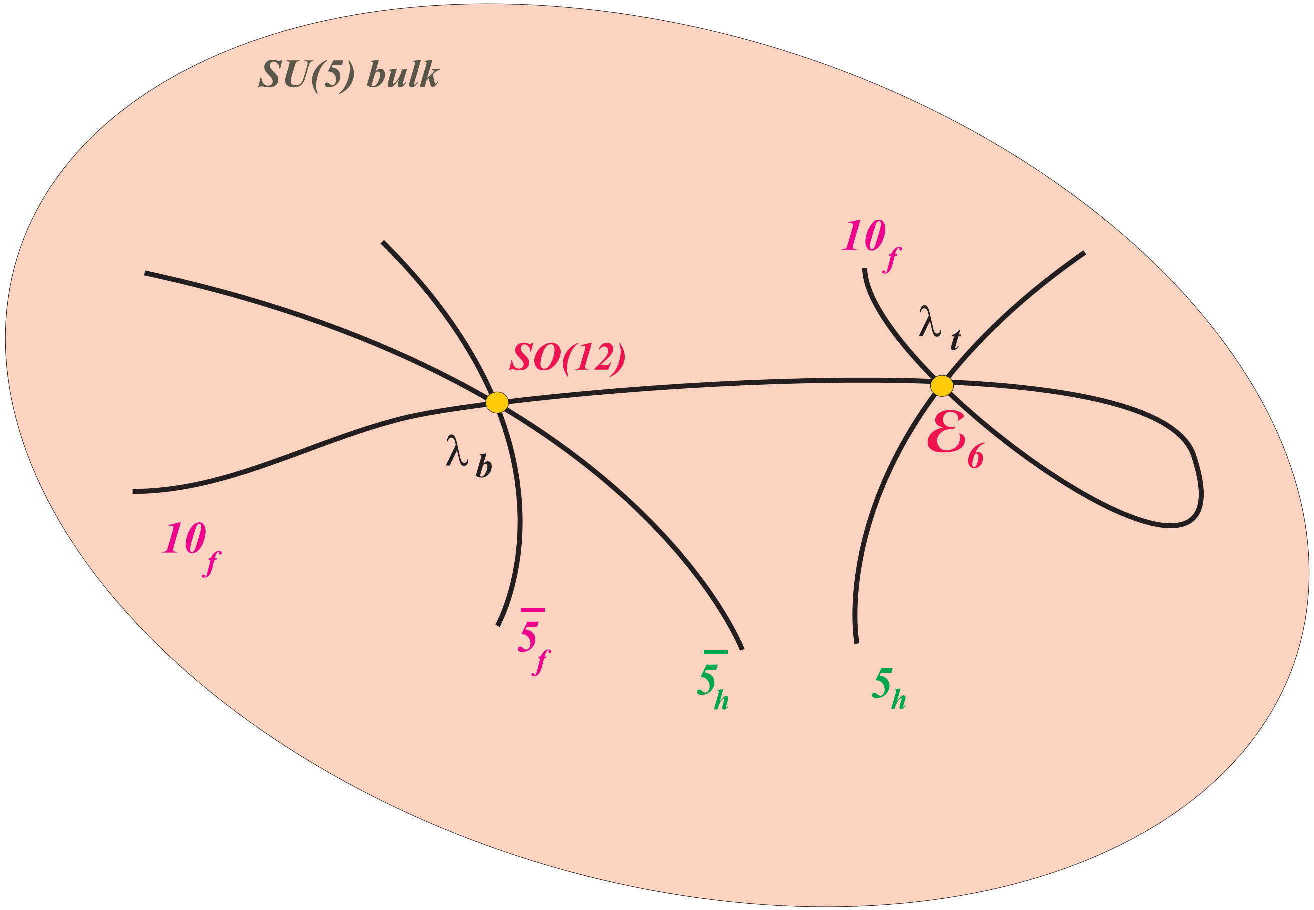
★ $\mathcal{C}$ *Commutant ...* $\Rightarrow$ *monodromies:*

$$U(1)^n, \text{or discrete symmetry } S_n,\ A_n,\ D_n,\ Z_n$$

... acting as family or discrete symmetries (*for interesting low energy implications see:)*

Karozas, King, GKL, Meadowcroft 1505.000937

Example: $SU(5)$: Matter along intersections with other 7-branes



$\lambda_{t,b}$-Yukawas at intersections and gauge symmetry enhancements

(Heckman et al 0811.2417; Font et al 0907.4895; GG Ross, GKL, 1009.6000);

(Cecotti et al 0910.0477; Camara et al, 1110,2206; Aparicio et al, 1104.2609,. . .)

Fluxes

$SU(5)$-Chirality

$$\#5 - \#\bar{5} = \begin{cases} n(3,1)_{-\frac{1}{3}} - n(\bar{3},1)_{+\frac{1}{3}} & = & M_5 \\ n(1,2)_{+\frac{1}{2}} - n(1,2)_{-\frac{1}{2}} & = & M_5 \end{cases} \tag{4}$$

$$\#10 - \#\overline{10} = \begin{cases} n(3,2)_{+1/6} - n(\bar{3},2)_{-1/6} & = & M_{10} \\ n(\bar{3},1)_{-2/3} - n(3,1)_{+2/3} & = & M_{10} \\ n(1,1)_{+1} - n(1,1)_{-1} & = & M_{10}. \end{cases} \tag{5}$$

Hypercharge flux

$SU(5)$ breaking and *Splitting of representations*

$$\#5 - \#\bar{5} = \begin{cases} n(3,1)_{-\frac{1}{3}} - n(\bar{3},1)_{+\frac{1}{3}} & = & M_5 \\ n(1,2)_{+\frac{1}{2}} - n(1,2)_{-\frac{1}{2}} & = & M_5 + N \end{cases} \tag{6}$$

$$\#10 - \#\overline{10} = \begin{cases} n(3,2)_{+1/6} - n(\bar{3},2)_{-1/6} & = & M_{10} \\ n(\bar{3},1)_{-2/3} - n(3,1)_{+2/3} & = & M_{10} - N \\ n(1,1)_{+1} - n(1,1)_{-1} & = & M_{10} + N. \end{cases} \tag{7}$$

$\not{R}$-parity:*a specific example 1505.000937*

eliminated by Y-flux

$$10 \to (\not{Q}, u^c, e^c) \to (-, u^c, e^c)$$

parity violating term $10\bar{5}\bar{5} \to \lambda_{dbu} u^c\, d^c\, d^c$ **only!**$\to$ **Neutron-antineutron oscillations**

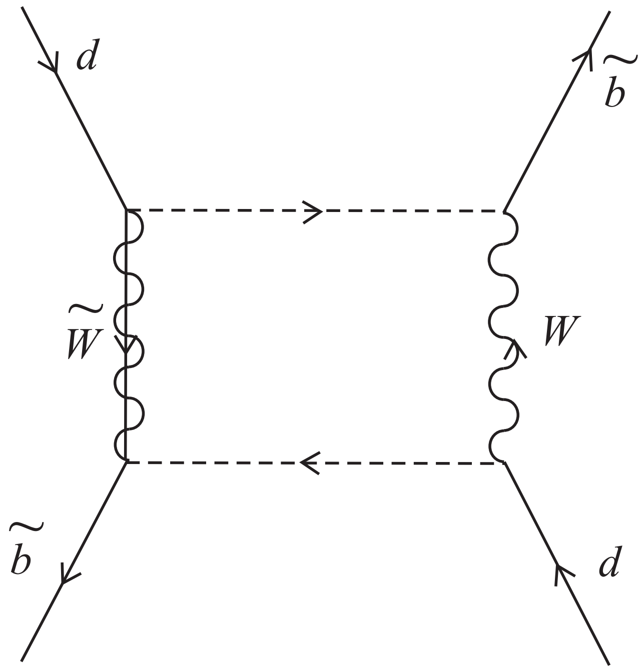


Figure 2: Feynman box graph for $n - \bar{n}$ oscillations (*Goity&Sher PLB 346(1995)69*)

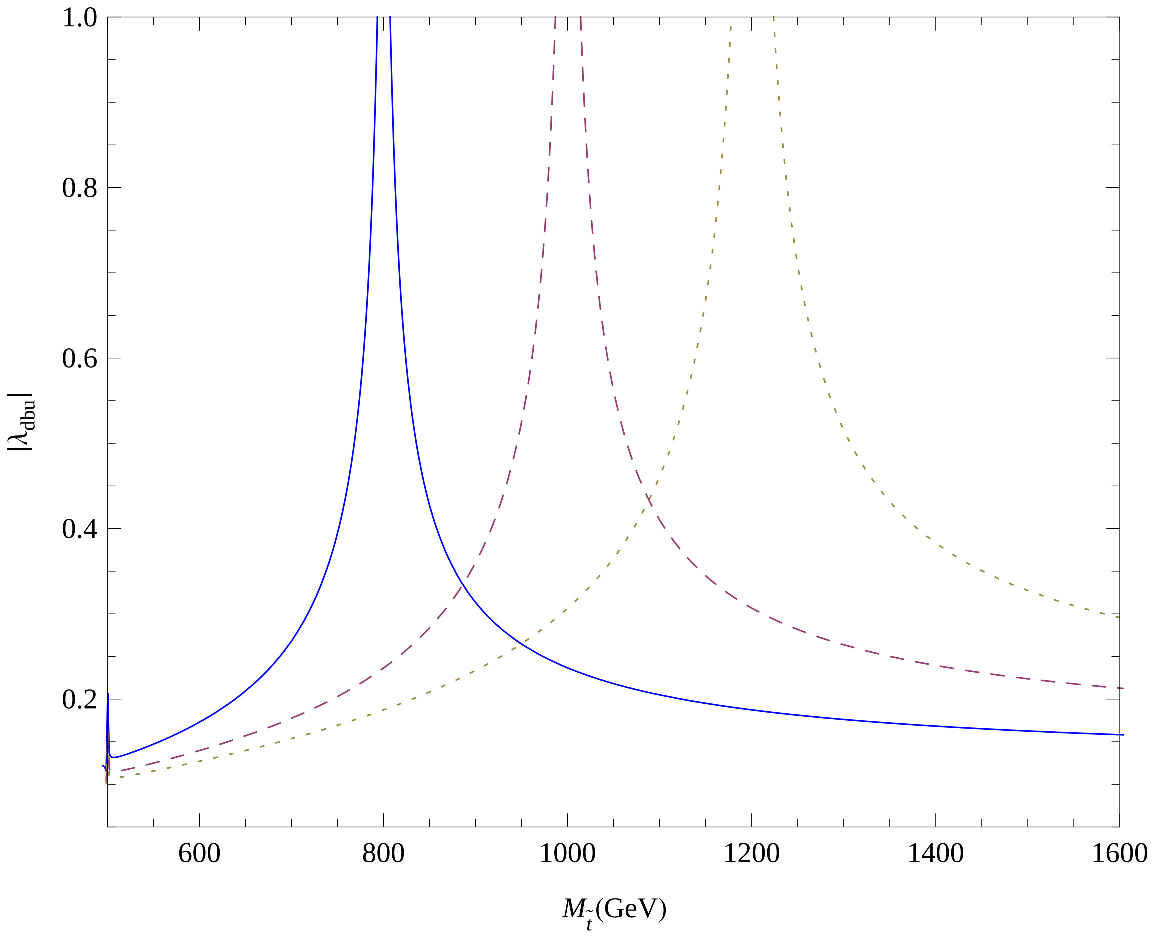


Figure 3: λ_{dbu} bounds for: Blue: $M_{\tilde{u}} = M_{\tilde{c}} = 0.8\,TeV$, Dashed: $M_{\tilde{u}} = M_{\tilde{c}} = 1\,TeV$, Dotted: $M_{\tilde{u}} = M_{\tilde{c}} = 1.2\,TeV$. ($M_{\tilde{b}_L} = M_{\tilde{b}_R} = 500GeV$, $\tau = 10^8 sec.$).

$\mathcal{E}$

Mordell-Weil $U(1)$ *and* $\mathcal{E}_6$ *GUT*

Antoniadis & GKL 1404.6720

★ *A new class of Abelian Symmetries associated to Rational Sections of elliptic curves*

Mordell-Weil group ... finitely generated:

$$\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r} \oplus \mathcal{G}$$

Abelian group: Rank - r (*unknown*)

Torsion part: $\mathcal{G} \rightarrow$:

$$\mathcal{G} = \begin{cases} \mathbb{Z}_n & n = 1, 2, \ldots, 10, 12 \\ \mathbb{Z}_k \times \mathbb{Z}_2 & k = 2, 4, 6, 8 \end{cases}$$

$\rightarrow$ ... models with new $U(1)$'s and *Discrete Symmetries from Mordell-Weil*

(*Cvetic et al 1210.6094,1307.6425; Mayhofer et al, 1211.6742; Borchmann et al 1307.2902; Krippendorf et al, 1401.7844*)

Simplest (*and perhaps most viable*) Case:

Rank-1 Mordell-Weil

To construct a model with Mordell-Weil $U(1)$'s, one starts with a line bundle.

- Let point P associated to holomorphic section
- point Q associated to rational section

★ $M = \mathcal{O}(P+Q)$ deg-2 line bundle.

Riemann-Roch theorem for genus-1 curves:

$\#$ of global sections $=$ to its degree $h^0(M) = d \rightarrow$

Sections required: $[u:v:w] = [1:1:2] \rightarrow$

$\mathbb{P}_{(1,1,2)}$-weighted projective space

... *described by the equation:* (*see Morrison & Park 1208.2695*)

$$w^2 + a_2 v^2 w = u(b_0 u^3 + b_1 u^2 v + b_2 u v^2 + b_3 v^3)$$

Need to obtain Standard form of Weierstrass model... to read off the non-Abelian singularity part

Birational Map

$$v = \frac{a_2 y}{b_3^2 u^2 - a_2^2 \left(b_2 u^2 + x\right)} \quad (8)$$

$$w = \frac{b_3 u y}{b_3^2 u^2 - a_2^2 \left(b_2 u^2 + x\right)} - \frac{x}{a_2} \quad (9)$$

$$u = z \quad (10)$$

These lead to the Weierstraß equation in Tate's form

$$
\begin{aligned}
y^2 + 2\frac{b_3}{a_2}xyz \pm b_1 a_2 yz^3 \quad &= \quad x^3 \pm \left(b_2 - \frac{b_3^2}{a_2^2}\right) x^2 z^2 \\
&\quad -b_0 a_2^2 xz^4 - b_0 a_2^2 \left(b_2 - \frac{b_3^2}{a_2^2}\right) z^6
\end{aligned}
$$

but now Tate's coefficients are not all independent !

$$y^2 + 2\frac{b_3}{a_2}xyz \pm b_1 a_2 yz^3 = x^3 \pm \left(b_2 - \frac{b_3^2}{a_2^2}\right) x^2 z^2 - b_0 a_2^2 xz^4 - b_0 a_2^2 \left(b_2 - \frac{b_3^2}{a_2^2}\right) z^6$$

... comparing with standard general Tate's form:

$$y^2 + \alpha_1 xyz + \alpha_3 yz^3 = x^3 + \alpha_2 x^2 z^2 - \alpha_4 xz^4 - \alpha_6 z^6$$

Observation:

$$\boxed{\alpha_6 = \alpha_2 \alpha_4}$$

Implications on the non-abelian structure

Assume local expansion of Tate's coefficients

$$\alpha_k = a_{k,0} + \alpha_{k,1}\xi + \cdots$$

Vanishing orders for $SU(2n)$:

$$\begin{aligned}
\alpha_2 &= a_{2,1}\,\xi + \cdots \\
\alpha_4 &= \alpha_{4,n}\xi^n + \cdots \\
\alpha_6 &= \alpha_{6,2n}\xi^{2n} + \cdots
\end{aligned}$$

$$\alpha_6 = \alpha_2\alpha_4 \to \alpha_{2,1}\alpha_{4,n}\xi^{n+1} = \alpha_{6,2n}\xi^{2n} \;\Rightarrow\; n = 1$$

...from $SU(n)$ series, compatible are Only for:

$$SU(2),\ \text{and}\ \ SU(3)$$

... extending the analysis to exceptional groups...

Viable non-Abelian GUTs with $U(1)_{MW}$

and the vanishing order of the coefficients $a_2 \sim a_{2,m}\xi^m, b_k \sim b_{k,n}\xi^n$

Group	a_2	b_0	b_1	b_2	b_3
$\mathcal{E}_6$	1	1	1	2	2
	0	3	1	2	1
$\mathcal{E}_7$	1	1	2	2	2
	0	3	3	2	1

$\mathcal{E}_6$ model: *Symmetry Explorations*

If:

$$b_0 = 0 \ ; \ b_2 = \frac{b_3^2}{a_2^2}$$

... Tate's form exhibits a Z_3 symmetry:

$$y^2 + \alpha_1 xyz + \alpha_3 yz^3 = x^3$$

Final Model

$$\mathcal{E}_6 \times U(1)_{MW}/Z_3$$

Remarks

Spectral Cover:

- Models with gauge symmetry

$$G_{GUT} \times G_{family} \in E_8$$

- Non-abelian discrete symmetries naturally incorporated

$$G_{family} \to S_n, A_n, U(1) \cdots$$

Mordell-Weil:

- *... gauge symmetries with one abelian Mordell-Weil:*

$$\mathcal{E}_6 \times U(1)_{MW}, \; \mathcal{E}_7 \times U(1)_{MW}$$

- ... extra $U(1)_{MW}$ might have interesting implications to Model building ...
- Torsion group: possible explanation of discrete symmetries...


STRING PHENO 2016
15th conference in the
String Phenomenology Conference series
Ioannina,Greece, June 20-24
http://stringpheno2016.physics.uoi.gr
e-mail: stringpheno2016@conf.uoi.gr

Additional Material

IIB - action leading to equs of motion:

(*see for example Denef 0803.1194*)

$$S_{IIB} \propto \int d^{10}x\sqrt{-g}\,R - \frac{1}{2}\int \frac{1}{(\mathrm{Im}\tau)^2} d\tau \wedge *d\bar{\tau}$$
$$+ \frac{1}{\mathrm{Im}\tau} G_3 \wedge *\overline{G}_3 + \frac{1}{2}\tilde{F}_5 \wedge *\tilde{F}_5 + C_4 \wedge H_3 \wedge F_3$$

Properties:

1. Invariant under $SL(2,Z)$ S-duality:

   $\tau \to \frac{a\tau+b}{c\tau+d}$ and $\begin{pmatrix} H \\ F \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix}$

2. This is the analogue of a 12-d. theory compactified on torus with modulus τ with $F_3,\ H_3$ *components of some* 12-d. $\hat{F}_4$ reduced along the 1-cycles of torus τ.

Topological Properties of Weierstraß Equation

▲ Scalings :

$$x \to \lambda^2 x,\; y \to \lambda^3 y,\; f(z) \to \lambda^4 f(z),\; g(z) \to \lambda^6 g(z)$$

$\Rightarrow \mathcal{WE}$:

$$\lambda^6 y^2 = \lambda^6 (x^3 + f(z)\, x + g(z))$$

Chern classes associated to bundle structure

▲ $c_1 \to 1^{st}$ Chern class of the **Tangent** Bundle to S_{GUT}

▲ $-t \to 1^{st}$ Chern class of the **Normal** Bundle to S_{GUT}

Then:

$z \to [z] = -t$

$$\text{If :} [x] = 2(c_1 - t); \quad [y] = 3(c_1 - t); \quad [b_k] = \eta - k c_1 = (6-k) c_1 - t$$

$\mathcal{WE}$ transforms as: $\boxed{6\,(c_1 - t)}$. *For example:*

$$[b_2 x z^3] \;=\; \{(6-2)c_1 - t\} + \{2(c_1 - t)\} - 3t \;=\; 6\,(c_1 - t)$$

Kodaira classification:

- Type of Manifold **singularity** is specified by the vanishing order of $f(z),\ g(z)$ polynomials
- **Singularities** are classified in terms of $\mathcal{A}\,\mathcal{D}\,\mathcal{E}$ Lie groups.

Interpretation of geometric singularities

$$\Downarrow$$

CY_4-**Singularities** $\rightleftarrows$ **gauge symmetries**

$$\mathbf{gauge\,symmetries} \quad \rightarrow \quad \begin{cases} SU(n) \\ SO(m) \\ \mathcal{E}_n \end{cases}$$

ord($f(z)$)	ord$g(z)$)	ord($\Delta(z)$)	fiber type	Singularity
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	$\mathcal{E}_6$
3	≥ 5	9	III^*	$\mathcal{E}_7$
≥ 4	5	10	II^*	$\mathcal{E}_8$

Table 1: Kodaira's classification of Elliptic Singularities with respect to the vanishing order of f, g, Δ with respect to z.

Tate's form

Useful algorithm for local description:

Procedure: (*see Katz et al 1106:3854*) Expand f, g

$$f(z) = \sum_n f_n z^n, \; g(z) = \sum_m g_m z^m$$

Then

$$\Delta = 4\left[f_0 + f_1 z + \cdots)\right]^3 + 27\left[g_0 + g_1 z + \cdots\right]^2$$

Demand $z/\Delta \;\Rightarrow$

$$f_0 = -\frac{1}{3}\,t^2, \quad g_0 = \frac{2}{27}\,t^3$$

while $\mathcal{WE}$ obtains Tate's $\mathbf{I_1}$ form:

$$y^2 = x^3 + t\,x^2 + (f_1 + f_2 z + \cdots)z\,x + (\tilde{g}_1 + \tilde{g}_2 z + \cdots)z$$

Tate's Form

$$y^2 + a_1 x\, y + a_3 y = x^3 + a_2\, x^2 + a_4 x + a_6$$

The algorithm (*Partial results*)

Group	a_1	a_2	a_3	a_4	a_6	Δ
$SU(2n)$	0	1	n	n	$2n$	$2n$
$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
$SU(5)$	0	1	2	3	5	5
$SO(10)$	1	1	2	3	5	7
$\mathcal{E}_6$	1	2	3	3	5	8
$\mathcal{E}_7$	1	2	3	3	5	9
$\mathcal{E}_8$	1	2	3	4	5	10

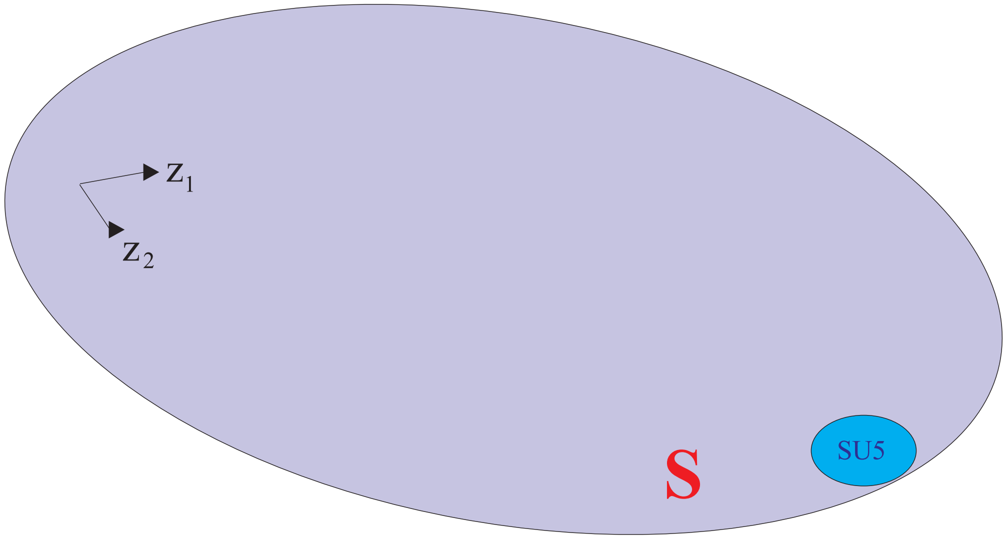
$\mathcal{G}$

Model Building

Basic ingredient in F-theory:

$D7$ - brane

GUTs are associated to 7-branes wrapping certain classes [a] of *'internal'* **2-complex dim.** surface **S** (*called a 'divisor'* $S \subset B_3$)



[a] del Pezzo, Enrique, Hirtzebruch

▲ The precise gauge group is determined by the singular fibers over the surface $\mathbf{S}$.

▲ Elliptic Fibration: Highest singularity is $\mathcal{E}_8$

▲ Gauge symmetry: *(in principle)* **Any** $\mathcal{E}_8$ subgroup $G \supset SM$:

$$\mathcal{E}_8 \to \mathbf{G_{GUT}} \times \mathcal{C}_{\mathrm{spectral\,cover}}$$

★ **Spectral Cover** $\Rightarrow$ useful local properties of G_{GUT}

▲ Sensible choice: $G_{GUT} = SU(5)$

(*a single condition* $c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) = -2$ *ensures absence of exotics*)

GUT example in this talk: $\mathbf{SU(5)}$

- SM representations are accommodated as follows:
- $SU(5)$ Chiral and Higgs Representations:

$$\begin{aligned} 10 &\rightarrow Q + u^c + e^c \\ \bar{5} &\rightarrow d^c + \ell \\ 5 + \bar{5} &\rightarrow (T + h_u) + (\bar{T} + h_d) \end{aligned}$$

- Yukawa Couplings:

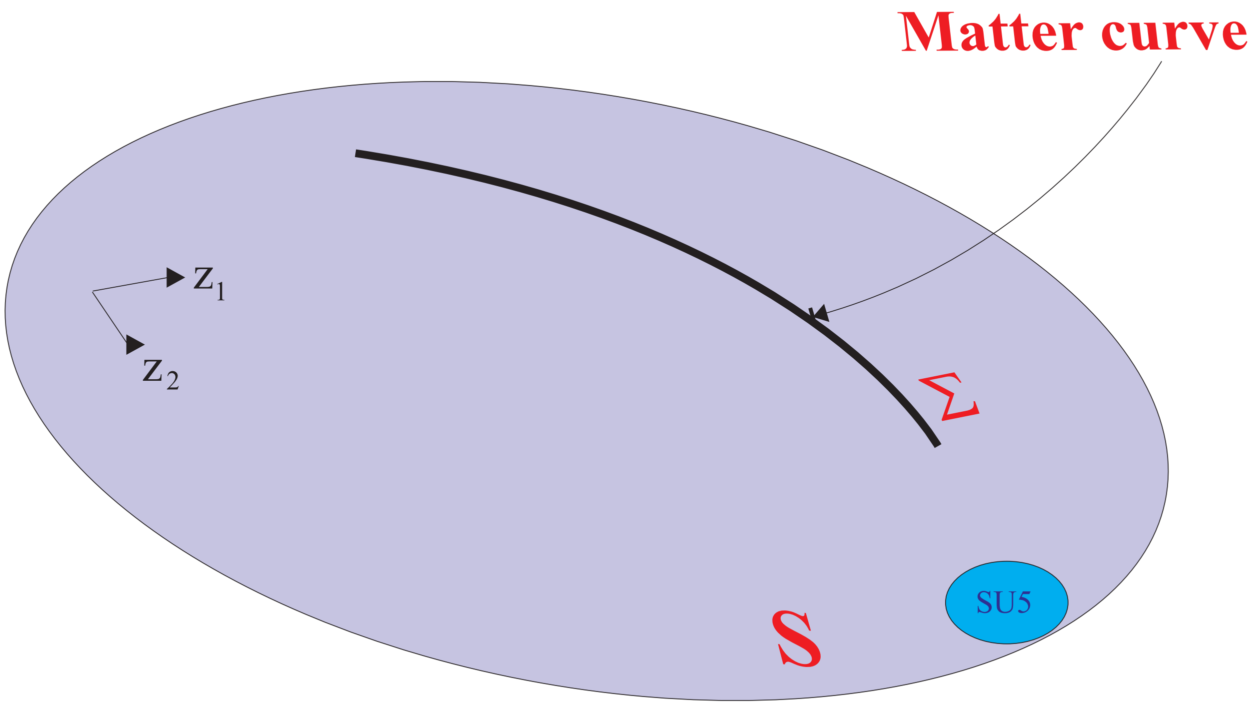
$$10 \cdot 10 \cdot 5 \rightarrow m_{top} \tag{11}$$

$$10 \cdot \bar{5} \cdot \bar{5} \rightarrow m_b \tag{12}$$

In top Yukawa-coupling 10*'s have to be the same!*

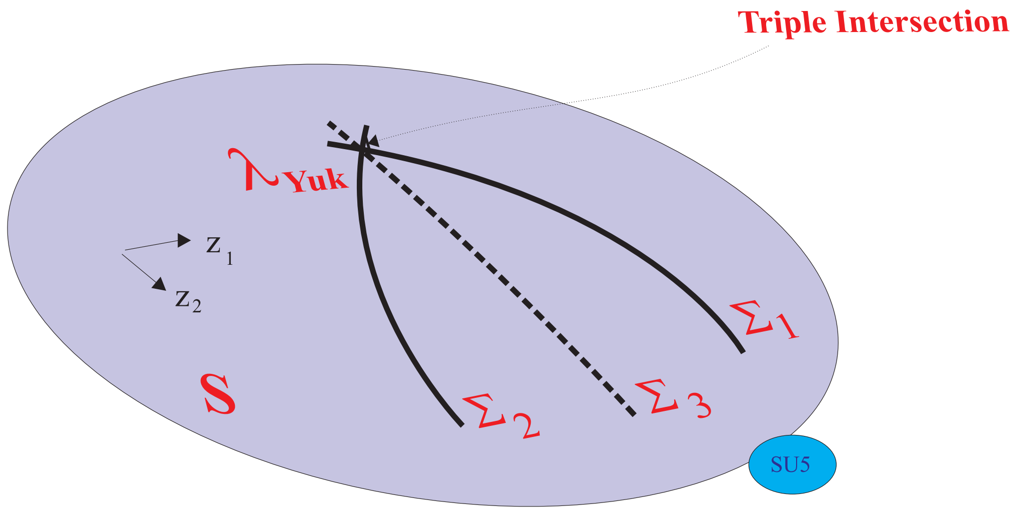
... back in F-theory:

Matter is localised along intersections with other 7-branes...



remember that when 7-branes intersect S, $\Delta = 0$, *therefore along a matter curve* Σ *gauge symmetry is enhanced*

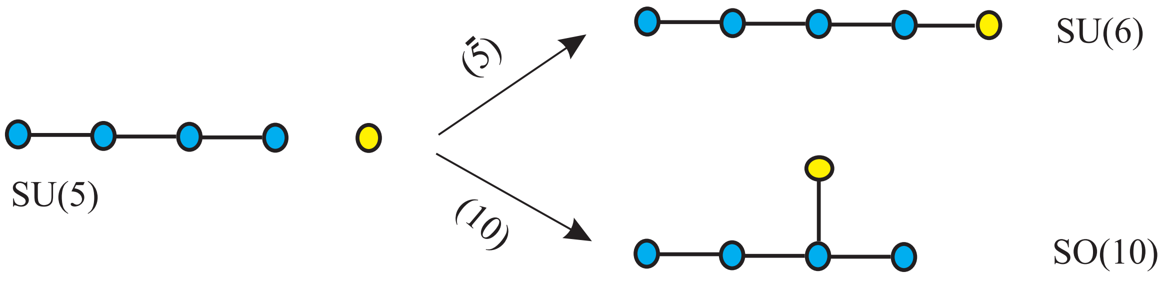
Yukawa couplings are formed at triple intersections...

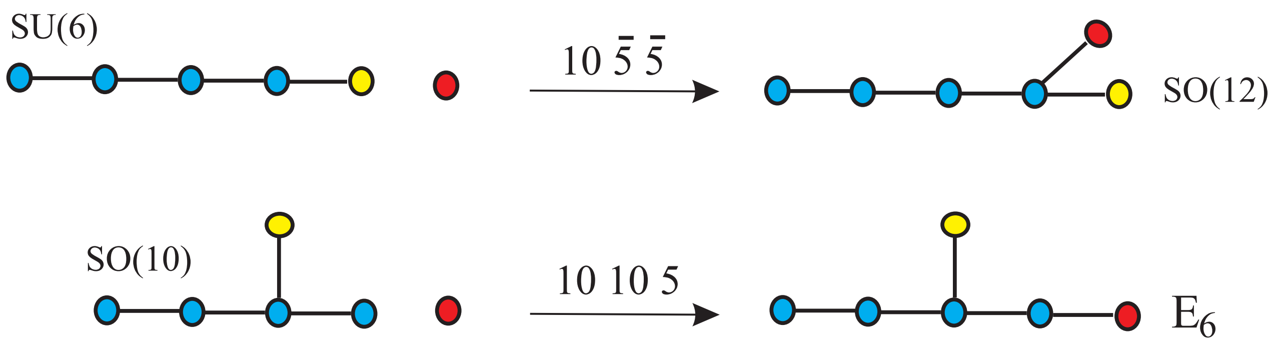


Now more 2 branes intersect, singularity increases and **gauge symmetry** is further enhanced. In particular:

$$\lambda_b \, 10 \cdot \bar{5} \cdot \bar{5} \;\in \mathbf{SO(12)}$$
$$\lambda_t \, 10 \cdot 10 \cdot 5 \in \mathbf{E_6}$$

Symmetry enhancements for $SU(5)$.





$G_S = SU(5)$: Singularity enhancement:

- Matter curves accommodating $\bar{\mathbf{5}}$ are associated with $SU(6)$

$$\Sigma_{\bar{5}} = S \cap S_{\bar{5}} \quad \Rightarrow \quad SU(5) \to SU(6)$$

$$\mathrm{ad}_{SU_6} = 35 \quad \to \quad 24_0 + 1_0 + 5_6 + \bar{5}_{-6}$$

- Matter curves accommodating $\mathbf{10}$ are associated with $SO(10)$

$$\Sigma_{10} = S \cap S_{10} \quad \Rightarrow \quad SU(5) \to SO(10)$$

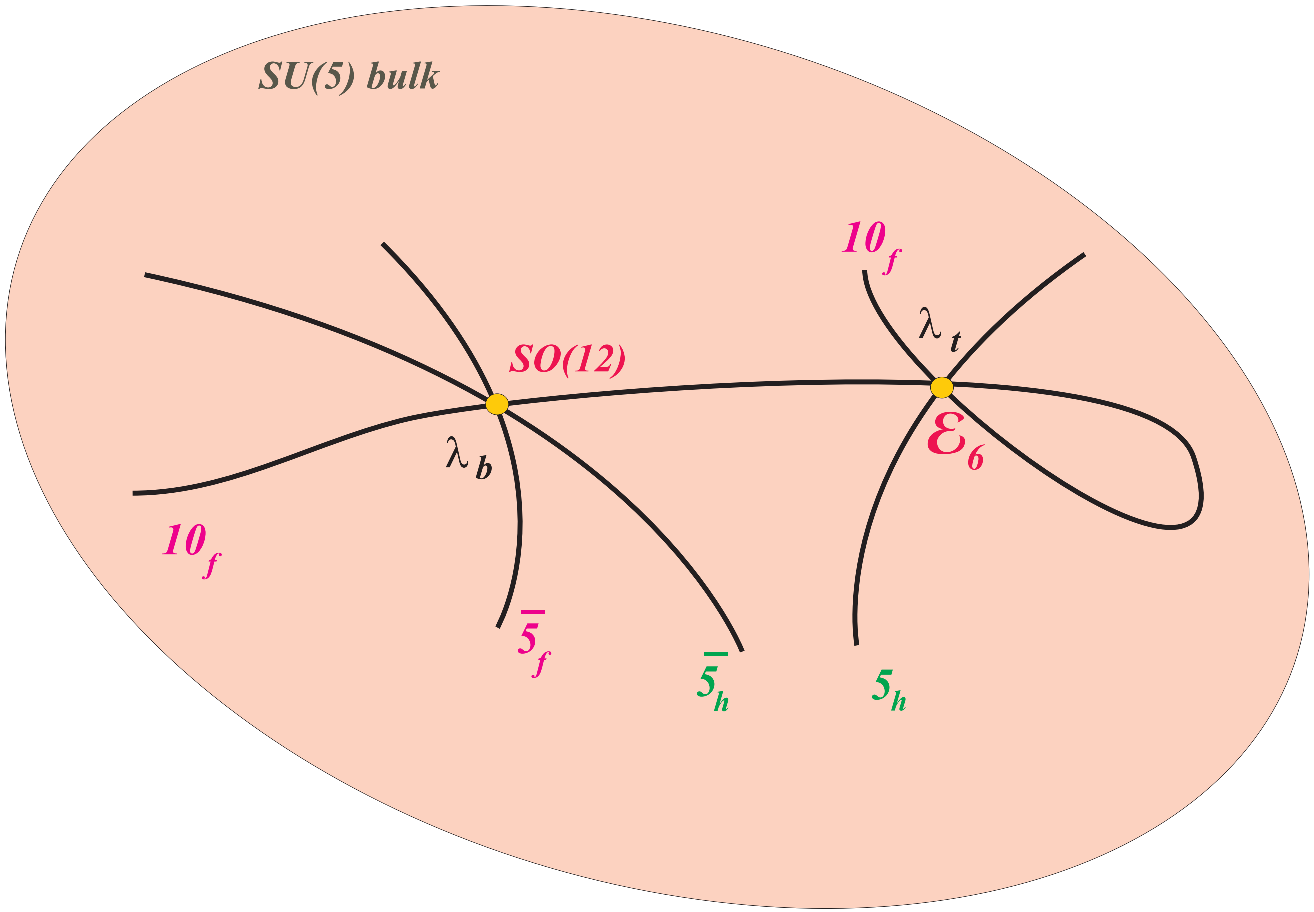
$$\mathrm{ad}_{SO_{10}} = 45 \quad \to \quad 24_0 + 1_0 + 10_4 + \overline{10}_{-4}$$

- Further enhancement in triple intersections$\to$ **Yukawas**:

$$SO(10) \equiv E_5 \quad \Rrightarrow \quad E_6 \to \mathbf{10} \cdot \mathbf{10} \cdot \mathbf{5}$$

$$SU(6) \quad \Rrightarrow \quad SO(12) \to \mathbf{10} \cdot \bar{\mathbf{5}} \cdot \bar{\mathbf{5}}$$

Matter is localised along intersections with other 7-branes...



$\lambda_{t,b}$-Yukawas at intersections and gauge symmetry enhancements

★ **F-Spectrum**

10-d Super YM theory :

$$\begin{cases} 10dim \text{ Gauge Field } A \\ \text{Adjoint fermions in } 16_+ \text{ of } SO(9,1) \end{cases}$$

Under Reduction $SO(9,1) \to SO(7,1) \times U(1)_R$ fields decompose to

$$\begin{cases} 8dim \text{ Gauge Field } A \\ \text{scalars } \varphi, \bar{\varphi} = A_8 \pm i\, A_9 \\ \text{fermions } \Psi_\pm = (S_\pm, \pm\frac{1}{2}) \end{cases}$$

F-theory described by 8-d YM Compactified on $R^{7,1} = R^{3,1} \times S$.

$$SO(7,1) \times U(1)_R \to SO(3,1) \times SO(4) \times U(1)_R$$

The 8-d spinor Ψ_+ decomposes ($O(4) \sim SU(2) \times SU(2)$)

$$\left(S_+, \frac{1}{2}\right) \to \left((2,1),(2,1),\frac{1}{2}\right) \oplus \left((1,2),(1,2),-\frac{1}{2}\right)$$

$\Rightarrow$ globally, NOT well defined!

TWIST:

$$J \sim U(1) \in U(2), \quad J_R \sim U(1)_R \to J_\pm = J \pm 2J_R$$

$\Rightarrow$

$$\left(S_+, \frac{1}{2}\right) \to \{(2,1) \otimes 2_1\} \oplus \{(1,2) \otimes (1_2 \oplus 1_0)\}$$

preserving $\mathcal{N} = 1$ SUSY.

(*Beasley, Heckmann, Vafa, 0802.3391*)

• Under twisting, scalars & fermions become **forms:**

$$\text{scalars} : \varphi \quad = \quad \varphi_{mn}\, dz^m \wedge dz^n$$

$$\text{fermions} : \quad = \quad \begin{cases} \eta_\alpha & (0,0) \\ \psi_{\dot\alpha} = \psi_{\dot\alpha m}\, dz^m & (1,0) \\ \chi_\alpha = \chi_{\dot\alpha mn}\, dz^m \wedge dz^n & (2,0) \end{cases}$$

The above can be organised in $\mathcal{N} = 1$ multiplets

$$(\mathbf{A}_\mu, \eta),\ (\mathbf{A}_{\bar{\mathbf{m}}}, \psi_{\bar{\mathbf{m}}}),\ (\phi_{\mathbf{12}}, \chi_{\mathbf{12}})$$

Action

$$\mathcal{S}_{\mathcal{F}} = \int_{R^{3,1}\times\mathcal{S}} d^4x\,\mathrm{Tr}\Big(\chi\wedge\partial_A\psi + 2i\sqrt{2}\omega\wedge\partial_A\eta\wedge\psi + \frac{1}{2}\psi\wedge[\varphi,\psi] + \sqrt{2}\eta[\bar{\varphi},\chi] + c.c.\Big) \tag{13}$$

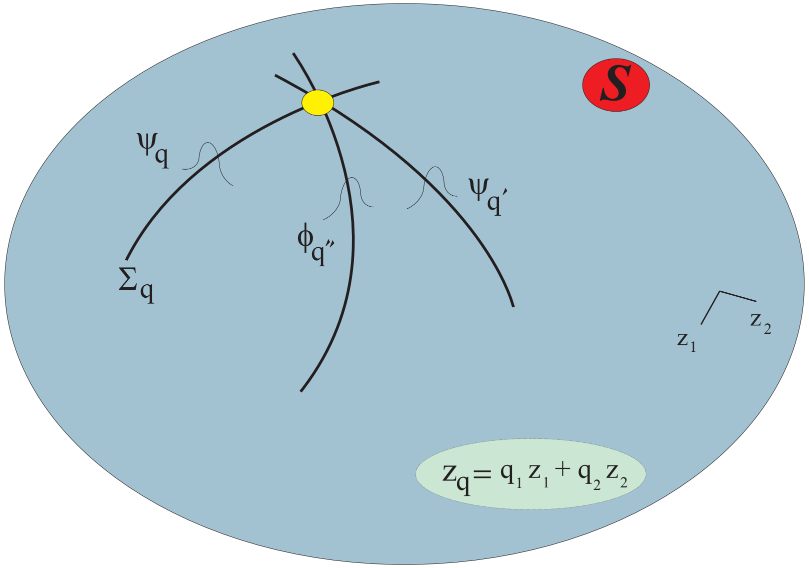
Variating the action $\longrightarrow$ Equations of motion

$$\bar{\partial}_A\chi - 2i\sqrt{2}\omega\wedge\partial_A\eta - [\varphi,\psi] = 0 \tag{14}$$

$$\bar{\partial}_A\psi - \sqrt{2}[\bar{\varphi},\eta] = 0 \tag{15}$$

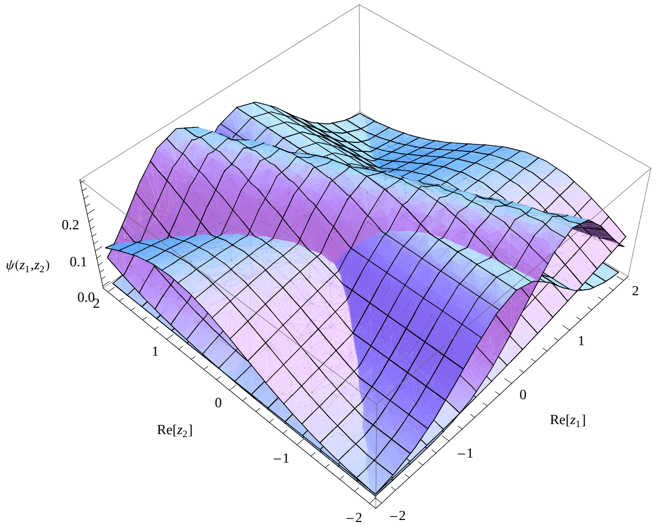
$$\omega\wedge\partial_A\psi + \frac{i}{2}[\bar{\varphi},\chi] = 0 \tag{16}$$

Matter fields are represented by wavefunctions ψ_i, ϕ on the intersections of 7-branes with $\mathbf{S}$.



$$\psi \propto f(z_{||})\,\exp(-|z_{\perp}|^2)$$

(*Font et al, 1211.6529, Camara et al, 1110.2206, GKL, GG Ross, 1009.6000*)



Yukawa coupling $\propto$ integral of overlapping wavefunctions

at the intersection

$$\lambda_{ij} \sim \int_S \psi_U^j \psi_Q^i \psi_H$$

Integral's main dependence is on local details near the intersection $\Rightarrow$ reliable λ_{ij}-estimation without knowing global geometry!

Mechanisms for Fermion mass hierarchy

▼ If all three families are on the same matter curve, masses to lighter families can be generated by:

i) non-commutative fluxes *Cecotti et al, 0910.0477*

ii) non-perturbative effects, *Aparicio et al, 1104.2609*

▼ If families are distributed on different matter curves:

Implementation of Froggatt-Nielsen mechanism (*Nucl.Phys. B147 (1979) 277*) in F-models:

Dudas and Palti, 0912.0853

GKL and G.G. Ross, 1009.6000

Callaghan, King, GKL, Ross 1109.1399

Callaghan and King, 12106913

▲▲ Combined mechanism:

Only two families on the same matter curve

$\mathcal{H}$

The Spectral Cover

Recall Weierstrass' equation for the $SU(5)$ singularity

$$y^2 = x^3 + b_0 z^5 + b_2 x z^3 + b_3 y z^2 + b_4 x^2 z + b_5 x y$$

$\to$ spectral cover obtained by defining homogeneous coordinates

$$z \to U, \; x \to V^2, \; y \to V^3$$

so Weierstrass becomes

$$V^6 = V^6 + b_0 U^5 + b_2 V^2 U^3 + b_3 V^3 U^2 + b_4 V^4 U + b_5 V^5$$

$$\text{Introduce Affine parameter}: s = \frac{U}{V}$$

Then, $SU(5)$ *spectra cover* linked to the equation:

$$\mathcal{C}_5 : \boxed{0 \; = \; b_0 s^5 + b_2 s^3 + b_3 s^2 + b_4 s + b_5}$$

Notice that: $b_1 = 0 \to \sum_i s_i = 0$ ($SU(N)$ *property*)

★ **Origin** and **Nature** of **Family Symmetries** ★

In F-theory all matter descends from the $\mathcal{E}_8$-adjoint decomposition

We already assumed

$$\mathcal{E}_8 \to SU(5)_{GUT} \times SU(5)_\perp$$

therefore

$$248 = (24, 1) + (1, 24_\perp) + (10, 5_\perp) + (\bar{5}, 10_\perp) + (5, \overline{10}_\perp) + (\overline{10}, \bar{5})_\perp$$

Interpretation from geometric point of view:

$SU(5)_{GUT}$ fields reside on matter curves:

$$\Sigma_{10_{t_i}} \quad : \quad n_{10} \times 10_{t_i} + \bar{n}_{\overline{10}} \times \overline{10}_{-t_i} \tag{17}$$

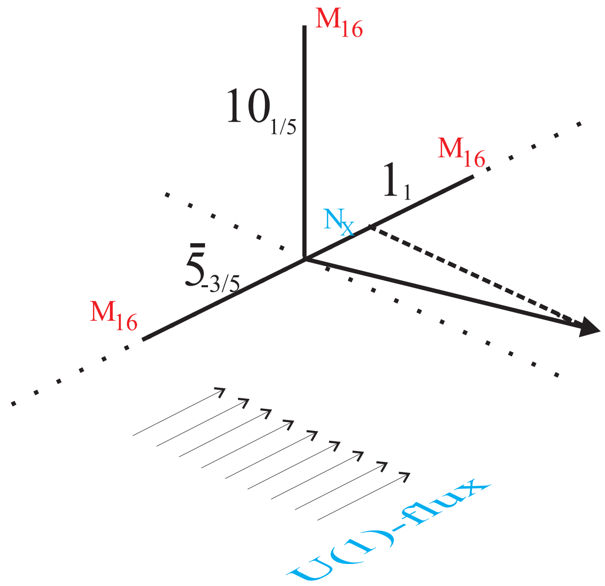
$$\Sigma_{5_{t_i+t_j}} \quad : \quad n_5 \times \bar{5}_{t_i+t_j} + \bar{n}_{\bar{5}} \times 5_{-t_i-t_j} \tag{18}$$

Families on different curves distinguished by roots $t_i, t_j \in SU(5)_\perp$

Chirality generated by fluxes... see depiction next page $\gg$

Example: $\boxed{SO(10) \to SU(5) \text{ breaking by } U(1)_X \text{ flux}}$

$$\mathbf{16} \Rightarrow \mathbf{10}_{1/5} + \bar{\mathbf{5}}_{-3/5} + \mathbf{1}_{1}$$



$$\# \, 10_{1/5} = M_{16}$$
$$\# \, \bar{5}_{-3/5} = M_{16} - N_X$$
$$\# \, 1_{1} = M_{16} + N_X$$

Monodromies

Roots of *Spectral Cover* equation $\sum_i s_i = 0$ are identified with $SU(5)_\perp$ Cartan subalgebra:

$$Q_t = \text{diag}\{t_1, t_2, t_3, t_4, t_5\}$$

★ *Matter curves* characterised by t_i's

★ Polynomial coefficients depend on t_i

$$b_k = b_k(t_i)$$

but: *Topological Properties are carried by* b_k $\Rightarrow$

t_i must be expressed in terms of them:

$$t_i = t_i(b_k)$$

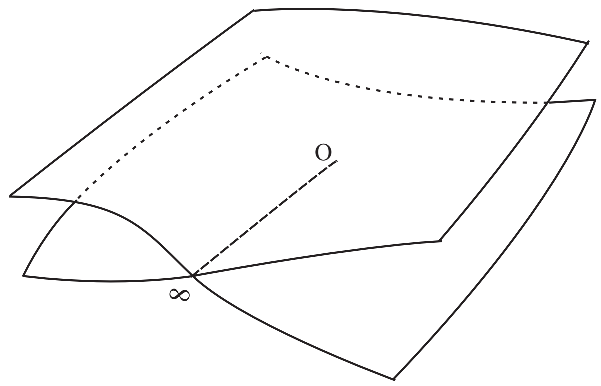
Inversion implies **brunchcuts**! $\Rightarrow$

EXAMPLE ..Simplest monodromy Z_2 : :

$$a_1 + a_2 s + a_3 s^2 = 0 \to s_{1,2} = \frac{-a_2 \pm \sqrt{\Delta}}{2a_3}$$

Under $\theta \to \theta + 2\pi \to \sqrt{\Delta} \to -\sqrt{\Delta}$ branes interchange locations

$$s_1 \leftrightarrow s_2 \text{ or } t_1 \leftrightarrow t_2$$



Two $\mathbf{U(1)}$'s related by monodromies , gauge symmetry reduces to:

$$SU(5) \times U(1)^4 \to \mathbf{SU(5)} \times \mathbf{U(1)^3}$$

Implications on Fermion Masses

- Monodromy $t_1 = t_2 \Rightarrow$ identification of matter curves

$$\Sigma_{10_{t_1}} = \Sigma_{10_{t_2}} \to \Sigma_{10_3}$$

- 3^{rd} Family assigned on them

$$10_{t_1} \cdot 10_{t_2} \cdot 5_H \to \lambda_t \, 10_3 \cdot 10_3 \cdot 5_H \to m_t$$

- Fermion mass Hierarchy organised by the remaining $U(1)$**'s** from underlying $\mathcal{E}_8$ via Singlet vevs $\langle \theta_{ij} \rangle$

$SU(5)_\perp$ breaking patterns may correspond to any of the possible spittings of the *Spectral Cover*:

$$\begin{array}{ccc} \mathcal{C}_5 & \to & \mathcal{C}_4 \times \mathcal{C}_1 \\ \mathcal{C}_5 & \to & \mathcal{C}_3 \times \mathcal{C}_2 \\ \cdots & & \cdots \end{array}$$

... with the roots respectively forming “finite groups” such as:

$$\begin{array}{rcl} \mathcal{S}_4, \mathcal{A}_4, \mathcal{Z}_4 & : & \{t_1, t_2, t_3, t_4\},\ \{t_5\} \\ \mathcal{S}_3, \mathcal{Z}_3 \times \mathcal{Z}_2 & : & \{t_1, t_2, t_3\},\ \{t_4, t_5\} \\ \mathcal{Z}_2 \times \mathcal{Z}_2 & : & \{t_1, t_2\},\ \{t_3, t_4\}, \{t_5\} \\ \mathcal{Z}_{\mathbf{2}} & : & \{t_1, t_2\},\ \{t_3\}, \{t_4\}, \{t_5\} \\ \cdots & & \cdots \end{array} \quad (19)$$

★ Examples ★

▲ Application: The $\mathcal{C}_4 \times \mathcal{C}_1$ case

▲ Motivation: The neutrino sector

▲ $\mathcal{C}_4 \times \mathcal{C}_1$ implies the splitting of the polynomial in two factors

$$\sum b_k s^{5-k} = \underbrace{(a_1 + a_2 s + a_3 s^2 + a_4 s^3 + a_5 s^4)}_{\mathcal{C}_4} \underbrace{(a_6 + a_7 s)}_{\mathcal{C}_1}$$

Topological properties of a_i are fixed in terms of those of b_k, by equating coefficients of same powers of s

$$b_0 = a_5 a_7, \; b_5 = a_1 a_6, \; etc...$$

Moreover:

▲ $\mathcal{C}_1$: associated to a $\mathcal{U}(1)$

▲ $\mathcal{C}_4$: reduction to (i) continuous $SU(4)$ subgroup, or

(ii) to Galois group $\in S_4$ (*see I. Antoniadis and GKL 1308.1581*)

Properties and Residual Spectral Cover Symmetry

- If $\mathcal{H} \in S_4$ the **Galois** group, final symmetry of the model is:

$$SU(5)_{GUT} \times \underbrace{\mathcal{H} \times \mathcal{U}(1)}_{family\ symmetry}$$

- The final subgroup $\mathcal{H} \in S_4$ is linked to specific topological properties of the polynomial coefficients a_i.

- a_i coefficients determine useful properties of the model, such as

i) **Geometric** symmetries $\rightarrow$ $\mathcal{R}$-parity

ii) **Flux** restrictions on the matter curves

- Fluxes determine useful properties on the matter curves including :

Multiplicities and **Chirality** of matter/Higgs representations

Determining the Galois group in $\mathcal{C}_4$-spectral cover

In order to find out which is the **Galois** group, we examine *partially symmetric* functions of roots t_i (*Lagrange method*)

$1.)$ **The Discriminant** Δ

$$\Delta = \delta^2 \quad \text{where} \quad \delta = \prod_{i<j}(t_i - t_j)$$

δ *is invariant under* S_4**-even** *permutations* $\Rightarrow \mathcal{A}_4$

Δ symmetric $\longrightarrow$ can be expressed in terms of coefficients $a_i \in \mathcal{F}$

$$\Delta(t_i) \ \rightarrow \ \Delta(a_i)$$

If $\Delta = \delta^2$, such that $\delta(a_i) \in \mathcal{F}$, then

$$\mathcal{H} \subseteq \mathcal{A}_4 \text{ or } V_4 \ \ (= Klein\ group)$$

If $\Delta \neq \delta^2$, (i.e. $\delta(a_i) \notin \mathcal{F}$), then

$$\mathcal{H} \subseteq \mathcal{S}_4 \text{ or } \mathcal{D}_4$$

$2.)$ *To study possible reductions of S_4, A_4 to their subgroups, another partially symmetric function should be examined:*

$$f(x) = (x - x_1)(x - x_2)(x - x_3)$$

$$x_1 = t_1 t_2 + t_3 t_4, \quad x_2 = t_1 t_3 + t_2 t_4, \quad x_3 = t_2 t_3 + t_1 t_4$$

$x_{1,2,3}$ are invariant under the three *Dihedral groups* $D_4 \in S_4$.

Combined results of Δ and $f(x)$:

	$\Delta \neq \delta^2$	$\Delta = \delta^2$
$f(x)$ irreducible	S_4	A_4
$f(x)$ reducible	D_4, Z_4	V_4

The induced restrictions on the coefficients a_i

1. Tracelessness condition $b_1 = 0$ demands

$$a_4 = a_0 a_6, \qquad a_5 = -a_0 a_7$$

2. The requirement that the discriminant is a square $\Delta = \delta^2$ imposes the following relations among a_i:

$$\left({a_2}^2 a_5 - {a_4}^2 a_1\right)^2 = \left(\frac{16 a_1 a_5 - a_2 a_4}{3}\right)^3$$

3. Reducibility of the function $f(x)$ is achieved if

$$f(0) = 4 a_5 a_3 a_1 - a_1 {a_4}^2 - a_5 {a_2}^2 = 0$$