Entanglement and anomalies

Amos Yarom

(Work in progress with T. Nishioka (Tokyo))
If our Hilbert space is separable, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, then the entanglement entropy of a state $|\psi\rangle$ is given by

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Thus,

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Which can be shown to lead to

\[ S_A = \frac{c}{3} \ln \left( \frac{L}{\epsilon} \right) + \ldots \]
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The metric on \( M_n \) is given by

\[ ds^2 = dr^2 + r^2 d\phi^2 + \sum_i (dx^i)^2 \]

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Thus:

\[ S_A = \frac{c_L + c_R}{12} \left( \ln \Lambda / \epsilon \right) \]
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\[ \partial_\theta S_A = \frac{c_L - c_R}{24} \quad \text{(A related result has been obtained previously by Castro, Detournay, Iqbal, Perlmutter, 2014)} \]
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Method 2: One can think of \( Z_n \) as the thermodynamic partition function on a semi-infinite line with non-uniform temperature.

\[ T^{-1} = 2\pi n r \]

\( W_n \) is the generating function for connected correlators in such a state.
Constructing $W_n$

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Then: (Jensen, Loganayagam, AY, 2012)

$$W = \int d^2x \sqrt{-g} \left( \frac{\pi}{12} (c_R + c_L) T^2 - \frac{\pi}{12} (c_R - c_L) \beta^{-1} T \epsilon^{0\nu} u_\nu ight.$$

$$+ \frac{c_L + c_R}{48\pi} u^\beta \partial_\beta u_\gamma u^\alpha \partial_\alpha u_\gamma + \frac{c_R - c_L}{96\pi} u_\alpha u^\beta \epsilon^{\mu\nu} \partial_\mu \Gamma^\alpha_{\beta\nu} \left. \right)$$
Constructing $W_n$

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We can compute

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S_A = - \lim_{n \to 1} (\partial_n - 1) W_n
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Unsurprisingly, we find:

\[
S_A = \frac{c_L + c_R}{12} \ln(\Lambda/\epsilon)
\]
Adding anomalies

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Entanglement entropy is given by

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$$S_A(\theta) = - \lim_{n \to 1} (\partial_n - 1) W_n(\theta)$$
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This happens because of the gravitational anomaly

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\[ \delta_\theta S_A = - \lim_{n \to 1} (\partial_n - 1) \delta_\theta W_n \]
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Acting with a coordinate transformation on \( \tilde{W} \) gives the derivative of the stress tensor

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In the presence of anomalies, the stress tensor is not conserved,

\[ \partial_\nu T^{\mu\nu} = \tau^\mu \]
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The form of \( \tau \) is completely fixed by the Wess-Zumino consistency conditions e.g., in 2d:

\[ \tau^\nu = -c_g g^{\mu \nu} \frac{1}{\sqrt{g}} \partial_\lambda \left( \sqrt{g} \epsilon^{\alpha \beta} \partial_\alpha \Gamma_\lambda^{\mu \nu} \right) \]
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Thus,

\[ \partial_\theta S_A \bigg|_{\theta=0} = - \int d^d x \sqrt{g} \tau^\theta \]
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but we can also compute the same for non-conformal theories:

\[ \partial_\theta S_A \big|_{\theta=0} = 4\pi c_g \]
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\[ \partial_\theta S_A \bigg|_{\theta=0} = 4\pi c_g \left( c_g = - \frac{2\pi}{4!(8\pi^2)} \sum_i \chi_i q_i \right) \]
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In 4 dimensions there isn’t a gravitational anomaly, but there is a mixed gauge-gravitational anomaly.
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$$B = F_{xy}$$
Adding anomalies

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We find:

$$\partial_{\theta} S_A \bigg|_{\theta=0} = -4\pi \alpha c_m B \text{vol}(\mathbb{R}^2)$$
Adding anomalies

In the presence of a gravitational anomaly,

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Summary

Entanglement entropy is given by

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and will be susceptible to gravitational anomalies, viz.

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Explicitly,

\[ \partial_\theta S_A \big|_{\theta=0} = 4\pi c_g \quad \text{(2d)} \]

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Thank you