A Hidden Symmetry of AdS Resonances

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Work(s) with

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Long Introduction

- Orienting question: Is AdS stable against (arbitrarily) small perturbations time-evolved over long periods? "Non-linear stabiliy".
- In Minkowski space, black holes form only if amplitude ($\sim$ localized energy) is big enough [Choptuik]. For low enough amplitude, energy leaks off to infinity, given enough time. Slogan: "Minkowski space is stable against non-linear weak prturbations" [Christodoulou and Klainerman].
- In AdS, if we choose reflective boundary cionditions, it is plausible that perturbations can get inside their own Schwarzschild radius after enough reflections.
- Bizon-Rostworowski did numerics and found that for low eneough amplitudes, if we scale the amplitude by ϵ, the timescale of the onset of collapse scales as $\sim \frac{1}{\epsilon^2}$.
- Turned out eventually that there are various subtleties in their claim. Still not 100% clear if the result is true or not for low enough amplitudes because numerics is (apparently) quite hard. My current view is that collapse does happen at that timescale, but I have changed my opinion twice before.

- Some related comments: black hole formation is dual to thermalization in the CFT.
- Not instability in the sense of AdS flowing to another lower energy state. More like (the gravity dual of) an excitation on the vaccum transferring energy to higher and higher modes. Forming a black hole is clssical gravity's way to avoid the UV catastrophe without an $\hbar$.
- Other authors whose work we will use: Craps, Vanhoof, Balasubramanian, Buchel, Liebling, Lehner, Yang.
- Many others also worked on closely related things, but our focus is somewhat non-standard, so we will not need/list them.

Why is the question of AdS stability hard to get clean intuition about?

My view: There are three main sources of physics whose interplay leads to the dynamics of AdS collapse.

- Effect 1: Gravity wants to clump and collapse. (Pro)
- Effect 2: Modes in AdS are (in a sense I will make precise) maximally resonant, so huge number of secular terms that one would expect to lead to uncontrolled growth. (Pro)
- Effect 3: A large class of secular terms are suppressed due to selection rules. (Con)

The relative importance of the last two items have been misjudged before.

In their original paper, Bizon and Rostowrowski do not talk about the third effect and claimed instability due to the second effect. (They might be right about the claim anyway.)

Third effect was first noticed by Craps-Evnin-Vanhoof and implicitly by Balasubramnain-Buchel-Green-Lehner-Liebling. The latter also presented some evidence that for some initial data, the collapse doesn't happen within the timescale $\sim 1/\epsilon^2$. The system exhibited Fermi-Pasta-Ulam like quasi-periodicity for a long while.

Bizon et al disagree with the no-collapse claim, but it is not entirely clear that even if the collapse happens, it happens in a time $\sim 1/\epsilon^2$.

Note that last two effects are crucially tied to the fact that we are in AdS, wile the first is a universal feature of gravity.

In this talk, I will try to make the mistake that has not been made yet – namely we will work with a model, where the first effect (clumping) is absent but the last two are captured. A simple choice: a probe massless interacting scalar field theory in AdS [Basu-CK-Saurabh, Basu-CK-Subramanian].

Essentially we are trying to replace the non-linearity of gravity by another (specific) non-linear system in AdS which does not have the clumping instability. It might not be a-priori obvious that this is a useful toy model.

However, it turns out that–

- (somewhat surprisingly) this model captures *some* non-trivial features of the full gravitational dynamics. In fact we found close parallels to the gravitational selection rules for secular terms and reproduced the FPU quasi-periodicity using the quartic scalar [Basu-CK-Saurabh], and anticipated some conserved charges which were later identified by the instability people [Craps et al., Buchel et al.] for gravity as well.
- There is a stochasticity threshold for the initial amplitudes above which the quartic scalar thermalizes fast [Basu-CK-Saurabh]. But for low amplitudes, thermalization is slow. In fact using a two-time-formalism one can show that at late times the energy in the modes is exponentially suppressed in the mode number, which can be taken as evidence for the absence of thermalization in this sytem [Basu-CK-Subramanian].
- This makes it tempting to speculate that there is no collapse also in AdS gravity. Of course the scalar knows nothing about clumping, but then again if the short distance modes are hard to excite, clumping might not become relevant.

Continued–

- Scalar serves as a clean set-up for studying resonances in AdS absracted away from the messy problem of gravitational instability. This enabled [Yang] to derive the resonance selection rules using manipulations of mode functions, without assuming spherical symmetry.
- Finally- the scalar enables us to uncover a hidden symmetry of modes in AdS, which is perhaps intriguing in itself [Evnin-CK].

This last item will be the focus of this talk - it is a simple observation and we can discuss it independently of the whole AdS instability problem. (But it seems evident that the hidden symmetry and selection rules are related.)

AdS Wave Equation

The global AdS metric (after setting the AdS radius to unity) is

$$ds^2_{AdS_{d+1}} = \sec^2 x \left(-dt^2 + dx^2 + \sin^2 x \, d\Omega^2_{d-1}\right) \equiv \sec^2 x \, (ds^2_{ES}), \tag{1}$$

where we have identifed the metric on (half of) the Einstein static universe ds^2_{ES} for future convenience. The equations of motion for the scalar field are given by

$$\Box_{AdS_{d+1}} \phi - m^2 \phi \equiv \cos^2 x \, \left(-\partial_t^2 \phi + \Delta_s^{(d)} \phi\right) - m^2 \phi = \frac{\lambda}{N!} \phi^N \tag{2}$$

where

$$\Delta_s^{(d)} \equiv \frac{1}{\tan^{d-1} x} \partial_x (\tan^{d-1} x \partial_x) + \frac{1}{\sin^2 x} \Delta_{\Omega_{d-1}}. \tag{3}$$

Here $\Delta_{\Omega_{d-1}}$ is the Laplacian on the $(d-1)$-sphere.

Mode Functions

The solution to the free theory (i. e., $\lambda = 0$), can be found by separating variables:

$$\phi^{(0)}(t, x, \Omega) = \sum_{n=0}^{\infty} \sum_{l,k} (A_{nlk} \; e^{-i\omega_{nlk} t} + \bar{A}_{nlk} \; e^{i\omega_{nlk} t}) e_{nlk}(x, \Omega), \tag{4}$$

where A_{nlk} are arbitrary complex amplitudes and

$$\omega_{nlk} = 2n + l + \Delta, \tag{5}$$

with $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$. The mode functions are a ortho-normalizable complete set of solutions to the equation

$$\left(\Delta_s^{(d)} - \frac{m^2}{\cos^2 x} \right) e_{nlk}(x, \Omega) = -\omega_{nlk}^2 \; e_{nlk}(x, \Omega), \tag{6}$$

Mode Functions (Contd.)

They take the explicit form

$$e_{nlk}(x,\Omega) = \cos^{\Delta} x \; \sin^{l} x \; P_n^{\left(\Delta - \frac{d}{2},\, l + \frac{d}{2} - 1\right)}(-\cos 2x) \; Y_{lk}(\Omega). \tag{7}$$

Y_{lk} are the $(d-1)$-dimensional spherical harmonics, and the set of 'azimuthal quantum numbers' on a general $(d-1)$-sphere is collectively indicated by the label k. Its details don't matter.

$P_n^{(a,b)}(y)$ are the Jacobi polynomials. and they are orthogonal with respect to a scalar product defined by

$$(f,g) = \int dx d\Omega \, \tan^{d-1} x \, f(x,\Omega) \, g(x,\Omega). \tag{8}$$

Hidden Symmetry

- The first hint of hidden symmetry is seen from the energy degeneracy

$$\omega_{nlk} = 2n + l + \Delta, \tag{9}$$

- The mode function operator has a manifest $SO(d)$ symmetry of rotations of the $(d-1)$-sphere, but this can only explain why the k quantum numbers do not show up in ω.
- One might have been tempted to look for isometries of AdS as the source of degeneracies. Since the mode functions are defined on a single spatial slice, rather than in the whole space-time, one might have tried to talk of the isometry group of a single spatial slice of AdS_{d+1}, which is $SO(d,1)$. This is a wrong perspective, however, since the Laplacian on a single spatial slice of AdS_{d+1} is $\cos^2 x\, \Delta_s^{(d)}$, which is different from the mode function operator (6), even for $m = 0$.
- Besides, the eigenvalues depend only on $2n + l$, which implies that the values of the angular momentum l entering each 'level' are either all even or all odd. This property is not shared by the $SO(d)$ decomposition of $SO(d,1)$ representations.

- How do we proceed?
- The conformal relation between AdS metric and the Einstein static metric implies:

$$\cos^{(d+3)/2} x \left(\Box_{ES} - \frac{(d-1)^2}{4} \right) \frac{\phi(t,x,\Omega)}{\cos^{(d-1)/2} x} = \left(\Box_{AdS_{d+1}} + \frac{d^2-1}{4} \right) \phi(t,x,\Omega). \tag{10}$$

- Now,

$$\Box_{ES} = -\partial_t^2 + \Delta_{\Omega_d}, \tag{11}$$

where the d-sphere Laplacian is explicitly

$$\Delta_{\Omega_d} \equiv \frac{1}{\sin^{d-1} x} \partial_x (\sin^{d-1} x \, \partial_x) + \frac{1}{\sin^2 x} \Delta_{\Omega_{d-1}}. \tag{12}$$

- Relating the AdS wave equation to the ES wave equation using this, one can re-write the mode function equation in the form of a Schrödinger equation

$$\left(-\Delta_{\Omega_d} + V(x)\right) \tilde{e}_{nlk} = E_{nlk}\tilde{e}_{nlk}, \tag{13}$$

with

$$V(x) = \frac{1}{\cos^2 x}\left(m^2 + \frac{d^2-1}{4}\right) \qquad \text{and} \qquad E_{nlk} = \omega_{nlk}^2 - \frac{(d-1)^2}{4}. \tag{14}$$

We have defined $\tilde{e}_{nlk} \equiv e_{nlk}/\cos^{(d-1)/2} x$.

- Note that the range of x is only a hemi-sphere, $x \in [0, \pi/2)$, since the potential $V(x)$ is unbounded and confines the 'particle' to this hemi-sphere.

Higgs-Leemon Oscillator

- At this point, we can declare victory (up to some caveats which we will mention), because it turns out that the enhanced symmetries of the Schrödinger equation for a particle on a d-sphere, with a potential $V(x) \sim 1/\cos^2 x$ have been noticed already, by Higgs and Leemon back in 1978.
- One way to motivate the existence of the enhanced symmetry is to consider the corresponding classical problem [Higgs].
- Fact: If the orbital shape of a trajectory in a central potential $V(r)$ in the ordinary d-dimensional flat space is $r = r(\varphi)$, then the orbital shape of the motion on a d-sphere in the potential $V(\tan x)$ is $x = \arctan(r(\varphi))$, with x being the polar angle on the sphere, as in our case.
- A corollary is that if the orbits close for a central potential $V(r)$ in the ordinary d-dimensional flat space, then they will close as well for the central potential $V(\tan x)$ on a d-sphere.

Connection to Bertrand's Theorem

- It is known that the orbits close in flat space only for two potentials: the Coulomb potential $V(r) \sim 1/r$ and the isotropic harmonic oscillator potential $V(r) \sim r^2$. This is the so-called Bertrand's theorem.
- The corresponding potentials on a d-sphere, for which the closure of orbits is guaranteed by the above consideration, are the sphere Coulomb potential $V(x) \sim \cot x$ and the sphere 'harmonic oscillator' potential $V(x) \sim 1/\cos^2 x$. It is the latter potential that appears in our case.
- In flat space, the closure of orbits is explained by enhanced symmetries and the corresponding conserved quantities (I have only found explicit case-by-case discussions, but perhaps a general argument can also be made).
- For the (flat space) Coulomb potential $V(r) \sim 1/r$, the conserved quantity is the Laplace-Runge-Lenz vector, which, together with the angular momentum, forms an $so(d+1)$ Lie algebra with respect to taking the Poisson brackets.
- For the (flat space) harmonic oscillator potential $V(r) \sim r^2$ the corresponding conserved quantity is a traceless symmetric second rank tensor, sometimes known as Elliott's quadrupole or the Fradkin tensor. Together with the angular momentum (antisymmetric second rank tensor), it forms an $su(d)$ Lie algebra under Poisson brackets.

- The situation on a d-sphere forms a close parallel to the one described above for flat space.
- The d-sphere Coulomb/Kepler problem was first discussed by Schrodinger and reveals an $SO(d+1)$ symmetry and the corresponding degeneracy pattern, exactly identical to the flat space case.
- For the (classical) sphere harmonic oscillator $SU(d)$ symmetry and conserved charges have been constructed by Higgs and Leemon.
- An explicit quantum version of this symmetry generators, has only been constructed for $d = 2$, which corresponds to AdS_3 in our setting, because of problems with resolving the ordering ambiguities. The explicit construction in the general case is not done yet, to my knowledge, even though it sounds doable.
- Nevertheless, the fact that the correct degeneracies of the energy levels and the correct breakdown of reps under $SO(d)$ demonstrates the existence of the hidden $SU(d)$ symmetry.

- The degeneracies and the multiplets of the eigenfunctions are exactly the same as for the isotropic harmonic oscillator in flat space, bcause the symmetries are the same.
- Of course, the symmetries are realized in a much more non-trivial way in the non-linear case of motion on a sphere.
- For flat space, the $SU(d)$ symmetry can be seen immediately by simply writing the Hamiltonian $H \sim \sum(p_i^2 + x_i^2)$ in terms of the creation-annihilation operators $H \sim \sum a_i^\dagger a_i$.

Representation Break-Up

- For each given n and l, the rotational properties of the mode functions are determined by the spherical harmonics $Y_{lk}(\Omega)$, which transform according to the traceless symmetric rank l tensor representation of $SO(d)$.
- A given representation of $SU(d)$ is formed by all such functions with the same value of 'energy', i. e., with the same value of $2n + l$. Since both n and l are positive, there will be a maximum possible value of l in each multiplet, which we can call L. Each level will then be composed of $SO(d)$ multiplets of the form: $(n = 0, l = L)$, $(n = 1, l = L - 2)$, etc. This is precisely the $SO(d)$ content of the fully symmetrized Lth power of the fundamental representation of $SU(d)$.
- Indeed, to separate irreducible representations of $SU(d)$ into irreducible representations of its $SO(d)$ subgroup, one must separate each tensor into its trace and traceless parts (group theory fact). Applied to a fully symmetric tensor of rank L, this will generate traceless fully symmetric tensors of ranks L, $L - 2$, etc (since two indices get contracted to produce each trace). These are precisely the rotational representations appearing for mode functions at each given level, with the angular momentum varying in steps of 2.

Comments

- This hidden symmetry should be understood as a phase space symmetry, not as a purely configuration space symmetry. Just as the flat space Hydrogen atom $SO(4)$ and the flat space harmonic oscillator $SU(d)$ are symmetries in phase space. These are sometimes called
- $SU(d)$ suggests the existence of a CP^{d-1} in the phase space. Indeed for the flat space oscillator this connection has been explored usefully. Higgs and Leemon have not used this structure for the sphere oscillator, they work with pretty ugly coordinates.
- To "manifest" the hidden symmetry, perhaps for classifying the selction rules systematically, we will probably need to take into account both these facts.

Thank you

Non-Linearities

One can then take the non-linearities into account perturbatively by expanding solutions to 2) as

$$\phi = \phi^{(0)} + \lambda\phi^{(1)} + \cdots \tag{15}$$

For $\phi^{(1)}$, one gets

$$-\partial_t^2\phi^{(1)} + \left(\Delta_s^{(d)} - \frac{m^2}{\cos^2 x}\right)\phi^{(1)} = \frac{\lambda}{N!}\frac{(\phi^{(0)})^N}{\cos^2 x}. \tag{16}$$

It is convenient to expand $\phi^{(1)}$ in the basis of e_{nlk}:

$$\phi^{(1)}(t, x, \Omega) = \sum_{nlk} c^{(1)}_{nlk}(t)e_{nlk}(x, \Omega). \tag{17}$$

Substituting (4), (6) and (17) in (16), and projecting on the eigenmodes e_{nlk} using (8) one gets

$$\ddot{c}^{(1)}_{nlk} + \omega^2_{nlk}c^{(1)}_{nlk} \sim \frac{\lambda}{N!}\sum_{n_1 l_1 k_1}\cdots\sum_{n_N l_N k_N} C_{nlk|n_1 l_1 k_1|\cdots|n_N l_N k_N} \tag{18}$$

$$(A^{(0)}_{n_1 l_1 k_1}\ e^{-i\omega_{n_1 l_1 k_1} t} + \bar{A}^{(0)}_{n_1 l_1 k_1}\ e^{i\omega_{n_1 l_1 k_1} t})\cdots(A^{(0)}_{n_N l_N k_N}\ e^{-i\omega_{n_N l_N k_N} t} + \bar{A}^{(0)}_{n_N l_N k_N}\ e^{i\omega_{n_N l_N k_N} t}).$$

The coefficients C are given by

$$C_{nlk|n_1 l_1 k_1|\cdots|n_N l_N k_N} = \int dx d\Omega\ \tan^{d-1} x \sec^2 x\ e_{nlk}e_{n_1 l_1 k_1}\cdots e_{n_N l_N k_N}. \tag{19}$$

Resonances

The right-hand side of (18) consists of a sum of simple oscillating terms of the form $e^{i\omega t}$ with

$$\omega = \pm\omega_{n_1 l_1 k_1} \pm \cdots \pm \omega_{n_N l_N k_N}, \tag{20}$$

where all the plus-minus sign choices are independent. If ω is different from $\pm\omega_{nlk}$, the corresponding term produces an innocuous oscillating contribution to $c^{(1)}$, and the corresponding contribution to ϕ remains bounded, with an amplitude proportional to λ for all times.

However, if $\omega = \pm\omega_{nlk}$, the corresponding term is in resonance with the left-hand side of (18), producing a secular term in $c^{(1)}$ and ϕ, a term that will grow indefinitely with time and invalidate the perturbation theory at times of order $1/\lambda$.

Some of these secular terms can be gotten rid of by resummation procedures and perturbation theory can be made to be valid at times of order $1/\lambda$ in which secular terms become replaced with (resummed) slow changes of the mode amplitudes. The picture of non-linearities inducing energy transfer between linear modes (this transfer being slow in the weakly non-linear regime) is physically very intuitive.

Selection Rules

Each resonant term in the sum on the right hand side of (18) produces the corresponding term in $c^{(1)}$ and, after resummation, a corresponding term in the flow equations describing the slow variations of the complex amplitudes, cf. (4), due to the energy transfer between the modes. The general resonance condition reads

$$\omega_{nlk} = \pm\omega_{n_1 l_1 k_1} \pm \cdots \pm \omega_{n_N l_N k_N}. \tag{21}$$

However in AdS, many of the plus-minus choices in the above expression do not in fact result in secular terms, because the corresponding C-coefficients in (18) vanish. This was noted for gravity [Craps et al] and scalar [Craps-CK-saurabh] under a spherically symmetric assumption first, but then generalized to the case without spherical symmetry, by [Yang]. These vanishing conditions are what we call Selection Rules.