The Geometry of Moduli Space and Trace Anomalies.

A.Schwimmer

(with J.Gomis,P-S.Nazgoul,Z.Komargodski ,
N.Seiberg ,S.Theisen)

Outline

I)General Features of CFT
II)Geometry of Moduli Space:the Classification of Moduli Anomalies
III)Application of Moduli Anomalies to Exact Partition Functions

I)General Features of CFT.

1)Symmetries

A Conformal Field Theory has a conserved and traceless energy momentum tensor $T_{\mu\nu}$:

$$\partial_\mu T^{\mu\nu} = 0 \qquad\qquad T^\mu_\mu = 0$$

The corresponding Ward identities are studied by coupling the energy momentum to an external metric $\gamma_{\mu\nu}$ and additional operators to sources J the generating functional $W \equiv log Z$ depending on them.

- The conservation is translated into invariance of W under diffeo transformations :

$$\delta_\zeta \gamma_{\mu\nu}(x) = \nabla_\mu \zeta_\nu(x) + \nabla_\nu \zeta_\mu(x)$$

-The tracelessness is translated into invariance of $W(\gamma, J_\Delta, ..)$ under "Weyl transformations":

a)the metric : $\delta_\sigma \gamma_{\mu\nu}(x) = 2\sigma(x)\gamma_{\mu\nu}(x)$

b) other operators: $\delta_\sigma \mathcal{O}_\Delta(x) = \Delta\sigma(x)\mathcal{O}_\Delta(x)$

and their sources: $\delta_\sigma J_\Delta(x) = (d-\Delta)\sigma(x)J_\Delta(x)$

2)Trace (Weyl) Anomalies

In even dimensions the generating functional has necessarily contributions which cannot preserve both symmetries . If one decides to preserve diffeo invariance then the Weyl variation is not vanishing:

$$\delta_\sigma W = \int d^d x \sqrt{\gamma}\sigma(x)\mathcal{A}$$

The anomaly $\mathcal{A}$ fulfils the conditions :

a)its variation obeys the (abelian) algebra of the Weyl transformations
b) it is local i.e. a polynomial in curvatures,covariant derivatives and sources
c)it cannot be obtained from the variation of a local term

3)Moduli

If the CFT has a set of scalar operators $\mathcal{O}_I$ of dimension d one can add them to the action :

$$\delta S = \sum_I \lambda_I \int d^d x \mathcal{O}_I$$

a)if their three point functions are vanishing no beta function will be produced and the theory stays conformal:truly marginal=Moduli
b)the 0-dimensional sources $\lambda_I(x)$ are reparametrization invariant in field space

II)The Geometry of Moduli Space . Classification of Moduli Anomalies .

Zamolodchikov proposed a metric $g_{IJ}(\lambda^K)$ on the moduli space:

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle_{\lambda^K} = \frac{g_{IJ}(\lambda^K)}{(x-y)^{2d}}$$

Kutasov clarified the role of the other geometric quantities i.e. connections,curvatures on moduli space. For d=2 if the CFT is the world sheet theory for a String Compactification the geometric quantities calculate terms in the target space effective action .

The essential feature of correlators of moduli in even dimensions is the presence of logarithms in momentum space . This leads to trace anomalies in joint correlators of moduli and energy momentum tensors i.e. in the generating functional depending on the metric and the moduli sources .

Therefore classifying the Weyl anomalies of this generating functional will give the general geometric structure on the moduli space. In addition to the usual rules for Weyl anomalies we should require also reparametrization invariance in the space of the sources $\lambda_I(x)$

In d=4 we get the list:

a) $$\delta_\sigma \log Z = -\frac{1}{192\pi^2}\int d^4x\sqrt{\gamma}\,\sigma\left(g_{IJ}\widehat{\Box}\lambda^I\widehat{\Box}\lambda^J - 2\,g_{IJ}\partial_\mu\lambda^I(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu}R)\partial_\nu\lambda^J\right)$$

where $$\widehat{\Box}\lambda^I = \Box\lambda^I + \Gamma^I_{JK}\partial^\mu\lambda^J\partial_\mu\lambda^K$$

The integrability condition is fulfilled and the tensor and connection correspond to the Zamolodchikov metric: a Ward identity connects the anomaly to the Zamolodchikov correlator.

b) $$\delta_\sigma \log Z = \int d^4x\,\sigma\sqrt{\gamma}\,c_{IJKL}\partial_\mu\lambda^I\partial^\mu\lambda^J\partial_\nu\lambda^K\partial^\nu\lambda^L$$

where c_{IJKL} is a new, in principle independent tensor on moduli space though a higher symmetry may relate it to g_{IJ}

c)

$$\delta_\sigma log Z = \int d^4x \sqrt{\gamma} \sigma e_I \hat{\Delta}_4 \lambda^I$$

where

$$\Delta_4 = \Box^2 + \frac{1}{3}\nabla^\mu R \nabla_\mu + 2\, R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3} R \Box$$

is the appropriately covariantized in source space FTPR operator transforming homogenously under Weyl transformations .

The difference between the structures a) and c) generates a new type of anomaly but its normalization vanishes in a unitary CFT.

d) In addition there are unnatural parity anomalies.

e) There are many cohomologically trivial terms e.g.:

$$\delta_\sigma log Z = \int d^4x \sqrt{\gamma} \Box \sigma(x) h_{IJ} \partial_\mu \lambda^I \partial^\mu \lambda^J \quad \text{or} \quad \int d^4x \sqrt{\gamma} \Box \sigma R f(\lambda^I)$$

related to the variation of the local terms

$$\int d^4x \sqrt{\gamma} R h_{IJ} \partial_\mu \lambda^I \partial^\mu \lambda^J \qquad \text{and} \qquad \int d^4x \sqrt{\gamma} R^2 f(\lambda^I)$$

III) Exact Partition Functions

Generically partition functions of a QFT on a compact manifold(e.g. with S_4 topology) are not universal . Regularization dependence shows up through local counterterms consistent with the symmetries with arbitrary coefficients .

With SUSY (N=2 in d=4) the moduli dependent part of the partition function becomes universal : the anomalous (nonlocal) action requires a completion by local terms fixed by the anomaly i.e. cohomologically trivial terms become nontrivial . The nonlocal action in superspace leads to local terms in ordinary space.

In order to study Weyl anomalies for a N=2 superconformal theory one has to couple it to a full N=2 supergravity background .

The moduli are in chiral(anti-chiral) multiplets as well as their sources $\Phi^A, \overline{\Phi}^{\overline{A}}$ which have 0- Weyl weight .

The Weyl parameter also is imbedded into chiral(anti-chiral) superfields Σ , $\overline{\Sigma}$.

In N=2 superspace the anomaly has a simple, unique form :

$$\delta_\sigma log Z = \int d^4x\, d^4\theta\, d^4\overline{\theta}\, E\, (\Sigma + \overline{\Sigma}) K(\Phi^A, \overline{\Phi}^{\overline{A}})$$

where E is the superfield analogue of the metric determinant and $K(\Phi^A, \overline{\Phi}^{\overline{A}})$ is the "Kahler potential" on moduli space defined through the anomaly. Using the Kahler potential the other geometric quantities are defined :

the Kahler metric : $g_{I\overline{J}} = \partial_I \partial_{\overline{J}} K$

connection : $\Gamma^I_{JK} = g^{I\overline{L}} \partial_J \partial_K \partial_{\overline{L}} K$

and curvature: $\mathcal{R}_{I\overline{J}K\overline{L}} = \partial_I \partial_{\overline{J}} g_{K\overline{L}} - g^{M\overline{N}}\, \partial_I g_{K\overline{L}}\, \partial_{\overline{J}} g_{M\overline{N}}$

The expansion of the anomaly in components, normalized to the Zamolodchikov metric gives:

$$
\begin{aligned}
\delta_\Sigma \log Z = +\frac{1}{192\pi^2}\int d^4x\sqrt{g}\Big\{ & \boxed{(\sigma+\overline{\sigma})\mathcal{R}_{I\overline{K}J\overline{L}}\nabla^\mu\phi^I\,\nabla_\mu\phi^J\,\nabla^\nu\overline{\phi}^{\overline{K}}\,\nabla_\nu\overline{\phi}^{\overline{L}}} \\
& + \boxed{(\sigma+\overline{\sigma})g_{I\overline{J}}\left(\widehat{\Box}\phi^I\,\widehat{\Box}\overline{\phi}^{\overline{J}} - 2\left(R^{\mu\nu}-\frac{1}{3}R\,\gamma^{\mu\nu}\right)\nabla_\mu\phi^I\,\nabla_\nu\overline{\phi}^{\overline{J}}\right)} \\
& + \boxed{\frac{1}{2}K\,\Box^2(\sigma+\overline{\sigma}) + \frac{1}{6}K\,\nabla^\mu R\,\nabla_\mu(\sigma+\overline{\sigma}) + K\left(R^{\mu\nu}-\frac{1}{3}\gamma^{\mu\nu}R\right)\nabla_\mu\nabla_\nu(\sigma+\overline{\sigma})} \\
& - 2\,g_{I\overline{J}}\,\nabla^\mu\phi^I\,\nabla^\nu\overline{\phi}^{\overline{J}}\,\nabla_\mu\nabla_\nu(\sigma+\overline{\sigma}) + g_{I\overline{J}}\left(\widehat{\nabla}^\mu\widehat{\nabla}^\nu\phi^I\,\nabla_\nu\overline{\phi}^{\overline{J}} - \widehat{\nabla}^\mu\widehat{\nabla}^\nu\overline{\phi}^{\overline{J}}\,\nabla_\nu\phi^I\right)\nabla_\mu(\sigma-\overline{\sigma}) \\
& - \frac{1}{2}\left(\widehat{\nabla}_I\widehat{\nabla}_J K\,\nabla^\mu\phi^I\nabla_\mu\phi^J - \widehat{\nabla}_{\overline{I}}\widehat{\nabla}_{\overline{J}}K\,\nabla^\mu\overline{\phi}^{\overline{I}}\nabla_\mu\overline{\phi}^{\overline{J}} + \nabla_I K\,\widehat{\Box}\phi^I - \nabla_{\overline{I}}K\,\widehat{\Box}\overline{\phi}^{\overline{I}}\right)\Box(\sigma-\overline{\sigma}) \\
& + \left(R^{\mu\nu}-\frac{1}{3}R\,\gamma^{\mu\nu}\right)\left(\nabla_I K\,\nabla_\mu\phi^I - \nabla_{\overline{I}}K\,\nabla_\mu\overline{\phi}^{\overline{I}}\right)\nabla_\nu(\sigma-\overline{\sigma})\Big\}
\end{aligned}
$$

One recognizes in □ the Zamolodchikov anomaly.

In □ the four-index anomaly appears but N=2 fixed the tensor to be the curvature of the Zamolodchikov metric.

In and after , all the terms are cohomologically trivial i.e. are variations of local terms. In particular the terms multiplying the Kahler potential integrate to :

$$\frac{1}{2}\Box^2\sigma + \frac{1}{6}\nabla^\mu R \nabla_\mu \sigma + \left(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu} R \right) \nabla_\mu \nabla_\nu \sigma = \frac{1}{2}\Delta_4 \sigma$$
$$= \delta_\sigma \left(\frac{1}{8} E_4 - \frac{1}{12}\Box R + c\, C^2 \right)$$

where C is the Weyl tensor, E_4 the Euler density and c an arbitrary constant.

Therefore if the metric on the compact manifold is conformally flat one can calculate exactly the partition function by taking the moduli sources to constants :

$$Z = e^{K/12}$$

The Kahler potential is defined modulo the addition of purely holomorphic or antiholomorphic functions of the moduli .
The calculation of the partition function has the same ambiguity since for a holomorphic Kahler potential the anomaly in superspace becomes cohomologically trivial i.e. it is the variation of the local term in superspace:

$$\int d^4x\, d^4\theta\, \mathcal{E} F(\Phi) \left(\Xi - W^{\alpha\beta} W_{\alpha\beta}\right) + \mathrm{c.c.}$$

Additional dependence on the moduli e.g. in the logR term cannot be there since it is in contradiction with the WZ condition .