The large-scale structure of the ambient boundary[1]

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[1] I. Antoniadis, S. Cotsakis, Eur. Phys. J. C. 75 (2015) 1, arXiv:1409.2220; Mod. Phys. Lett. A. (to appear), arXiv:1505.04737.

Physics in higher dimensions

Steven Weinberg (1985) gives three pieces of motive:

- Arena to develop math tricks to solve problems in 3+1 dimensions
- 'Grand tour' (like for *the sons of the English gentry and nobility* (late 1800's, early 1900's))
- We really do live in more than 4 spacetime dimensions ('...*as soon as you start moving in one of those directions, you immediately come back to where you started*...')

AdS_5/CFT_4 GEOMETRY

THE SIMPLEST AMBIENT METRIC I

BASIC FACT $\mathcal{M}_4$ is the boundary of AdS_5
(Witten (1998) gives three proofs of it.)

- Symmetry groups of bulk and brane agree on boundary of bulk
- Construct (AdS_5, g_+) metric as the Poincaré (hyperbolic) metric on unit open ball $\mathcal{B}_5$ of $\mathbb{R}^5$ ($\sum_{i=0}^4 y_i^2 < 1$ there) ,

$$g_+ := \frac{4|y|^2}{(1-|y|^2)^2}, \quad |y|^2 = \sum_{i=0}^{4} dy_i^2,\ y_0, y_1, \cdots, y_4 \text{ coordinates of } \mathbb{R}^5.$$

- g_+ does not extend everywhere on $\mathcal{B}_5 \cup S^4$ because it is singular on the boundary $\partial\mathcal{B}_5 = S^4$

AdS_5/CFT_4 GEOMETRY

THE SIMPLEST AMBIENT METRIC II

- Pick a function $\Omega = 1 - |y|^2 > 0$ on $\mathcal{B}_5$, and $\Omega = 0$ on S^4
- g_+ is conformal to a complete metric $\mathring{g} = \Omega^2 g_+$ that extends smoothly on $\partial AdS_5 = S^4$
- $\mathring{g}|_{S^4}$ is a metric in $[g_4]$. The conformal infinity $\mathscr{I}_{AdS_5} = S^4$, that is its boundary
- While $\mathcal{B}_5$ has a unique, well-defined metric, its boundary $\partial\mathcal{B}_5 = S^4$ has only a conformal structure (both preserved under the actions of their symmetry groups)
- Any function on S^4 extends uniquely to AdS_5 that has the given boundary values and satisfies the field eqn.
- A CFT on $(S^4, [g_4])$ should be well-behaved

AdS_5/CFT_4 geometry

The simplest ambient metric III

- Maldacena conjecture: A string theory on $AdS_5 \times S^5$ is equivalent to a certain SUSY YM defined on $\mathscr{I}_{AdS_5} \times S^5$.
- A black hole is here defined as a thermal state on the boundary, easier calculations because S^4 is conformally flat

The Fefferman-Graham fundamental theorem

FG, Asterisque (1985), The Ambient Metric (PUP, 2012) I

A basic problem in conformal geometry:

- From the Poincaré metric $g_{AdS_5} = 4(1-|y|^2)^{-2} g_E$, to $\mathring{g} = \Omega^2 g_{AdS_5}$, then restrict $\mathring{g}|_{S^4}$, and finally get a conformal structure on the boundary.
- inverse problem: Starting from a conformal space-time manifold $(M, [g_4])$, is there a metric on V such that when we perform the construction we get the given conformal structure that we started with?

The Fefferman-Graham metric: There exists a well-defined ambient metric g_+ on $M \times \mathbb{R}$ (points (x^μ, y)) with the following properties:

The Fefferman-Graham fundamental theorem

FG, Asterisque (1985), The Ambient Metric (PUP, 2012) II

- Locally around $M \times \{0\}$ in $M \times \mathbb{R}$, there is a smooth (non-unique) function Ω with $\Omega > 0$ on V, $\Omega = 0$ on M: $\Omega^2 g_+$ extends smoothly on V
- $(\Omega^2 g_+)|_{TM}$ is non-degenerate on M (that is its signature remains $(- + + + +)$ on M)
- $(\Omega^2 g_+)|_{TM} \in [g_4]$. ($M$ is the conformal infinity of V)
- g_+ satisfies the Einstein eqns with Λ to infinite order on M

The Fefferman-Graham fundamental theorem

FG, Asterisque (1985), The Ambient Metric (PUP, 2012) III

- g_+ is in normal form wrt g_4:

$$g_+ = y^{-2}(g_y + dy^2).$$

  Here, g_y stands for a suitable formal power series with $g_0 = g_4$. (We may also use y as Ω.)

- g_+ is unique: Given any two ambient metrics g_+^1, g_+^2 for $(M, [g_4])$, their difference $g_+^1 - g_+^2$ vanishes to infinite order everywhere along $M \times \{0\}$.

Brane solutions, asymptotic limits I

Consider (I. Antoniadis, S.C. & I. Klaoudatou (2005-15)):

- bulk space (V, g_5) (coordinates $A = (x^\mu, y)$) containing embedded 4-dimensional braneworld (M, g_4) (coordinates x^μ, signature $(-+++)$)
- filled with analogue of perfect fluid $p(y) = \gamma\rho(y)$
- satisfying the 5-dim Einstein equations in the bulk, $G_{AB} = \kappa_5 T_{AB}$.
- assume the ansatz,

$$g_5 = a^2(y) g_4 + dy^2, \quad \text{for } g_5 \text{ solutions on } \; V,$$

- look for solutions with (M, g_4) being Minkowski, de Sitter, or Anti-de Sitter

Brane solutions, asymptotic limits II

- Einstein eqns reduce to $\dot{x} = f(x)$, f smooth vector field and solution vector $x = (a, \dot{a}, \rho)$
- Solutions are then of the generic form given by method of asymptotic splittings (S.C. et al, (1990-2014)):

$$a(y) = y^p \sum_{i=0}^{\infty} c_i y^{i/s}, \quad y \to 0, \quad p \in \mathbb{Q}, s \in \mathbb{N}, c_i \in \mathbb{R},$$

and similarly for the density ρ.

Generic properties of braneworlds

(I. Antoniadis & S.C., arXiv:1409.1220)

The following conclusions apply to entire literature of brane cosmology (Randall-Sundrum, Arkani-Hamed, et al, Förste et al, Binetruy et al, etc.):

- all flat brane solutions are singular at a finite, arbitrary $y-$distance from the position of the brane located at $y = 0$; The generic curved problem is under investigation.
- g_5 cannot be continued to arbitrary values in the y-dimension;
- properties of the metric g_4 do not follow from those of the bulk metric g_5
- there is no conformal infinity for the 5-dimensional geometry (brane is a kind of boundary but never a *conformal* boundary).
- No holographic interpretation & no way to realize a boundary CFT.

Ambient algorithm I

AIM: A generic spacetime with improved properties: A bounding hypersurface, the conformal infinity, of a new cosmological metric in 5-dimensional 'ambient' space.

1. Take a 4-dimensional, non-degenerate 'initial' metric $g_{\mathrm{IN}}(x^\mu)$ on spacetime M. This step essentially involves the Penrose conformal method.
2. Conformally deform g_{IN} to a new metric $g_4 = \Omega^2 g_{\mathrm{IN}}$ by choosing a suitable conformal factor Ω. This step connects the 'bad' metric g_{IN} with the 'nice', non-degenerate, and non-singular metric $g_4(x^\mu)$.

Ambient algorithm II

3. Using the method of asymptotic splittings for the 5-dimensional Einstein equations with an arbitrary (with respect to the fluid parameter γ) fluid, solve for the 5-dimensional metric $g_5 = a^2(y)g_4 + dy^2$ and the matter density ρ_5.
4. Transform the solutions of step 3 to suitable factored forms of the general type, (divegent part) $\times$ (smooth part).
5. Construct the 'ambient' metric in normal form, g_+, for the 5-dim Einstein equations with a fluid. It is:

$$g_+ = w^{-n}\left(\sigma^2(w)g_4(x^\mu) + dw^2\right),$$

$n \in \mathbb{Q}^+$, as $w \to 0$, with $\sigma(w)$ a smooth (infinitely differentiable) function such that $\sigma(0)$ is a nonzero constant.

Ambient algorithm III

6. $(M, [g_4])$ is the conformal infinity of (V, g_+), that is $\mathscr{I} = \partial V = M$.
7. The metric g_+ is conformally compact. This means that a suitable metric $\mathring{g}$ constructed from g_+ extends smoothly to V, and its restriction to M, $\mathring{g}|_M$, is non-degenerate (i.e., maintains the same signature also on M).
8. The conformal infinity M of the ambient metric g_+ of any metric in the conformal class $[g_4]$ is controlled by the behaviour of a constant rescaling of the 'nice' metric g_4: For any two conformally related 4-metrics g_1, g_2 in the conformal geometry of M, g_1 being the 'good' (roughly meaning 'regular') and g_2 the 'bad' metric on the boundary, their

Ambient algorithm IV

ambient metrics $\mathring{g}_1|_M, \mathring{g}_2|_M$ differ by a homothetic transformation,

$$\mathring{g}_2|_M(0) = c\,\mathring{g}_1|_M(0), \quad c: \text{ const.}$$

Therefore: Starting from a conformal geometry on the spacetime M, the ambient cosmological metric returns a 4-geometry on M (its conformal infinity metric $\mathring{g}|_M$) that has a homothetic symmetry.

9. As a conformal manifold, $(M, [g_4])$ has no singularities. This means that there is always a regular metric on M: the metric $\mathring{g}|_M$ belonging to $[g_4]$ is regular.
10. Cosmic censorship on $(M, [g_4])$ is equivalent to the validity of the asymptotic condition satisfied by the ambient metric $\mathring{g}|_M$.

Zeeman topology I

Theorem

For Minkowski spacetime M, the group of homothetic symmetries (that is Lorentz transformations with dilatations) coincides with the group of all homeomorphisms of M provided that its topology is not the usual Euclidean metric topology (that is M is locally Euclidean) but a new one, called the fine topology $\mathcal{Z}$.

The Zeeman topology has the following properties:

- It is strictly finer than the Euclidean topology
- It possesses improved properties
- It extends to curved spacetimes

Description:

Zeeman topology II

- For $x \in M$, an open ball in $\mathcal{Z}$ has the form $B_{\mathcal{Z}}(x;r) = (B_{\mathcal{E}}(x;r) \setminus N(x)) \cup \{x\}$, where $B_{\mathcal{E}}(x;r)$ is the Euclidean-open ball, and $N(x)$ the null cone at x (we remove $N(x)$ and put back only the point x).
- Then $B_{\mathcal{Z}}(x;r)$ is $\mathcal{Z}$-open, but not $\mathcal{E}$-open.
- Hence, a set $A \subset M$ is $\mathcal{Z}$-open if $A \cap B$ as a subset of B is $\mathcal{E}$-open, for every spacelike plane and timelike line B.

Zeeman-Göbel theorem

Theorem

For a general spacetime, the homothetic group is isomorphic to the group of homeomorphisms of the Zeeman topology.

- For any spacetime M in general relativity we have the freedom to choose either the standard Euclidean metric topology, giving M the usual manifold topology, or the Zeeman topology.
- It is of course the former that is used in all standard discussions of relativity.
- For our bounding spacetime M, the conformal infinity of the ambient space V, we do not have this freedom.

Convergence of causal curves I

- For the convergence of a sequence of causal curves to a limit curve, one uses in an essential way the Euclidean balls with their Euclidean metric and their compactness to extract the necessary limits.
- Since the Zeeman topology is strictly finer, such sequences will be Zeno sequences, that is their convergence in the Euclidean topology will not guarantee the existence of a limit curve in the Zeeman topology.

Convergence of causal curves II

- In the proofs of the singularity theorems, a contradiction appears when assuming the existence of a curve of length greater than some maximum starting from a spacelike Cauchy surface Σ (on which the mean curvature is negative) downwards to the past.
- One extracts a limit curve γ (which locally maximizes the length between Σ and an event p), and no curve can have length greater than that of γ.
- Here we cannot extract such a limit.

This result opens the way for the construction of complete spacetimes as the conformal infinities of physical theories in higher dimensions.