Torsional Newton-Cartan geometry in Field Theory, Gravity and Holography

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based on work with:

Jelle Hartong and Elias Kiritsis: 1409.1519 (PLB), 1409.1522, 1502.00228, & to appear

Jelle Hartong 1504.0746 (JHEP)

and

Morten Holm Christensen, Jelle Hartong, Blaise Rollier

1311.4794 (PRD) & 1311.6471 (JHEP)

Outline

- Why Newton-Cartan (NC) ? (-> non-relativistic space-time)
  - holography,
  - field theory
  - gravity

- What is NC (& its torsionful generalization TNC) geometry ?
  - NC from gauging the Bargmann algebra

- How do non-relativistic field theories couple to NC ?

- What theory of gravity does one get when making TNC dynamical ?
  - connection to Horava-Lifshitz gravity

- Outlook

Motivation (Holography)

AdS/CFT has been very successful tool in studying strongly coupled (conformal) relativistic systems
-> holography beyond original AdS-setup ?

- How general is holographic paradigm ?
  (nature of quantum gravity, black hole physics, cosmology)

- Examples of potentially holographic descriptions based on non-AdS space-times:
  Lifshitz, Schrödinger, warped AdS3 (Kerr/CFT), flat space-time.

- simplest example appears to be Lifshitz spacetimes

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dr^2 + d\vec{x}^2\right)$$

characterized by anisotropic (non-relativistic) scaling between time and space

$$t \to \lambda^z t\,, \qquad \vec{x} \to \lambda \vec{x}$$

[Kachru,Liu,Mulligan]

* introduced originally to study strongly coupled systems with critical exponent z

Motivation (Holography) cont'd

- for standard AdS setup: boundary geometry is Riemannian just like the bulk geometry

- not generic: in beyond-AdS holography bdry. geometry typically non-Riemannian

Christensen,Hartong,Rollier,NO (1311)
Hartong,Kiritsis,NO (1409)

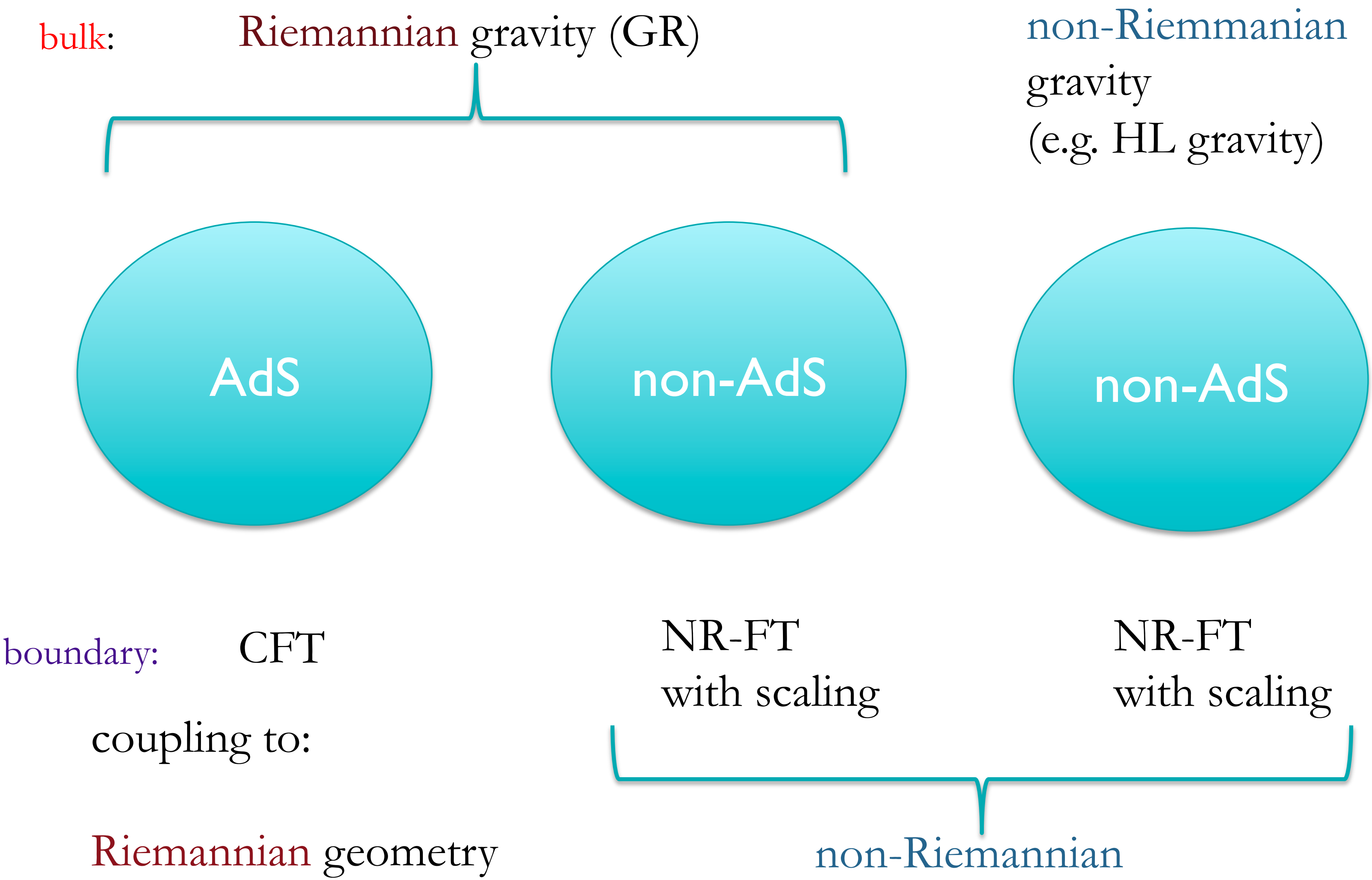
-> need new approach: prime (simplest) example to gain traction = Lifshitz
(lessons can subsequently be applied to other cases)

Result: torsional Newton-Cartan geometry is the boundary geometry found in large class of examples in EPD model

-> -> Lifshitz holography dual to field theories on TNC space-time

by making the resulting non-Riemannian geometry dynamical one gains access to other bulk theories of gravity (than those based on Riemannian gravity)
- apply holography (e.g. HL gravity)
- interesting in their own right

Different Holographic setups



Motivation (Field Theory)

- in relativistic FT: very useful to couple to background (Riemannian) geometry
  -> compute EM tensors, study anomalies, Ward identities, etc.

- background field methods for systems with non-relativistic (NR) symmetries require NC geometry (with torsion)

  -> there is full space-time diffeomorphism invariance when coupling to the right background fields

- Recent examples

  * Son's approach to the effective field theory for the FQHE [Son, 2013], [Geracie, Son, Wu, Wu, 2014]

  * non-relativistic (NR) hydrodynamics [Jensen,2014]

Motivation (Gravity)

- interesting to make NC geometry dynamical
  -> "new" theories of gravity

Hartong,NO (1504)

will see: **dynamical Newton-Cartan (NC) = Horava-Lifshitz (HL) gravity**

⇨ natural geometric framework with full diffeomorphism invariance
& possibly non-trivial consequences for HL gravity

such theories of gravity interesting as
- other bulk theories of gravity in holographic setups
- effective theories (cosmology)

Newton-Cartan makes Galilean local

- NC geometry originally introduced by Cartan to geometrize Newtonian gravity

→ both Einstein's and Newton's theories of gravity admit geometrical formulations which are diffeomorphism invariant

- NC originally formulated in "metric" formulation
  more recently: vielbein formulation (shows underlying sym. principle better)

Andringa,Bergshoeff,Panda,de Roo

Riemannian geometry: tangent space is Poincare invariant

Newton-Cartan geometry: tangent space is Bargmann (central ext. Gal.) invariant

- gives geometrical framework and extension to include torsion
i.e. as geometry to which non-relativistic field theories can couple
(boundary geometry in holographic setup is non-dynamical)

* will next consider dynamical (torsional) Newton-Cartan

From Poincare to GR

- make Poincare local (i.e. gauge the translations and rotations)

vielbein → e^a_μ; spin connection → $\omega_\mu{}^{ab}$

$$A_\mu = P_a e^a_\mu + \frac{1}{2} J_{ab}\omega_\mu{}^{ab}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,, A_\nu] = P_a R_{\mu\nu}{}^a(P) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J)$$

$$\delta A_\mu = \partial_\mu \Lambda + [A_\mu\,, \Lambda]\,, \qquad \Lambda = \xi^\mu A_\mu + \frac{1}{2} J_{ab}\lambda^{ab}$$

$R_{\mu\nu}{}^a(P) = 0$ →

- spin connection expressed in terms of vielbein
- $\delta A_\mu \to \delta e^a_\mu = \mathcal{L}_\xi e^a_\mu + e^b_\mu \lambda_b{}^a$
- covariant derivative defined via vielbein postulate
- $R_{\mu\nu}{}^{ab}(J)$ = Riemann curvature 2-form

- GR is a diff invariant theory whose tangent space invariance group is the Poincaré group

* Einstein equivalence principle -> local Lorentz invariance

Gauging the Bargmann algebra

Galilean

$H, P_a, J_{ab}, G_a \quad N$

Bargmann

(Galilean algebra is c to infinity limit of Poincare)

$[H, G_a] = P_a \qquad [P_a, G_b] = 0$

$[P_a, G_b] = N\delta_{ab}$

gauge Bargmann and impose curvature constraints

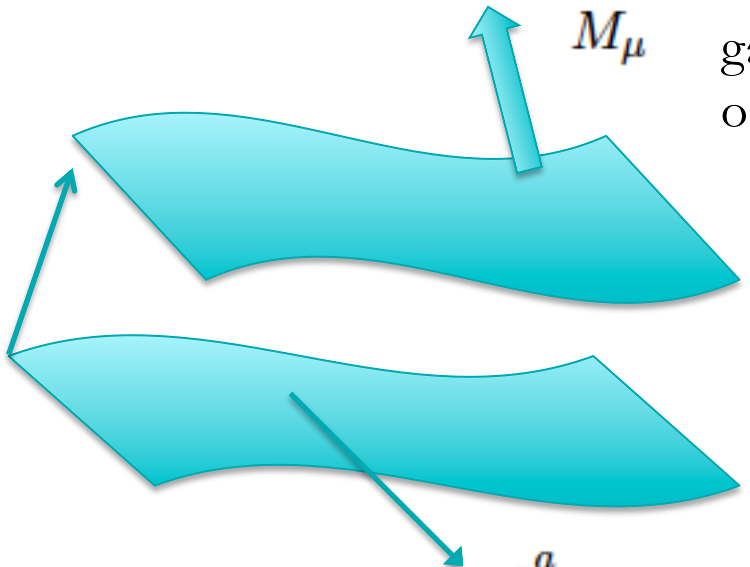
$$R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = R_{\mu\nu}(N) = 0.$$

independent fields: $\tau_\mu,\ e_\mu^a,\ m_\mu$

$h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}$

= gauge fields of Hamiltonian, spatial translations and central charge

Newton-Cartan geometry



gauge field of central extension of Galilean algebra (Bargmann)

spatial metric:

$$h_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$$

NC geometry = no torsion	→ $\tau_\mu = \partial_\mu t$	notion of absolute time
TTNC geometry = twistless torsion	→ $\tau_\mu = \text{HSO}$	preferred foliation in equal time slices
TNC geometry	no condition on τ_μ	

- in TTNC: torsion measured by $a_\mu = \mathcal{L}_{\hat{v}} \tau_\mu$
  geometry on spatial slices is Riemannian

Adding torsion to NC

Christensen,Hartong,Rollier,NO
Hartong,Kiritsis,NO/Hartong,NO
Bergshoeff,Hartong,Rosseel

- inverse vielbeins (v^μ, e^μ_a)

$$v^\mu \tau_\mu = -1\,, \qquad v^\mu e^a_\mu = 0\,, \qquad e^\mu_a \tau_\mu = 0\,, \qquad e^\mu_a e^b_\mu = \delta^b_a$$

can build Galilean boost-invariants

$$\begin{aligned} \hat{v}^\mu &= v^\mu - h^{\mu\nu} M_\nu\,, \\ \bar{h}_{\mu\nu} &= h_{\mu\nu} - \tau_\mu M_\nu - \tau_\nu M_\mu\,, \\ \tilde{\Phi} &= -v^\mu M_\mu + \tfrac{1}{2} h^{\mu\nu} M_\mu M_\nu\,, \end{aligned}$$

-introduce Stueckelberg scalar chi
(to ensure N-invariance):

$$M_\mu = m_\mu - \partial_\mu \chi.$$

→ affine connection of TNC (inert under G,J,N)

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}\right)$$

with torsion $\Gamma^\rho_{[\mu\nu]} = -\frac{1}{2}\hat{v}^\rho(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu)$

$$\nabla_\mu \tau_\nu = 0\,, \qquad \nabla_\mu h^{\nu\rho} = 0\,,$$

analogue of metric compatibility

torsion in NC (recent activity)

- NC introduced in problem of FQH — Son (1306)

- TNC first observed as bdry geometry in z=2 Lifshitz holography — Christensen,Hartong,Rollier,NO (1311)
  & generalized to large class with general z — Hartong,Kiritsis,NO (1409)

- TTNC introduced in FQH — Geracie,Son,Wu,Wu(1407))

- TNC from gauging Schrödinger algebra — Bergshoeff,Hartong,Rosseel (1409)
- TNC from gauging Bargmann (with torsion) — Hartong,NO (1504)

- coupling of non-relativistic field theories to TNC (independent of holography) — Jensen (1408), Hartong,Kiritsis,NO (1409)

- TNC related to warped geometry that couples to 2D WCFT — Hofmann,Rollier (1411)

- other approaches — Banerjee,Mitra,Mukherjee (1407), Brauner,Endlich,Monin,Penco(1407), Bekaert,Morand (1412)

- recent activity using NC/TNC in CM (strongly-correlated electron system, FQH) — Gromov,Abanov][Moroz,Hoyos][Geracie,Son] [Wu,Wu],[Geracie,Golkar,Roberts] ,….

- (T)NC from non-rel limits — Jensen,Karch (1412) , Bergshoeff,Rosseel,Zojer (1505)

Coupling FTs to TNC

[Hartong,Kiritsis,NO]

- action functional $S = S[\hat{v}^\mu, h^{\mu\nu}, \tilde{\Phi}]$.

EM tensor: $T^\mu{}_\nu$

mass current T^μ

energy current (density +flux)

momentum current

$$\delta S \sim \int d^{d+1}x e[\mathcal{E}^\mu \delta\tau_\mu + \mathcal{P}_\mu h^\mu{}_\nu \delta v^\nu + \mathcal{T}_{\mu\nu} h^\mu{}_\rho h^\nu{}_\sigma \delta h^{\rho\sigma} + T^\mu \delta m_\mu]$$

spatial stress

mass density

* important to have torsion in order to describe the most general energy current !

- from the various local symmetries:
  - particle number conservation (if extra local U(1))
  - mass current= momentum current (local boosts)
  - symmetric spatial stress (local rotations)

intermezzo: geodesics on NC space-time

- worldline action of non-rel particle of mass m on NC background

$$S = \int d\lambda L = \frac{m}{2} \int d\lambda \frac{\bar{h}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho}$$

[Kuchar], [Bergshoeff et al]

- gives the geodesic equation with NC connection $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$,

* reduces to Newton's law $\frac{d^2 x^i}{dt^2} + \delta^{ij} \partial_j \Phi = 0$,

provided we take

$$M_t = \partial_t M + \Phi\,, \qquad M_i = \partial_i M\,,$$

for flat NC space-time: zero Newtonian potential

symmetries of flat NC = conformal Killing vectors (spanning Lifshitz) + extra

Global symmetries for non-rel FTs on flat NC

- novel phenomenon:

notion of global symmetries depends on type of matter fields (and their couplings)

two scenarios when coupling non-rel FT to TNC background
i) theory has internal local U(1) related to particle #
ii) not

one finds for non-rel. FTs on flat NC:
-> i) mechanism that enhances Lif with:
particle # + Galilean boosts (+ special conformal)
example: Schrödinger model (+ deformations)

-> ii) no sym. enhancement (only Lif symmetry)
example: Lifshitz model

* interplay between conserved currents and space-time isometries is different compared to relativistic case: same mechanism seen in Lifshitz holography !

Dynamical Newton-Cartan geometry

so far: (T)NC geometry was non-dynamial:
- what happens when we allow it to fluctuate ?

- what is the theory of gravity that incorporates local Galilean symmetry ?

(Einstein equivalence principle, but applied to Galilean instead of Lorentz)

recently shown that: [Hartong,NO]

- dynamical NC geometry = projectable HL gravity
- dynamical TTNC geometry = non-projectable HL gravity

* Horava-Lifshitz gravity was originally introduced as non-Lorentz invariant and renormalizable UV completion of gravity
  - phenomenologically viable ?
  - interesting theoretically as alternate bulk gravity theories relevant to i) holography for strongly coupled non-relativistic systems
    ii) alternate theories in cosmology

NC/TTNC gravity

TNC geometry is a natural geometrical framework underlying HL gravity

- NC quantities combine into: $g_{\mu\nu} = -\tau_\mu \tau_\nu + \hat{h}_{\mu\nu}$
- ADM parametrization of metric used in HL gravity:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right)$$

relation:

$$\tau_\mu \sim \text{lapse} \quad , \quad \hat{h}_{\mu\nu} \sim \text{spatial metric} \quad , \quad m_\mu \sim \text{shift} + \text{Newtonian potential} \, ,$$

some features:

- khronon field of BPS appears naturally $\tau_\mu = \psi \partial_\mu \tau$ Blas,Pujolas,Sibiryakov(2010)

  NC (no torsion): $N = N(t)$ projectable HL gravity

  TTNC: $N = N(t, x)$ non-projectable HL gravity
- U(1) extension of HMT emerges naturally as Bargmann U(1)
- new perspective (via chi field) on nature of U(1) symmetry

Horava,Melby-Thompson(2010)

Effective actions reproduce HL

- covariant building blocks:

- extrinsic curvature: $\hat{h}_{\nu\rho}\nabla_\mu \hat{v}^\rho = -K_{\mu\nu}$ spatial curvature $\check{R}_{\mu\nu\sigma}{}^\rho$.

- covariant derivative, torsion vector a_μ , inverse spatial metric $h^{\mu\nu}$

- tangent space invariant integration measure $e = \det(\tau_\mu, e^a_\nu)$

-> construct all terms that are relevant or marginal (up to dilatation weight d+z)

- in 2+1 dimensions for $1 < z \leq 2$

$$S = \int d^3x e \left[C\left(h^{\mu\rho}h^{\nu\sigma}K_{\mu\nu}K_{\rho\sigma} - \lambda\left(h^{\mu\nu}K_{\mu\nu}\right)^2\right) - \mathcal{V}\right]$$

kinetic terms (2[nd] order)

potential:

$$-\mathcal{V} = 2\Lambda + c_1 h^{\mu\nu}a_\mu a_\nu + c_2\mathcal{R} + \delta_{z,2}\left[c_{10}\left(h^{\mu\nu}a_\mu a_\nu\right)^2 + c_{11}h^{\mu\rho}a_\mu a_\rho \nabla_\nu\left(h^{\nu\sigma}a_\sigma\right)\right.$$
$$\left. + c_{12}\nabla_\nu\left(h^{\mu\rho}a_\rho\right)\nabla_\mu\left(h^{\nu\sigma}a_\sigma\right) + c_{13}\mathcal{R}^2 + c_{14}\mathcal{R}\nabla_\mu\left(h^{\mu\nu}a_\nu\right) + c_{15}\mathcal{R}h^{\mu\nu}a_\mu a_\nu\right]$$

Perspectives for HL gravity

new perspectives on HL:

- different vacuum (flat NC space-time): reexamine issues with HL gravity
- IR effective theory for non-relativistic field theories
- insights into non-relativistic quantum gravity corner of $(\hbar, G_N, 1/c)$ cube ?

- relevance for cosmology ?

alternate theories of gravity in cosmological scenarios, effective theories for inflation

- examine TNC gravity (general torsion)

  * relation with vector khronon of [Janiszweski,Karch]

TNC in NR hydro & fluid/gravity correspondence

- TNC of growing interest in cond-mat (str-el, mes-hall) literature

developments in Lifshitz holography can drive development of tools to study dynamics and hydrodynamics of non-rel. systems

Lifshitz hdyro: [Hoyos,Kim,Oz]
Galilean: [Jensen]

(in parallel to progress in the last many years in relativistic fluids and superfluids inspired from the fluid/gravity correspondence in AdS)

TNC right ingredients to start constructing effective TNC theories and their coupling to matter (e.g. QH-effect)

- organizing principle for derivative expansion of stress tensor/mass current (transport coefficients)
- consider boosted Lifshitz black branes & perturb

in progress: [Kiritsis,Matsuo[,[Hartong,NO,Sanchioni]

Outlook

- employ similar techniques to Schrödinger, warped AdS, flat space holography

  [Andrade,Keeler,Peach,Ross,].[Hofman,Rollier][Armas,Blau,Hartong(in progress)]

- adding charge (Maxwell in the bulk)

  adding other exponents (hyperscaling, matter scaling)

  [Kiritsis,Goutereaux][Gath,Hartong,Monteiro,NO]
  [Khveshchenko][Karch][Hartnoll,Karch]

- applications to non-rel. hydrodynamics:

  fluid/gravity: black branes with zero/non-zero particle no. ? Galilean perfect fluids

  Lifsthiz: [Hoyos,Kim,Oz] Galilean: [Jensen]
  in progress: [Kiritsis,Matsuo[,[Hartong,NO,Sanchioni]

- flat space holography: gauging of Caroll group and ultra-relativistic gravity

  [Hartong]

- NC supergravity, NC in string theory [Bergshoeff et al]

- revisit HL gravity using TNC language/connections with NR String Theory

- effective TNC theories and their coupling to matter (e.g. QH-effect)

  [Son] et al

The end

From Bargmann to NC

Andringa, Bergshoeff, Panda, de Roo

Newtonian gravity is a diff invariant theory whose tangent space is Bargmann (make Bargmann local)

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$R_{\mu\nu}(H)$
space translations	P_a	$e_\mu{}^a$	$\zeta^a(x^\nu)$	$R_{\mu\nu}{}^a(P)$
boosts	G_a	$\omega_\mu{}^a$	$\lambda^a(x^\nu)$	$R_{\mu\nu}{}^a(G)$
spatial rotations	J_{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x^\nu)$	$R_{\mu\nu}{}^{ab}(J)$
central charge transf.	N	m_μ	$\sigma(x^\nu)$	$R_{\mu\nu}(N)$

curvature constraints $\quad R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = R_{\mu\nu}(N) = 0.$

leaves as independent fields: $\quad \tau_\mu,\ e_\mu^a,\ m_\mu$

$$h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}$$

transforming as

$$\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu \\
\delta e_\mu^a &= \mathcal{L}_\xi e_\mu^a + \lambda^a \tau_\mu + \lambda^a{}_b e_\mu^b \\
\delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e_\mu^a
\end{aligned}$$

brief intermezzo on TNC and FQH

- TNC used by Son et al to construct effective actions for external gauge field and background "metric" source
  (incorporating full diffeo inv. seen e.g. for non-relativistic particles in external electromagnetic field)

-> can be used to find electromagnetic/gravitational response of quantum Hall fluid

- captures universal features of FQHE (beyond Hall conductivity):

e.g. Wen-Zee term, which encodes Hall viscosity, naturally appears:
coupling between gauge field and spatial curvature

- can get in streamlined way WIs for non-zero spin and g-factor
- rederive viscosity-conductivity relations in easier way

ADM decomposition

relate NC to metric decomposition in HL

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

construction of NC quantities into Galilean inv. symmetric 2-tensor:

$$g_{\mu\nu} = -\tau_\mu \tau_\nu + \hat{h}_{\mu\nu}$$
$$g^{\mu\nu} = -\hat{v}^\mu \hat{v}^\nu + h^{\mu\nu}$$

TTNC: $\tau_\mu = \psi \partial_\mu \tau$ (tau is khronon field) Blas,Pujolas,,Sibiryakov (2010)

Fix foliation $\tau = t$

$$\tau_t = N \quad , \quad \hat{h}_{ti} = \gamma_{ij}N^k \quad , \quad \hat{\gamma}_{ij} = \gamma_{ij} \quad , \quad m_i = -N^{-1}\gamma_{ij}N^j$$

NC: $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0.$ $N = N(t)$ projectable HL gravity

TTNC: $N = N(t,x)$ non-projectable HL gravity

ADM decomposition (cont'd)

- relevance of Newton potential:

$$m_t = -\frac{1}{2N}\gamma_{ij}N^i N^j + N\tilde{\Phi}$$

additional field denoted by A in HMT

- Bargmann U(1) is same U(1) as in HMT
(including introduction of Stueckelberg field, called Newtonian prepotential in HMT)

Details on the action

G, J invariant	τ_μ	$\hat{h}_{\mu\nu}$	$\hat{v}^\mu$	$h^{\mu\nu}$	e	$\tilde{\Phi}$	χ
dilatation weight	$-z$	-2	z	2	$-(z+2)$	$2(z-1)$	$z-2$

U(1) invariant action

$$S = \int d^3x e \left[C \left(h^{\mu\rho} h^{\nu\sigma} K^\chi_{\mu\nu} K^\chi_{\rho\sigma} - \lambda \left(h^{\mu\nu} K^\chi_{\mu\nu} \right)^2 - \tilde{\Phi}_\chi \left(\mathcal{R} - 2\Omega \right) \right) - \mathcal{V} \right]$$

$$\tilde{\Phi}_\chi = \tilde{\Phi} + \hat{v}^\mu \partial_\mu \chi + \frac{1}{2} h^{\mu\nu} \partial_\mu \chi \partial_\nu \chi$$

Diffeomorphism and scale Ward identities

- diffeos -> on-shell WI

$$\nabla_\nu T^\nu{}_\mu + \text{torsion terms} + \rho \nabla_\mu \tilde{\Phi} = 0$$

* conserved currents $\partial_\nu \left(e K^\mu T^\nu{}_\mu \right) = 0\,.$

for K a TNC Killing vector:

extra force term

- if theory has scale invariance:

can use TNC analogue of dilatation connection

$$z\mathcal{E} + \mathrm{Tr}\, T_{\text{spatial}} + 2(z-1))\rho\Phi = 0$$

z-deformed trace WI

Flat NC space-time

- Riemannian:

symmetries of flat space (i.e. Minkowski) = Poincare (or conformal)
-> relativistic field theories on Minkowski are Poincare invariant

notion of flat NC:
- use global inertial coordinates (t, x^i) $\Gamma^{\rho}_{\mu\nu} = 0 \to M_\mu = \partial_\mu M \, .$

symmetries of flat NC = conformal Killing vectors (spanning Lifshitz) + extra

- flat NC space should include M=const.
  * turns out that one can allow for more general choices that are equivalent to M=const by local syms of the theory
  -> defines the notion of orbit of M (depends on matter couplings, see later)

* in spirit: analogous to fact that relativistic field theories that are also conformal need special couplings

Schrödinger model & Lifshitz model

- simplest toy model for coupling non-rel. scale-inv theory to TNC (z=2)

$$S = \int d^{d+1}x e \left(-i\phi^\star \hat{v}^\mu \partial_\mu \phi + i\phi \hat{v}^\mu \partial_\mu \phi^\star - h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^\star - 2\tilde{\Phi} \phi \phi^\star - V_0 (\phi\phi^\star)^{\frac{d+2}{d}} \right)$$

-> gives Schr. equation

* can also consider deformations preserving local scale inv

- other possibility: do not couple to $\tilde{\Phi}$ -> e.g. z=2 Lifshitz model

$$S = \int d^{d+1}x e \left[\frac{1}{2} (\hat{v}^\mu \partial_\mu \phi)^2 - \frac{\lambda}{2} (h^{\mu\nu} \nabla_\mu \partial_\nu \phi)^2 \right]$$

The Schrödinger model and deformations

- simplest toy model for coupling non-rel scale-inv theory to TNC (z=2)

$$S = \int d^{d+1}xe\left(-i\phi^\star\hat{v}^\mu\partial_\mu\phi + i\phi\hat{v}^\mu\partial_\mu\phi^\star - h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^\star - 2\tilde{\Phi}\phi\phi^\star - V_0(\phi\phi^\star)^{\frac{d+2}{d}}\right)$$

* consider deformations preserving local scale inv

$$\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$$

- change the potential $V_0\varphi^{\frac{2(d+2)}{d}}(1+b\theta^2)$

- adding the term: $-a\int d^{d+1}xe\varphi^2 h^{\mu\nu}\tilde{\nabla}_\mu\partial_\nu\theta \quad - a\int d^{d+1}xe\varphi^2 e^\mu_a \mathcal{D}_\mu M^a$

- can show that a-deformed model has local symmetry

$\delta M_\mu = \partial_\mu\alpha\,, \qquad \delta\theta = -\alpha\,,$ giving on-shell WI $\partial_\mu\left(eT^\mu\right) = 0\,,$

* diffeos + local boosts (+ possibly local scale) induce trafos of type:

$\tilde{N}\;:\qquad \delta v^\mu = 0\,,\qquad \delta h^{\mu\nu} = 0\,,\qquad \delta M_\mu = \partial_\mu\tilde{\sigma}\,,$

-> posssibility of extra global symmetries (intimately connected to vector field)

Scale invariant FTs on flat NC

- role of M is non-trivial: consider the toy FT models

  - (deformed) Schrödinger model:

$$S = \int d^{d+1}x \left(-\varphi^2 \left[\partial_t (\theta + M) + \frac{1}{2}\partial_i (\theta + M)\, \partial^i (\theta + M) + a \partial_i \partial^i (\theta + M) \right] - \frac{1}{2}\partial_i \varphi \partial^i \varphi - V_0 \varphi^{\frac{2(d+2)}{d}} \left(1 + b\theta^2\right) \right),$$

  - Lifshitz model:

$$S = \int d^{d+1}x \left[\frac{1}{2} \left(\partial_t \phi + \partial^i M \partial_i \phi\right)^2 - \frac{\lambda}{2} \left(\partial_i \partial^i \phi\right)^2 \right]$$

can we remove M by local transformations (field redefinitions) ?
and get M=const. : depends on the model in question

b=0: $\tilde{\theta} = \theta + M$ → Sch-invariant for a=0
Lif + Galilean boost for a not zero

(consequence of local U(1) symmetry)

b not zero
& Lifsthiz model → only Lif invariance

More on the b=0 model

- in terms of the physical field $\tilde{\theta} = \theta + M$

$$S = \int d^{d+1}x \left(-\varphi^2 \left[\partial_t \tilde{\theta} + \frac{1}{2} \partial_i \tilde{\theta} \partial^i \tilde{\theta} + a \partial^2 \tilde{\theta} \right] - \frac{1}{2} \partial_i \varphi \partial^i \varphi - V_0 \varphi^{\frac{2(d+2)}{d}} \right)$$

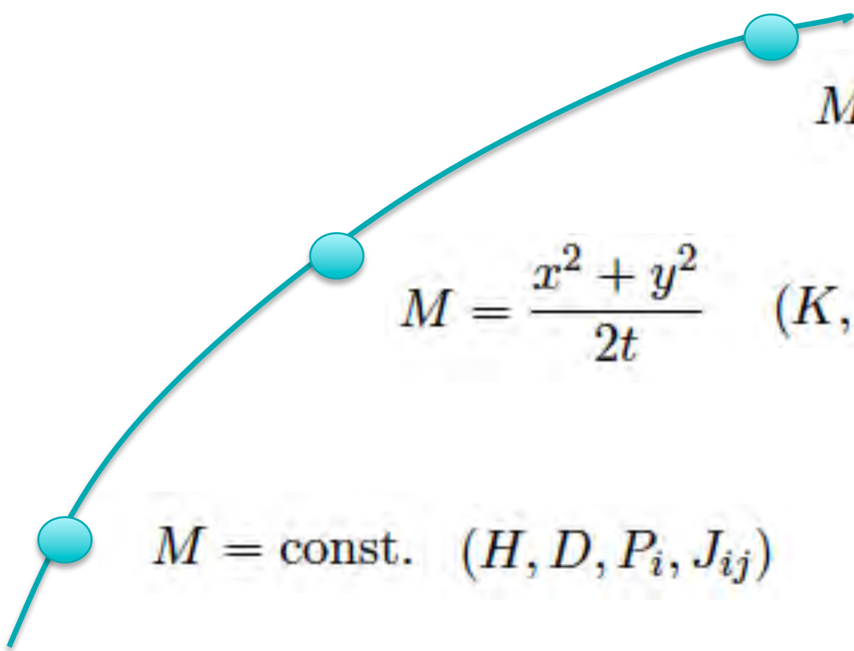
Lifshitz invariance
+ Galilean boost

$$t = t' , \qquad x^i = x'^i - v^i t' ,$$

$$\tilde{\theta} = \tilde{\theta}' + \frac{1}{2} v^i v^i t' - v^i x'^i ,$$

Orbits of M

- the M functions related to M=const by residual trafos define orbit
    * maximal orbit underlies Sch symmetry (as in undeformed Sch model)



$$M = -\frac{1}{2}V^iV^it + V^ix^i$$

$$M = \frac{x^2+y^2}{2t} \quad (K, D, G_i, J_{ij})$$

$$M = \text{const.} \quad (H, D, P_i, J_{ij})$$

- for each choice of M: CKVs form a Lifshitz subalgebra
    - residual trafos of flat NC with $\delta M = 0$

-> will be useful next when we look at Lif vacuum (in holography)

Orbits of M

- the M functions related to M=const by residual trafos are characterized by

$$\tilde{\Phi} = \partial_t M + \frac{1}{2}\partial_i M \partial^i M = 0 \,.$$

$$0 = \partial_i \partial_j \partial^j M \,,$$

$$0 = \partial_i \partial_j M - \frac{1}{d}\delta_{ij}\partial_k \partial^k M \,.$$

* maximal orbit underlies Sch symmetry (as in undeformed Sch model)

-> will be useful later when we look at Lif vacuum (in holography)

- families of M solutions:

$$M = C + \frac{(x^i - x_0^i)(x^i - x_0^i)}{2(t - t_0)}$$

$$M = C - \frac{1}{2}V^i V^i t + V^i x^i$$

TNC Killing vectors

[Kiritsis,Hartong,NO]2,3

- consider residual trafos with $\delta M = 0$

* correspond to conformal Killing vectors

$$
\begin{aligned}
\mathcal{L}_K \tau_\mu &= -z\Omega\tau_\mu\,, & \mathcal{L}_K \hat{v}^\mu &= z\Omega\hat{v}^\mu\,, & \mathcal{L}_K \bar{h}_{\mu\nu} &= -2\Omega\bar{h}_{\mu\nu} \\
\mathcal{L}_K h^{\mu\nu} &= 2\Omega h^{\mu\nu}\,, & \mathcal{L}_K \Phi_N &= 2(z-1)\Omega\Phi_N\,, & \mathcal{A}\Omega &= 0
\end{aligned}
$$

$$
\begin{aligned}
M &= \text{cst} && H, D, P_i, J_{ij}\,, \\
M &= \frac{x^2+y^2}{2t} && K, D, G_i, J_{ij}\,, \\
M &= -\frac{1}{2}V^iV^it + V^ix^i && H, D, P_i, J_{ij}\,,
\end{aligned}
$$

- for each choice of M: CKVs form a Lifshitz subalgebra

$$
\begin{aligned}
H &= \partial_t\,, & P_i &= \partial_i\,, \\
G_i &= t\partial_i\,, & J_{ij} &= x_i\partial_j - x_j\partial_i\,, \\
D &= zt\partial_t + x^i\partial_i\,, & K &= t^z\partial_t + t^{z-1}x^i\partial_i\,,
\end{aligned}
$$

Local realization of Schr on M

CKVs can be used to generate maximal orbit: of Sch sym

$$H = \partial_t \,, \qquad P_i = \partial_i \,,$$
$$G_i = t\partial_i + x_i \tilde{N} \,, \qquad J_{ij} = x_i\partial_j - x_j\partial_i \,,$$
$$D = zt\partial_t + x^i\partial_i \,,$$

$\tilde{N}$ shifts of M

$$K = t^2\partial_t + tx^i\partial_i + \frac{1}{2}x^i x^i \tilde{N} \,.$$

Symmetries of the Lifshitz vacuum (back to holography)

[Kiritsis,Hartong,NO]3

- what is bulk realization of residual syms of flat NC ?

Lif metric for any M in flat NC-orbit

$$ds^2 = \left(\frac{dr}{r} - \frac{1}{d}\partial_i\partial^i M dt\right)^2 - \frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dx^i - \partial^i M dt\right)^2 .$$

- sources for Lif vacuum transform under Sch group (via bulk PBH trafos)
  - trafos for given M in M=const orbit: Lifshitz
  - delta M trafos lie in Sch algebra

- in suitable bulk coords this is dual to:

flat NC with CKVs spanning Lif and Sch realized locally on $M_\mu = \partial_\mu M$.

- possible to have conserved particle number associated to local shifts in M (generated by Galilean and special conformal)

Schrödinger invariant probe actions

have seen: FTs on flat NC realize Sch with mechanism in which
M is ``eaten" up
(generators outside Lif are realized as projective transformations)

-> projective realizations of space-time syms cannot be predicted
by looking at Killing vectors

- can construct z=2 probe actions on Lifshitz bulk geometries that are invariant under Sch (in same manner as in FT setting)

use covariant characterization of Lif

$$ds^2 = (-B_M B_N + \gamma_{MN})\, dx^M dx^N$$

$B^2 = -1$ $\quad$ γ_{MN} is orthogonal to B^M

$$S = \int d^4x \sqrt{-g}\, (\gamma^{MN}\partial_M\phi^\star\partial_N\phi + iq\phi^\star B^M\partial_M\phi - iq\phi B^M\partial_M\phi^\star - (m^2 - q^2)\phi^\star\phi)$$

eat up M: $\quad \phi = \exp[-iqM - \frac{i}{4}qr^2\partial^2 M]\tilde{\phi}$ $\quad$ + use all props of M

$$r^2\left(\partial_i\partial^i\tilde{\phi} + 2iq\partial_t\tilde{\phi}\right) + r^2\partial_r^2\tilde{\phi} - 3r\partial_r\tilde{\phi} - (m^2 - q^2)\tilde{\phi} = 0$$

Particle number current

local transformations of source M -> WI for $\partial_\mu T^\mu$

$$\delta S^{\text{ren}}_{\text{on-shell}}[M] = -\int d^{d+1}x \partial_\mu T^\mu \delta M$$

can show

$$\begin{aligned}\partial_\mu T^\mu &= -\partial_t \lambda_1 - \partial_i(\lambda_1 \partial^i M) - \partial_i \partial_j \partial^j \lambda^i + \left(\partial_i \partial_j - \frac{1}{d}\delta_{ij}\partial_k \partial^k\right)\lambda^{ij} \\ &= -\partial_t \lambda_1 - \partial_i(\lambda_1 \partial^i M) + \left(\partial_i \partial_j \Lambda^{ij} - \frac{1}{d}\partial_i \partial^i \Lambda^k{}_k\right) \equiv \partial_\mu J^\mu \,,\end{aligned}$$

-> local Sch inv of on-shell action with flat NC bcs can lead to conserved current

$$\partial_\mu \left(T^\mu - J^\mu\right) = 0 \,.$$

so possible to have conserved particle number associated to local shifts in M (generated by Galilean and special conformal)

Lessons from z=2 holographic Lifshitz model

[Christensen,Hartong,NO,Rollier]

considered first a specific z=2 example (in 4D) that can be obtained by Scherk-Schwarz dim. reduction (null on bdry) froma 5D AlAdS solution

[Donos,Gauntlet][Cassani,Faedo][Chemissany,Hartong]

counterterms and reduction: [Papadimitriou][Chemissany,Geisbuehler,Hartong,Rollier]

important lessons:

- use of vielbeins highly advised (see also [Ross])
- identification of sources requires appropriate lin. combo of timelike vielbein and bulk gauge field (-> crucial for boundary gauge field)
- bdr. geometry is torsional Newton-Cartan
- can compute unique gauge and tangent space inv. bdry stres tensor
- WIs take TNC covariant form
- conserved quantities from WIs and TNC (conformal) Killing vectors

EPD model and AlLif spacetimes

bulk theory

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}W(\Phi)B^2 - V(\Phi)\right)$$

- admits Lifshitz solutions with z>1

For AlLif BCs useful to write:

$$ds^2 = \frac{dr^2}{R(\Phi)r^2} - E^0E^0 + \delta_{ab}E^aE^b\,, \qquad B_M = A_M - \partial_M\Xi$$

[Ross],[Christensen,Hartong,NO,Rollier]
[Hartong,Kiritsis,NO]1

then AlLif BCs

$$\begin{array}{rclcrcl} E^0_\mu & \propto & r^{-z}\tau_\mu + \dots & \quad & E^a_\mu & \propto & r^{-1}e^a_\mu + \dots \\ A_\mu - \alpha(\Phi)E^0_\mu & \propto & r^{z-2}\tilde{m}_\mu + \dots & & A_r & = & (z-2)r^{z-3}\chi + \dots \\ \Xi & = & r^{z-2}\chi + \dots & & \Phi & = & r^\Delta\phi + \dots \end{array}$$

Transformation of sources

use local bulk symmetries:
local Lorentz, gauge transformations and diffs preserving metric gauge

these symmetries induce an action on sources: $\tau_\mu,\ e^a_\mu,\ \tilde{m}_\mu,\ \chi$

= action of Bargmann algebra plus local dilatations = Schrödinger

there is thus a Schrödinger Lie algebra valued connection given by

$$A_\mu = H\tau_\mu + P_a e^a_\mu + M m_\mu + \frac{1}{2} J_{ab}\omega_\mu{}^{ab} + G_a \omega_\mu{}^a + D b_\mu$$

with $\tilde{m}_\mu = m_\mu - (z-2)\chi b_\mu$

with appropriate curvature constrains that reproduces trafos of the sources

Torsional Newton-Cartan (TNC) geometry

source	ϕ	τ_μ	e_μ^a	v^μ	e_a^μ	$\tilde{m}_0$	$\tilde{m}_a$	χ
scaling dimension	Δ	$-z$	-1	z	1	$2z-2$	$z-1$	$z-2$

includes inverse veilbeins (v^μ, e_a^μ)

$$v^\mu \tau_\mu = -1\,, \qquad v^\mu e_\mu^a = 0\,, \qquad e_a^\mu \tau_\mu = 0\,, \qquad e_a^\mu e_\mu^b = \delta_a^b$$

from (inverse) vielbeins and vector: $M_\mu = \tilde{m}_\mu - \partial_\mu \chi$

can build Galilean boost-invariants

$$\begin{aligned} h^{\mu\nu} &= \delta^{ab} e_a^\mu e_b^\nu\,, & \hat{v}^\mu &= v^\mu - h^{\mu\nu} M_\nu \\ \bar{h}_{\mu\nu} &= \delta_{ab} e_\mu^a e_\nu^b - \tau_\mu M_\nu - \tau_\nu M_\mu\,, & \Phi_N &= -v^\mu M_\mu + \tfrac{1}{2} h^{\mu\nu} M_\mu M_\nu \end{aligned}$$

affine connection of TNC

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu} \right)$$

with torsion $\Gamma^\rho_{[\mu\nu]} = -\frac{1}{2} \hat{v}^\rho (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu)$