Hyperscaling-violating Lifshitz hydrodynamics from black holes

Yoshinori Matsuo

University of Crete

In collaboration with Elias Kiritsis

arXiv:1507.nnnnn

Introduction

The holography gives the correspondences between black holes and fluids.

[Son-Starinets, 02] [Policastro-Son-Starinets, 02] [Bhattacharyya-Hubeny-Minwalla-Rangamani, 07]

Lifshitz spacetimes give holographic description of the Lifshitz scaling invariant theories.

[Son, 08] [Balasubramanian-McGreevy, 08]

[Kachru-Liu-Mulligan, 08] [Taylor, 08]

$$ds^2 = -r^{2z}dt^2 + r^2(dx^i)^2 + \frac{dr^2}{r^2}$$

Boundary theories in the Lifshitz spacetimes have the Newton-Cartan geometry as a background.

[Christensen-Hartong-Obers-Rollier, 14]

[Hartong-Kiritsis-Obers, 14, 14, 15]

Here we would like to see that

Fluid/gravity correspondence for the Lifshitz spacetime give a holographic description of fluids in Newton-Cartan background.

Einstein-Maxwell-Dilaton model

The action

$$S = \frac{1}{16\pi G}\int d^{d+1}x\,\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}\mathrm{e}^{\lambda\phi}F^2 - \frac{1}{2}(\partial\phi)^2\right)$$

Lifshitz spacetime solution

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + \sum_i r^2\left(dx^i\right)^2$$

$$A = ar^{z+d-1}dt \qquad e^{\lambda\phi} = \mu r^{2(1-d)}$$

$$\lambda^2 = 2\,\frac{d-1}{z-1}$$

$$\Lambda = -\frac{(z+d-1)(z+d-2)}{2}$$

$$\mu a^2 = \frac{2(z-1)}{z+d-1}$$

The metric has Lifshitz scaling symmetry

$$t \to c^z t \qquad x^i \to c\,x^i \qquad r \to c^{-1}r$$

The gauge field A and dilaton ϕ breaks the scaling symmetry

Hydrodynamic ansatz

The black hole solution in the Eddington-Finkelstein coordinate

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i\right)^2$$

$$A = a r^{z+d-1} f(r) dt - a r^2 dr \qquad e^{\lambda\phi} = \mu r^{2(1-d)}$$

$$f(r) = 1 - \frac{r_0^{z+d-1}}{r^{z+d-1}}$$

Introduce Galilean boost ($x^i \to x^i - v^i t$) and x^μ-dependence

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i(x) dt\right)^2$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_i(x)\left(dx^i - v^i(x) dt\right)$$

$$e^{\lambda\phi} = \mu(x) r^{2(1-d)} \qquad f(r) = 1 - \frac{r_0^{z+d-1}(x)}{r^{z+d-1}}$$

This is not a solition of EOM → We have to introduce correction terms

Derivative expansion

We introduce the correction terms

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i(x) dt\right)^2 + h_{\mu\nu} dx^\mu dx^\nu$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_i(x) \left(dx^i - v^i(x) dt\right) + a_\mu dx^\mu$$

$$e^{\lambda\phi} = \mu(x) r^{2(1-d)} e^{\lambda\varphi}$$

Hydrodynamic regime: variation of x^μ-dependence is very slow.

→ derivative expansion (with respect to ∂_μ)

$$r_0(x) = r_0(0) + x^\mu \partial_\mu r_0(0) + \cdots$$

$$v^i(x) = v(0) + x^\mu \partial_\mu v^i(0) + \cdots$$

$$a(x) = a(0) + x^\mu \partial_\mu a(0) + \cdots$$

Derivative expansion

We introduce the correction terms

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i(x) dt \right)^2 + h_{\mu\nu} dx^\mu dx^\nu$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_i(x) \left(dx^i - v^i(x) dt \right) + a_\mu dx^\mu$$

$$e^{\lambda \phi} = \mu(x) r^{2(1-d)} e^{\lambda \varphi}$$

First order EOM

Correction terms $\psi = (h_{\mu\nu}, a_\mu, \varphi)$ are treated as first order in ∂_μ.

$$\psi''(r) + F_1(r)\psi'(r) + F_2 \psi(r) = F_3(\partial_\mu r_0, \partial_\mu v^i, \cdots)$$

Linear terms of correction terms $h_{\mu\nu}, a_\mu, \varphi$

Linear terms of source terms $\partial_\mu r_0, \partial_\mu v^i, \partial_\mu a, \cdots$

We solve these linear inhomogeneous differential equations

First order solution (for $d = 4, z = 2$)

$$ds^2 = -r^4 f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i dt\right)^2$$
$$+\frac{2}{3} r^2 \partial_i v^i dt^2 - r^2 F(r) \sigma_{ij} \left(dx^i - v^i dt\right)\left(dx^j - v^j dt\right)$$

$$A = a\left(r^5 f(r) - \frac{1}{3} r^3 \partial_i v^i\right) dt - a r^2 dr + \mathcal{A}_i \left(dx^i - v^i dt\right)$$

$$e^{\lambda\phi} = \mu r^{2(1-d)} \qquad f(r) = 1 - \frac{r_0^5}{r^5}$$

$$F(r) = \int dr \frac{r^3 - r_0^3}{r(r^5 - r_0^5)} \qquad \sigma_{ij} = \partial_i v^j + \partial_j v^i - \frac{2}{3}\delta_{ij} \partial_k v^k$$

It must satisfy the following constraints

$$0 = \partial_t a + v^i \partial_i a - a \partial_i v^i$$

$$0 = \partial_t r_0 + v^i \partial_i r_0 + \frac{1}{3} r_0 \partial_i v^i \qquad \mu(x) a^2(x) = \frac{2(z-1)}{z+d-1}$$

$$0 = \partial_t \mathcal{A}_i + v^j \partial_j \mathcal{A}_i + \mathcal{A}_j \partial_i v^j + a \partial_i r_0^5$$

Boundary stress-energy tensor for Lifshitz

Induced metric in terms of the vielbein

$$\gamma_{\mu\nu} = -r^{2z} f \tau_\mu \tau_\nu + r^2 e_\mu^a e_\nu^a \qquad \gamma^{\mu\nu} = -r^{-2z} f^{-1} \hat{v}^\mu \hat{v}^\nu + r^{-2} e_a^\mu e_a^\nu$$

Gauge field in this frame: $\hat{A}_0 = \hat{v}^\mu A_\mu \qquad \hat{A}_a = e_a^\mu A_\mu$

Variation of the action in these variables

$$\delta S = \int d^d x \left(-\hat{S}_\mu^0 \delta \hat{v}^\mu + \hat{S}_\mu^a \delta e_a^\mu + \hat{J}^0 \delta \hat{A}_0 + \hat{J}^a \delta \hat{A}_a + \mathcal{O}_\phi \delta \phi \right)$$

The stress-energy tensor and current are defined by

$$\hat{T}^\mu{}_\nu = \hat{S}_\nu^0 \hat{v}^\mu - \hat{S}_\nu^a e_a^\mu \qquad J^\mu = \hat{J}^0 \hat{v}^\mu + \hat{J}^a e_a^\mu$$

The stress-energy tensor is related to Brown-York tensor $T^\mu{}_\nu$ as

$$\hat{T}^\mu{}_\nu = T^\mu{}_\nu + J^\mu A_\nu \qquad \text{where} \qquad T^{\mu\nu} = \frac{1}{8\pi G}\left(\gamma^{\mu\nu} K - K^{\mu\nu}\right)$$

$\hat{T}^\mu{}_\nu$ is asymmetric tensor

$K_{\mu\nu}$: extrinsic curvature

Stress-energy tensor for Lifshitz fluid

We introduce the counter term

$$S_{\mathrm{ct}} = \int d^4x\sqrt{-\gamma}\left[-(5+z) + \frac{z+d-1}{2}e^{\lambda\phi}\gamma^{\mu\nu}A_\mu A_\nu\right]$$

The renormalized stress-energy tensor

$$\tilde{T}^0{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{3}{2}r_0^{z+3} - \frac{z-1}{a}v^i\mathcal{A}_i\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{z+3}{2}r_0^{z+3}v^i + \frac{z(z+3)}{4(z-1)}r_0^{2z}\partial_i r_0 - \frac{z-1}{a}v^i v^j\mathcal{A}_j + \frac{1}{2}r_0^3\sigma_{ij}v^j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^0{}_i = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\frac{z-1}{a}\mathcal{A}_i + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_j = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[\frac{z}{2}r_0^{z+3} - \frac{1}{2}r_0^3\sigma_{ij} + \frac{z-1}{a}v^i\mathcal{A}_j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

Stress-energy tensor for Lifshitz fluid

Since the volume form behaves as

$$\sqrt{-\gamma} \sim r^{z+3}$$

The leading terms of the following stress-energy tensor gives regular contributions

$$\tilde{T}^0{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{3}{2}r_0^{z+3} - \frac{z-1}{a}v^i\mathcal{A}_i\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{z+3}{2}r_0^{z+3}v^i + \frac{z(z+3)}{4(z-1)}r_0^{2z}\partial_i r_0 - \frac{z-1}{a}v^i v^j\mathcal{A}_j + \frac{1}{2}r_0^3\sigma_{ij}v^j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^0{}_i = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\frac{z-1}{a}\mathcal{A}_i + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_j = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[\frac{z}{2}r_0^{z+3} - \frac{1}{2}r_0^3\sigma_{ij} + \frac{z-1}{a}v^i\mathcal{A}_j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

Constraints and conservation law

The following constraints in EOM gives the conservation law on the boundary

$$n^\mu \gamma^{\nu\rho} R_{\mu\nu} = 8\pi G n^\mu \gamma^{\nu\rho} T_{\mu\nu} \qquad n_\nu \nabla_\mu \left(e^{\lambda\phi} F^{\mu\nu}\right) = 0$$

This should agree with the conservation law for $\tilde{T}^\mu{}_\nu$.

The constraints do not agree with $D_\mu \tilde{T}^\mu{}_\nu = 0$,
but agree with the conservation law in the Newton-Cartan theory.

$$D_\mu \mathcal{E}^\mu = -\frac{1}{2}(D^\mu \hat{v}^\nu + D^\nu \hat{v}^\mu)\mathcal{T}_{\mu\nu}$$

$$D_\mu \mathcal{T}^\mu{}_i = \hat{v}^\mu D_i \mathcal{P}_\mu - D_\mu(\hat{v}^\mu \mathcal{P}_i)$$

$$D_\mu \mathcal{J}^\mu = 0 \qquad \text{where} \qquad \mathcal{J}^\mu = \frac{1}{a}\hat{v}^\mu$$

Energy flow $\mathcal{E}^\mu$, momentum density $\mathcal{P}_\mu$ and stress tensor $\mathcal{T}^\mu{}_\nu$ are defined as

$$\mathcal{E}^\mu = -\tilde{T}^\mu{}_\nu \hat{v}^\nu \qquad \mathcal{P}_\mu = \tilde{T}^\rho{}_\nu \tau_\rho P^\nu_\mu \qquad \mathcal{T}^\mu{}_\nu = \tilde{T}^\rho{}_\sigma P^\mu_\rho P^\sigma_\nu$$

$\hat{v}^\mu = \left(1, v^i\right)$ $\qquad$ $P^\mu_\nu = e^\mu_a e^a_\nu$: Projection to spatial direction

Newton-Cartan geometry

Non-relativistic theory of gravity

Described by Newton-Cartan data $\left(\tau_\mu\,, h^{\mu\nu}, \hat{v}^\mu, \hat{A}_\mu\right)$

1-form τ defines time direction.

$h^{\mu\nu}$ is inverse metric on time slice.

} Galilei data

Galilei data $(\tau_\mu, h^{\mu\nu})$ satisifies

$$\tau_\mu h^{\mu\nu} = 0$$

Galilei data is constant. For covariant derivative D_μ

$$D_\mu \tau_\nu = 0 \qquad\qquad D_\rho h^{\mu\nu} = 0$$

Galilei connection is not unique.

In order to define connection, we have to introduce $\hat{v}^\mu$ and $\hat{A}_\mu$.

Conservation law in Newton-Cartan theory

In the Newton-Cartan theory variation of the action is

$$\delta S = \int d^d x \sqrt{-\gamma} [J^\mu \delta \hat{A}_\mu - \mathcal{P}_\mu \delta \hat{v}^\mu - \mathcal{E}^\mu \delta \tau_\mu - \frac{1}{2} \mathcal{T}_{\mu\nu} \delta h^{\mu\nu}]$$

Under the infinitesimal coordinate transformation ξ^μ (for torsionless case)

$$\delta_\xi \tau_\mu = D_\mu(\xi^\nu \tau_\nu) \qquad \delta_\xi h^{\mu\nu} = h^{\mu\rho} D_\rho \xi^\nu + h^{\nu\rho} D_\rho \xi^\mu$$

$$\delta_\xi \hat{v}^\mu = \xi^\rho D_\rho \hat{v}^\nu - \hat{v}^\rho D_\rho \xi^\nu \qquad \delta_\xi \hat{A}_\mu = -\hat{F}_{\mu\nu} \xi^\nu$$

Since the covariant derivative of $\hat{v}^\mu$ does not vanish,
the conservation law is different from ordinary cases.

$$D_\mu \mathcal{E}^\mu = -\hat{F}_{\mu\nu} \hat{v}^\mu J^\nu - \frac{1}{2} (D^\mu \hat{v}^\nu + D^\nu \hat{v}^\mu) \mathcal{T}_{\mu\nu}$$

$$D_\mu \mathcal{T}^\mu{}_\nu = \hat{v}^\mu D_\nu \mathcal{P}_\mu - D_\mu(\hat{v}^\mu \mathcal{P}_\nu) - \hat{F}_{\mu\nu} J^\mu$$

Fluid variables and fluid equations

Energy flow, momentum density and stress tensor takes the following form

$$\mathcal{E}^0 = \mathcal{E} \qquad \mathcal{E}^i = \mathcal{E}v^i - \kappa\partial_i T$$

$$\mathcal{P}_i = q\mathcal{A}_i \qquad \mathcal{T}_{ij} = P\delta_{ij} - \eta\sigma_{ij}$$

The energy density, pressure and charge density are

$$\mathcal{E} = \frac{3}{16\pi G} r_0^{z+3} \quad P = \frac{z}{16\pi G} r_0^{z+3} \qquad q = \frac{z-1}{a} \qquad T = \frac{z+3}{4\pi} r_0^z$$

The shear viscosity and thermal conductivity are

$$\eta = \frac{1}{16\pi G} r_0^3 \qquad \kappa = \frac{1}{8G(z-1)} r_0^{z+1}$$

Bulk viscosity vanishes.

Energy density and pressure satisfy the Lifshitz scaling condition

$$z\mathcal{E} = (d-1)P$$

Fluid variables and fluid equations

Energy flow, momentum density and stress tensor takes the following form

$$\mathcal{E}^0 = \mathcal{E} \qquad \mathcal{E}^i = \mathcal{E} v^i - \kappa \partial_i T$$

$$\mathcal{P}_i = q\mathcal{A}_i \qquad \mathcal{T}_{ij} = P\delta_{ij} - \eta\sigma_{ij}$$

The conservation law gives the fluid equations

$$D_\mu \mathcal{E}^\mu = -\frac{1}{2}(D^\mu \hat{v}^\nu + D^\nu \hat{v}^\mu)\mathcal{T}_{\mu\nu} \qquad D_\mu \mathcal{J}^\mu = 0$$

$$D_\mu \mathcal{T}^\mu{}_i = \hat{v}^\mu D_i \mathcal{P}_\mu - D_\mu(\hat{v}^\mu \mathcal{P}_i)$$

The fluid equation is expressed as

$$0 = \partial_t \mathcal{E} + v^i \partial_i \mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa \partial_i T)$$

$$0 = \partial_i P + q\partial_t \mathcal{A}_i + q v^j \partial_j \mathcal{A}_i + q\mathcal{A}_j \partial_i v^j - \partial_j\left(\eta\sigma_{ij}\right)$$

$$0 = \partial_t q + \partial_i\left(q v^i\right)$$

Non-relativistic fluid equations

The fluid equations from the Lifshitz black hole are

$$0 = \partial_t \mathcal{E} + v^i \partial_i \mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa\partial_i T)$$

$$0 = \partial_i P + q\partial_t \mathcal{A}_i + q v^j \partial_j \mathcal{A}_i + q\mathcal{A}_j \partial_i v^j - \partial_j\left(\eta\sigma_{ij}\right)$$

$$0 = \partial_t q + \partial_i\left(q v^i\right)$$

Ordinary fluid equations are

$$0 = \partial_t \mathcal{E} + v^i \partial_i \mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa\partial_i T)$$

$$0 = \partial_i P + \rho\partial_t v^i + \rho v^j \partial_j v^i - \partial_j\left(\eta\sigma_{ij}\right)$$

$$0 = \partial_t \rho + \partial_i\left(\rho v^i\right)$$

Energy conservation (1st line) and continuity equation (3rd line) agree.
But the Navier-Stokes does not agree.

Fluid equations in Newton-Cartan theory

The Newton-Cartan theory gives ordinary fluid equations.

Torsionless Newton-Cartan connection is invariant under Milne boost

$$v^i \to v^i + V^i \qquad \hat{A} \to \hat{A} + V^i\left(dx^i - v^i dt\right) - \frac{1}{2}V^2 dt$$

If $\hat{A}_\mu = 0$, for $v^i = 0$, we obtain after the boost to $v^i \neq 0$

$$\hat{A} = v^i dx^i - \frac{1}{2}v^2 dt$$

Then, the Navier-Stokes equation

$$0 = \partial_i P + \rho\partial_t v^i + \rho v^j \partial_j v^i - \partial_j\left(\eta\sigma_{ij}\right)$$

is expressed as

$$\partial_i P - \partial_j\left(\eta\sigma_{ij}\right) = \hat{F}_{i\mu} J^\mu \qquad \text{where} \qquad J^\mu = \rho\hat{v}^\mu$$

Navier-Stokes from Lifshitz black hole

The fluid equation from the Lifshitz black hole

$$0 = \partial_i P + q\partial_t \mathcal{A}_i + q v^j \partial_j \mathcal{A}_i + q\mathcal{A}_j \partial_i v^j - \partial_j(\eta\sigma_{ij})$$

can also be expressed as

$$\partial_i P - \partial_j(\eta\sigma_{ij}) = \mathcal{F}_{i\mu} J^\mu \quad \text{where} \quad \mathcal{A} = \mathcal{A}_i(dx^i - v^i dt)$$

The Navier-Stokes equation in the Newton-Cartan theory is

$$\partial_i P - \partial_j(\eta\sigma_{ij}) = \hat{F}_{i\mu} J^\mu \quad \text{where} \quad \hat{A} = v^i dx^i - \frac{1}{2} v^2 dt$$

They are in the same form but have different gauge fields in r.h.s.

Another ansatz for gauge field

What happens if we use $\mathcal{A} = v^i dx^i - \frac{1}{2} v^2 dt$

instead of $\mathcal{A} = \mathcal{A}_i (dx^i + v^i dt)$ from the beginning?

We generalize $\mathcal{A}_t$ to arbitrary

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i(x) dt\right)^2$$

$$e^{\lambda\phi} = \mu(x) r^{2(1-d)} \qquad f(r) = 1 - \frac{r_0^{z+d-1}(x)}{r^{z+d-1}}$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_i(x)\left(dx^i - v^i(x) dt\right)$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_t(x) dt + \mathcal{A}_i(x) dx^i$$

The solution does not change but constraint for $\mathcal{A} = v^i dx^i - \frac{1}{2} v^2 dt$

$$\partial_i P - \partial_j \left(\eta \sigma_{ij}\right) = \mathcal{F}_{i\mu} J^\mu = -q \partial_t v^i - q v^j \partial_j v^i$$

gives the non-relativistic Navier-Stokes equation.

The stress-energy tensor becomes

$$\tilde{T}^0{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{3}{2}r_0^{z+3} - \frac{z-1}{a}v^i\mathcal{A}_i\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{z+3}{2}r_0^{z+3}v^i + \frac{z(z+3)}{4(z-1)}r_0^{2z}\partial_i r_0 + \frac{z-1}{a}v^i\mathcal{A}_t + \frac{1}{2}r_0^3\sigma_{ij}v^j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^0{}_i = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\frac{z-1}{a}\mathcal{A}_i + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_j = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[\left(zr_0^{z+3} - \frac{z-1}{a}\left(\mathcal{A}_t + v^k\mathcal{A}_k\right)\right)\delta_{ij} - \frac{1}{2}r_0^3\sigma_{ij} + \frac{z-1}{a}v^i\mathcal{A}_j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$\tilde{T}^\mu{}_\nu$ cannot be identified with the fluid stress-energy tensor.

We define new stress-energy tensor $T^{\mu}{}_{\nu}$ as

$$\tilde{T}^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} + J^{\mu}\mathcal{A}_{\nu} - \delta^{\mu}{}_{\nu}J^{\rho}\mathcal{A}_{\rho}$$

Energy density $\mathcal{E}^{\mu}$, pressure P and stress tensor $\mathcal{T}^{i}{}_{j}$ does not change but momentum density vanishes $\mathcal{P}_i = 0$, for new $T^{\mu}{}_{\nu}$.

Now, we take $F_{\mu\nu}J^{\nu}$ term in the conservation into account.

$$D_{\mu}\mathcal{E}^{\mu} = -\mathcal{F}_{\mu\nu}\hat{v}^{\mu}J^{\nu} - \frac{1}{2}(D^{\mu}\hat{v}^{\nu} + D^{\nu}\hat{v}^{\mu})\mathcal{T}_{\mu\nu}$$

$$D_{\mu}\mathcal{T}^{\mu}{}_{\nu} = \hat{v}^{\mu}D_{\nu}\mathcal{P}_{\mu} - D_{\mu}(\hat{v}^{\mu}\mathcal{P}_{\nu}) - \mathcal{F}_{\mu\nu}J^{\mu}$$

Then, these equation gives the non-relativistic fluid equations

$$0 = \partial_t\mathcal{E} + v^i\partial_i\mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa\partial_i T)$$

$$0 = \partial_i P + q\partial_t v^i + qv^j\partial_j v^i - \partial_j\left(\eta\sigma_{ij}\right)$$

$$0 = \partial_t q + \partial_i\left(qv^i\right)$$

Holographic entropy current

Entropy current J_S^μ ⇦ Volume form on time slice at the horizon

$$\epsilon_{\mu_1 \cdots \mu_d} J_S^{\mu_1} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_d}$$

In terms of the normal vector to the horizon $n_\mu = \partial_\mu \mathcal{S}$ where $\mathcal{S} = r - r_0(x)$.

$$J_S^\mu = \frac{\sqrt{h}}{4G} \frac{n^\mu}{n^0}$$

The entropy current for Lifshitz black hole is calculated as

$$J_S^0 = \frac{1}{4G} r_0^3 \qquad J_S^i = \frac{1}{4G} r_0^3 v^i - \frac{z}{8G(z-1)} r_0^z \partial_i r_0$$

Thermodynamic relation

Energy flow, pressure and temperature

$$\mathcal{E}^0 = \frac{1}{16\pi G} r_0^{z+3} \qquad \mathcal{E}^i = \frac{1}{16\pi G}\left(r_0^{z+3} v^i - \frac{z(z+3)}{2(z-1)} r_0^{2z} \partial_i r_0\right)$$

$$P = \frac{z}{16\pi G} r_0^{z+3} \qquad T = \frac{z+3}{4\pi} r_0^z$$

Entropy current

$$J_S^0 = \frac{1}{4G} r_0^3 \qquad J_S^i = \frac{1}{4G} r_0^3 v^i - \frac{z}{8G(z-1)} r_0^z \partial_i r_0$$

They satisfy the thermodynamic relation

$$T J_S^\mu = -\tilde{T}^\mu{}_\nu \hat{v}^\nu + P \hat{v}^\nu = \mathcal{E}^\mu + P \hat{v}^\mu$$

Second law of thermodynamics

Fluid equaitons give

$$0 = 3\partial_t r_0^{z+3} + 3v^i \partial_i r_0^{z+3} + (z+3) r_0^{z+3} \partial_i v^i$$
$$-\sigma_{ij}\sigma_{ij} - \frac{z(z+3)}{2(z-1)} \partial_i (r_0^{2z} \partial_i r_0)$$

Divergence of the entropy current becomes

$$\partial_\mu J_S^\mu = \frac{1}{4G}\left[\partial_t r_0^3 + \partial_i \left(r_0^3 v^i - \frac{z}{2(z-1)} r_0^z \partial_i r_0\right)\right]$$
$$= 2\pi \frac{\eta}{r_0^z} \sigma_{ij}\sigma_{ij} + \frac{z^2}{8G(z-1)} r_0^{z-1} (\partial_i r_0)^2 \geq 0$$

Divergence of the entropy current is non-negative

⇒ 2nd law of thermodynamics

Conclusion

- We have considered fluid/gravity correspondence for the Lifshitz black hole geometry.
- Naïve ansatz gives the stress-energy tensor which satisfy the conservation law of the Newton-Cartan theory.
- Energy conservation and continuity equations agree with those for ordinary non-relativistic fluids.
- The Navier-Stokes equation is different from ordinary one, but in terms of the gauge field, it agrees with that in the Newton-Cartan theory.
- If we take $\mathcal{A} = v^i dx^i - \frac{1}{2} v^2 dt$, the constraints agrees with the ordinary non-relativistic fluid equations.
- But the boundary stress-energy tensor cannot simply be identified to the fluid stress-energy tensor. We have to subtract gauge field terms.
- Entropy current is defined from the horizon area and satisfies the local thermodynamic relation and second law.

Thank you

This work was supported in part by European Union's Seventh Framework Programme under grant agreements (FP7-REGPOT-2012-2013-1) no 316165, the EU program "Thales" MIS 375734 and was also cofinanced by the European Union (European Social Fund, ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) under "Funding of proposals that have received a positive evaluation in the 3rd and 4th Call of ERC Grant Schemes."

European Union
European Social Fund

OPERATIONAL PROGRAMME
EDUCATION AND LIFELONG LEARNING
investing in knowledge society
MINISTRY OF EDUCATION & RELIGIOUS AFFAIRS, CULTURE & SPORTS
MANAGING AUTHORITY

NSRF
2007-2013
programme for development
EUROPEAN SOCIAL FUND

Co-financed by Greece and the European Union