Quantum Field Theory with Random Coupling Constants

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Conformal Field Theories in d dimensions play an important role in many branches of Physics. For example, in statistical physics, they describe second-order phase transitions as we dial the temperature.

In the Ising model in $d > 1$-dimensions, there is a critical temperature T_c for which the correlation length and various other quantities diverge

$$\xi \sim (T - T_c)^{-\nu} \ ,$$

$$C \sim (T - T_c)^{-\alpha}$$

ν, α and other similar critical exponents can be understood in terms of CFT data:

$$\nu = \frac{1}{d - \Delta_\epsilon}$$

$$\alpha = \frac{d - 2\Delta_\epsilon}{d - \Delta_\epsilon}$$

At the critical point we have the d-dimensional Ising CFT given by the Ginzburg-Landau description

$$S = \int d^d x \left((\nabla \Phi)^2 + \lambda \Phi^4 \right) \ .$$

The exponents can be computed exactly in $d = 2$ and for $d \geq 4$ they take the mean-field values. In between, one can use the epsilon expansion, Monte-Carlo simulations, Bootstrap Equations, large N expansions, etc.

There are two relevant operators, Φ and Φ^2. The coupling of Φ corresponds to an external magnetic field and the coupling of Φ^2 is proportional to $T - T_c$, thus it corresponds to changing the temperature.

If $T > T_c$ then we have a gapped theory with a single vacuum and if $T < T_c$ we have two gapped vacua and there is nonzero magnetization.

We would like to explore what happens when we study an **ensemble of Quantum Field Theories**. Is there is any possible critical behavior if we study a collection of many QFTs with some probability distribution connecting them?

This question is very well motivated experimentally.

Real systems are never exactly pure. For example, we can't really set the coefficients of Φ and Φ^2 to be exactly zero. This corresponds to having some magnetic and non-magnetic impurities. Suppose, as an example, that the coefficients are taken to be unknown variables, with a Gaussian probability distribution, but with a very small variance:

$$\delta S = h(x)\Phi + h'(x)\Phi^2 \ ,$$
$$\overline{h(x)} = 0 \ , \qquad \overline{h(x)h(y)} = c^2\delta^{(d)}(x-y) \ ,$$
$$\overline{h'(x)} = 0 \ , \qquad \overline{h'(x)h'(y)} = c'^2\delta^{(d)}(x-y) \ .$$

We can ask what does such a tiny randomness in the coupling constants do to the phase transition (and the CFT). It may seem surprising, but such tiny randomness may have very dramatic consequences.

Let the QFT degrees of freedom be Φ and the sources h. We have an action $S[\Phi, h]$. The probability to find the system in some configuration Φ for a given h

$$P\left(\Phi \middle| h\right) = \frac{e^{-S[\Phi;h]}}{\int [D\Phi] e^{-S[\Phi;h]}}$$

We integrate over h with some probability distribution $P(h)$, e.g.

$$P(h) = e^{\frac{-1}{2c^2}\int d^d x h^2(x)}$$

and find the probability distribution of the ensemble QFT:

$$e^{-\mathcal{H}(\Phi)} = \int [Dh] e^{\frac{-1}{2c^2}\int d^d x h^2(x)} \frac{e^{-S[\Phi;h]}}{\int [D\Phi] e^{-S[\Phi;h]}}$$

Using

$$e^{-\mathcal{H}(\Phi)} = \int [Dh] e^{\frac{-1}{2c^2}\int d^d x h^2(x)} \frac{e^{-S[\Phi;h]}}{\int [D\Phi] e^{-S[\Phi;h]}}$$

we can compute the ensemble averages in the usual way

$$\overline{\langle \mathcal{O}_1(\Phi)...\mathcal{O}_n(\Phi)\rangle} = \int [D\Phi] \mathcal{O}_1(\Phi)...\mathcal{O}_n(\Phi) e^{-\mathcal{H}(\Phi)}$$

$$= \int [D\Phi][Dh] \mathcal{O}_1(\Phi)...\mathcal{O}_n(\Phi) \frac{e^{-S[\Phi;h]-\frac{1}{2c^2}\int d^d x h^2(x)}}{\int [D\Phi] e^{-S[\Phi;h]}}$$

Let us consider the case of free field theory, with the tadpole disorder

$$S[\Phi, h] = \int d^d x \left(\frac{1}{2} (\partial\Phi)^2 + h\Phi \right) \ .$$

A straightforward computation yields

$$\overline{\langle \Phi^2(0) \rangle} = \langle \Phi^2(0) \rangle + c^2 \int d^d p \frac{1}{p^4} \ .$$

This is infrared convergent only if $d > 4$ [Imry-Ma].

Very similar to the Coleman theorem; No symmetry breaking can occur in systems with disorder in the magnetic field at or below $d = 4$. The free CFT with disordered tadpole does not exist at or below $d = 4$.

Take an abstract CFT with "action" S_0. Let $\mathcal{O}$ be a scalar primary operator of dimension Δ. We define

$$S = S_0 + \int d^d x h(x)\mathcal{O}(x) \ .$$

We can use this to define arbitrary correlation functions with the prescription we explained before

$$\overline{\langle \mathcal{O}_1(\phi)...\mathcal{O}_n(\phi)\rangle}$$

$$= \int [D\phi][Dh]\mathcal{O}_1...\mathcal{O}_n \frac{e^{-S_0-\int d^d x\left[h(x)\mathcal{O}(x)-\frac{1}{2c^2}h^2(x)\right]}}{\int [D\phi] e^{-S_0-\int d^d x h(x)\mathcal{O}(x)}}$$

In the general setup

$$S = S_0 + \int d^d x h(x)\mathcal{O}(x) \ ,$$

with probability distribution for h given by $e^{-\frac{1}{2c^2}\int d^d x h^2(x)}$ we immediately see that

If $\Delta > d/2$ disorder is irrelevant since c has negative mass dimension. Then the critical exponents are those of the pure theory. (This criterion is the [Harris] bound.)

If $\Delta < d/2$ then disorder is relevant and

- One can get new critical exponents. (Is the theory even scale invariant?)
- Criticality may disappear (as in the free field example).

It would be very nice to

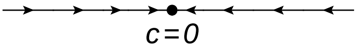
- Find good theoretical and numerical methods to evaluate the new critical exponents.
- Understand under what general conditions disorder can make the critical theory disappear altogether.

There is a lot of experimental data that one can hope to compare to...

We would like to study the case $\Delta = d/2$. It is the marginal case (zero heat capacity exponent). We will now explain that

$$\frac{dc^2}{d\log\mu} = c^4\left(2 - \frac{1}{2}C^2_{OOO}\right) + \mathcal{O}(c^6)\ .$$

C_{OOO} is the OPE coefficient in $O(x)O(0) \sim \frac{1}{x^d} + \frac{1}{x^{d/2}} C_{OOO} O(0)$. As long as the OPE coefficient is not too large, disorder is infrared free.



The disordered free energy

$$F_D[c] = -\int [Dh] e^{\frac{-1}{2c^2}\int d^d x h^2(x)} \log Z[h]$$

Which we can study using the replica trick

$$F_D[c] = -\frac{d}{dn}\bigg|_{n=0} \int [Dh] e^{\frac{-1}{2c^2}\int d^d x h^2(x)} Z^n[h]$$

So we can study n copies of the original theory coupled through

$$\sum_{A=1}^{n} L_0[\Phi_A] + h \sum_{A=1}^{n} O_A - \frac{1}{2c^2} h^2(x)$$

This is now a pretty much standard exercise in conformal perturbation theory. We compute the beta function for c^2 and then analytically continue to $n = 0$. It boils down to integrating out h and computing the term in the OPE that goes like $1/x^d$.

$$\sum_{A\neq B} O_A O_B(x) \sum_{C\neq D} O_C O_D(x) \sim \frac{1}{x^d}\left(4(n-2) + 2C^2_{OOO}\right) \sum_{A\neq B} O_A O_B(0)\,.$$

Single contraction: Insert the unit operator

Double Contraction: Use the OPE coefficient twice.

This case of $\Delta = d/2$ is interesting since the simplest phase transitions in 3d are very close:

- $O(1)$ model (3d Ising): $\Delta_\epsilon = 1.41...$
- $O(2)$ model (He_3): $\Delta_\epsilon = 1.51...$

In particular, in the 3d Ising it is slightly relevant classically and we obtain thus a Wilson-Fisher like beta function for the random coupling theory

$$\frac{dc^2}{d\log\mu} = \left(-d + 2\Delta_\epsilon\right)c^2 + \left(2 - \frac{1}{2}C_{OOO}^2\right)c^4 + \ldots\ ,$$

$$c_*^2 = \frac{d - 2\Delta_\epsilon}{2 - \frac{1}{2}C_{OOO}^2} \sim 0.2$$

We can thus compute observables in the random exchange Ising model in d=3. For example we find for the operator $\epsilon(x)$ in the infrared

$$\Delta^{IR} = \Delta_\epsilon + \frac{d - 2\Delta_\epsilon}{2 - \frac{1}{2}C^2_{OOO}} \ .$$

From this we can infer some of the critical exponents of the disordered theory

$$\alpha^{IR} = -0.13 \pm 0.03 \ , \qquad \nu^{IR} = 0.7 \pm 0.1 \ ,$$

Compare with experiment [e.g. Belanger]:

$$\alpha^{IR}_{Exp} = -0.10 \pm 0.02 \ , \qquad \nu^{IR}_{Exp} = 0.69 \pm 0.01 \ .$$

This is very good compared to existing theoretical approaches.

Large N and a 't Hooft-like Limit

At large N we have a special set of operators, $\mathcal{O}_i$, which are generalized free fields

$$\langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}\rangle = \langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\rangle\cdots\langle \mathcal{O}_{i_{n-1}}\mathcal{O}_{i_n}\rangle + permutations$$

For example, in vector models we can take $\mathcal{O} = \vec{\phi}^2$ and in adjoint theories $\mathcal{O} = Tr(...)$. Let us couple disorder to such a generalized free field

$$S = S_0 + \int d^d x h \mathcal{O} \ .$$

We normalize the generalized free field such that $\langle \mathcal{O}(x)\mathcal{O}(x)\rangle = \frac{1}{x^{2\Delta}}$ and we take $\overline{h(x)h(y)} = c^2\delta^{(d)}(x-y)$. c is order 1 in this normalization in the large N limit.

Large N and a 't Hooft-like Limit

We will see that this gives a nontrivial 't Hooft-like limit with critical exponents and beta functions that do not contain factors of N.

For example, we can compute the disordered one-point function exactly. It is

$$\overline{\langle \mathcal{O}^2(0) \rangle} = \langle \mathcal{O}^2(0) \rangle + c^2 \int d^d z \frac{1}{z^{2\Delta} z^{2\Delta}}$$

It is infrared divergent for

$$\Delta \leq d/4$$

In this case we lose the critical point!

Large N and a 't Hooft-like Limit

At least at large N we thus have a very concrete condition

$$\Delta \leq d/4$$

which determines whether or not the phase transition disappears. If the original pure theory is unitary, then $\Delta \geq d/2 - 1$. So we can only have $\Delta \leq d/4$ if $d \leq 4$.

Consequently, (at least at large N) above four dimensions a unitary CFT cannot be completely destabilized by disorder.

Large N and a 't Hooft-like Limit

If $\Delta > d/4$ then we find that some critical exponents change due to disorder and some remain intact

$$\overline{\langle \mathcal{O}(x)\mathcal{O}(0)\rangle} - \overline{\langle \mathcal{O}(x)\rangle\langle \mathcal{O}(0)\rangle} = \frac{1}{x^{2\Delta}}$$

Therefore the dimension of $\mathcal{O}$ stays Δ. But

$$\overline{\langle \mathcal{O}^2(x)\mathcal{O}^2(0)\rangle} - \overline{\langle \mathcal{O}^2(x)\rangle\langle \mathcal{O}^2(0)\rangle} \sim \frac{1}{x^{4\Delta}} + \frac{1}{x^{2\Delta}} c^2 \int d^d z \frac{1}{(x-z)^{2\Delta}} \frac{1}{z^{2\Delta}}$$

$$\longrightarrow_{x\to\infty} c^2 \frac{1}{x^{6\Delta - d}}$$

Therefore the infrared dimension of $\mathcal{O}^2$ is $3\Delta - d/2$.

Interplay with Double-Trace Deformations

Let us recall that if we have a double trace deformation of a pure Large N CFT,

$$\delta S = \frac{\lambda}{2} \int d^d x \mathcal{O}^2 \ ,$$

and if the UV dimension of $\mathcal{O}$ is $\Delta < d/2$, then the double trace deformation is relevant and leads to an infrared CFT where

$$\Delta_{IR}(\mathcal{O}) = d - \Delta$$

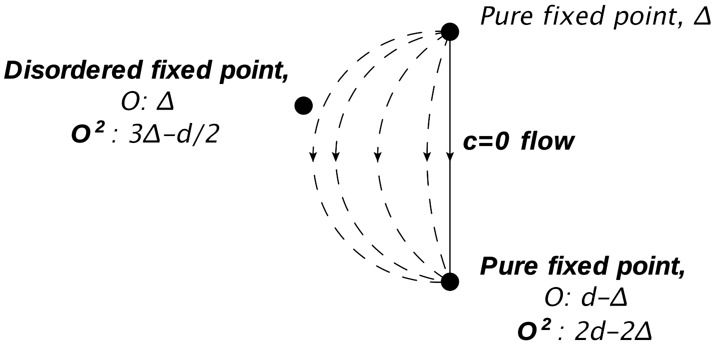
Thus, if we introduce disorder $\int d^d x h \mathcal{O}$ it would be relevant in the UV and irrelevant in the infrared (because $\Delta_{IR} > d/2$).

Interplay with Double-Trace Deformations

One can solve for $\overline{\langle\mathcal{O}(x)\mathcal{O}(y)\rangle} - \overline{\langle\mathcal{O}(x)\rangle\langle\mathcal{O}(y)\rangle}$ and $\overline{\langle\mathcal{O}^2(x)\mathcal{O}^2(0)\rangle} - \overline{\langle\mathcal{O}^2(x)\rangle\langle\mathcal{O}^2(0)\rangle}$ exactly in this theory.
These computations uphold the intuition in the previous slide, suggesting the following picture of the RG flow:

(It turns out that the situation is more complicated, but at least at the level of two-point functions this is true.)

Interplay with Double-Trace Deformations



In the marginal case of $\Delta = d/2$ one finds nonzero beta functions. The presence of double trace operators changes the previous result (for $C_{OOO} = 0$) $\frac{dc^2}{d\log\mu} = 2c^4$. One finds, instead, that

$$\frac{dc^2}{d\log\mu} = 2c^2\lambda \ ,$$

$$\frac{d\lambda}{d\log\mu} = \lambda^2 \ .$$

So the double trace operators exactly cancel the beta function if $\lambda = 0$. This leads at low energies to a decay of disorder as

$$c^2 \to \frac{1}{\log^2\mu} \ .$$

Disorder Beta Functions From Holography

Disorder averaging is actually very natural in theories with a holographic dual. We need to study the classical action as a functional of the boundary conditions and then average over the boundary conditions.

Recall that in the presence of double trace operators [Witten]

$$\phi(x,z) = z^{d/2}\left([\lambda\beta(x) + h(x)]\log(z\mu) + \beta(x)\right) + \cdots ~,$$

for our application we treat $h(x)$ as a random variable

$$\overline{h(x)} = 0~, \qquad \overline{h(x)h(y)} = c^2\delta^{(d)}(x-y)~.$$

Disorder Beta Functions From Holography

From this we find that

$$O(k) = \beta(k) = \frac{h(k)\log(k/\mu)}{1-\lambda\log(k/\mu)} \ ,$$

which is consistent with the fact that O does not pick anomalous dimension due to disorder. This also leads to $\lambda \to -\frac{1}{\log(\mu)}$ as usual. Additionally, matching the boundary conditions we find that the effective sources obeys

$$h(k,\mu) = h(k,\mu_0)\frac{1-\lambda(\mu)\log(k/\mu)}{1-\lambda_0\log(k/\mu_0)}$$

and hence, squaring this,

$$c^2(\mu) = \frac{c_0^2}{\lambda_0^2}\lambda^2(\mu) \to 1/\log^2(\mu) \ .$$

Conclusions and Outlook

- Disordered fixed point are very common.
- Can be treated analytically in a variety of ways, e.g. the heat capacity expansion we presented here.
- The symmetries of disordered fixed points are currently unclear. In some examples we find that even scale invariance is not obeyed (though one does have power law correlators).
- A 't Hooft limit exists (in fact there are two nontrivial limits; we discussed only one). The beta functions for disorder receive interesting corrections from double trace operators.
- One can follow the flows in the 't Hooft limit all the way. For example, we found that if $\Delta < d/4$ then the coupling to disorder would be disastrous for the critical theory.
- Due to lack of space/time we only discussed very briefly what holography has to say about disordered fixed points.

Thank You