On the quantization of the Hadronic String

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Introduction

- The holographic duality is an equivalence between certain bulk string theories and boundary field theories.
- Practically most of the applications of holography is based on relating bulk fields (not strings) and operators on the dual boundary field theory.
- This is based on the usual limit of $\alpha' \longrightarrow 0$ with which we go for instance from a closed string theory to a gravity theory .
- However, to describe hadrons in reality it seems that we need strings since after all in reality the string tension is not very large (λ of order one)

Introduction

- The major argument against describing the hadron spectra in terms of fluctuations of fields like bulk fields or modes on probe branes is that they generically do not admit properly the Regge behavior of the spectra.
- For M^2 as a function of J we get from flavor branes only J=0,J=1 mesons and there will be a big gap of order λ in comparison to high J mesons if we describe the latter in terms of strings.
- The attempts to get the linearity between M^2 and n basically face problems whereas for strings it is an obvious property.

Introduction

- Rotating open strings in a holographic background can be mapped into strings in flat four dimensional space-time with massive endpoints.
- Strangely enough so far a full quantization of such a string has not been done .
- I will show that a reasonably good fit to the experimental data of hadrons can be achieved using a classical picture plus a constant intercept (for each trajectory) associated with quantization.
- The intercept plays an important role in the fit.
- Thus one cannot relay on the semi-classical picture and the quantization of the hadronic string is essential.

Outline:

- Introduction
- From strings in holographic background to strings with massive endpoints.
- A brief review of fitting hadronic data.
- The classical actions, e.o.ms, symmetries, Noether charges and gauge fixings
- The classical relation between the energy and angular momentum
- On the quantum relation in the static and orthogonal gauges
- Computing the Casimir energy for a static string
- The map to the rotating case (transverse directions)
- The fluctuations in the plane of rotation.

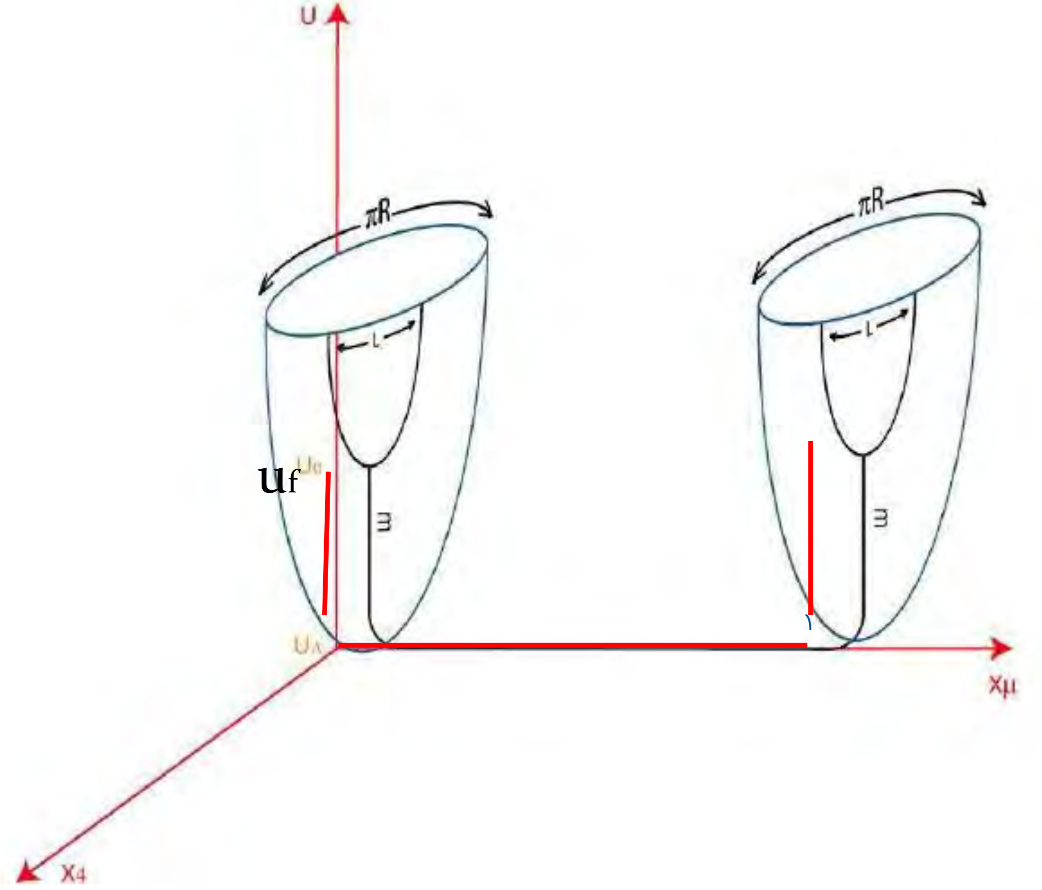
Outline

- The Polchinski Strominger term
- The quantization of the non-critical rotating string without massive endpoints.
- The full intercept in the large mass limits
- Summary

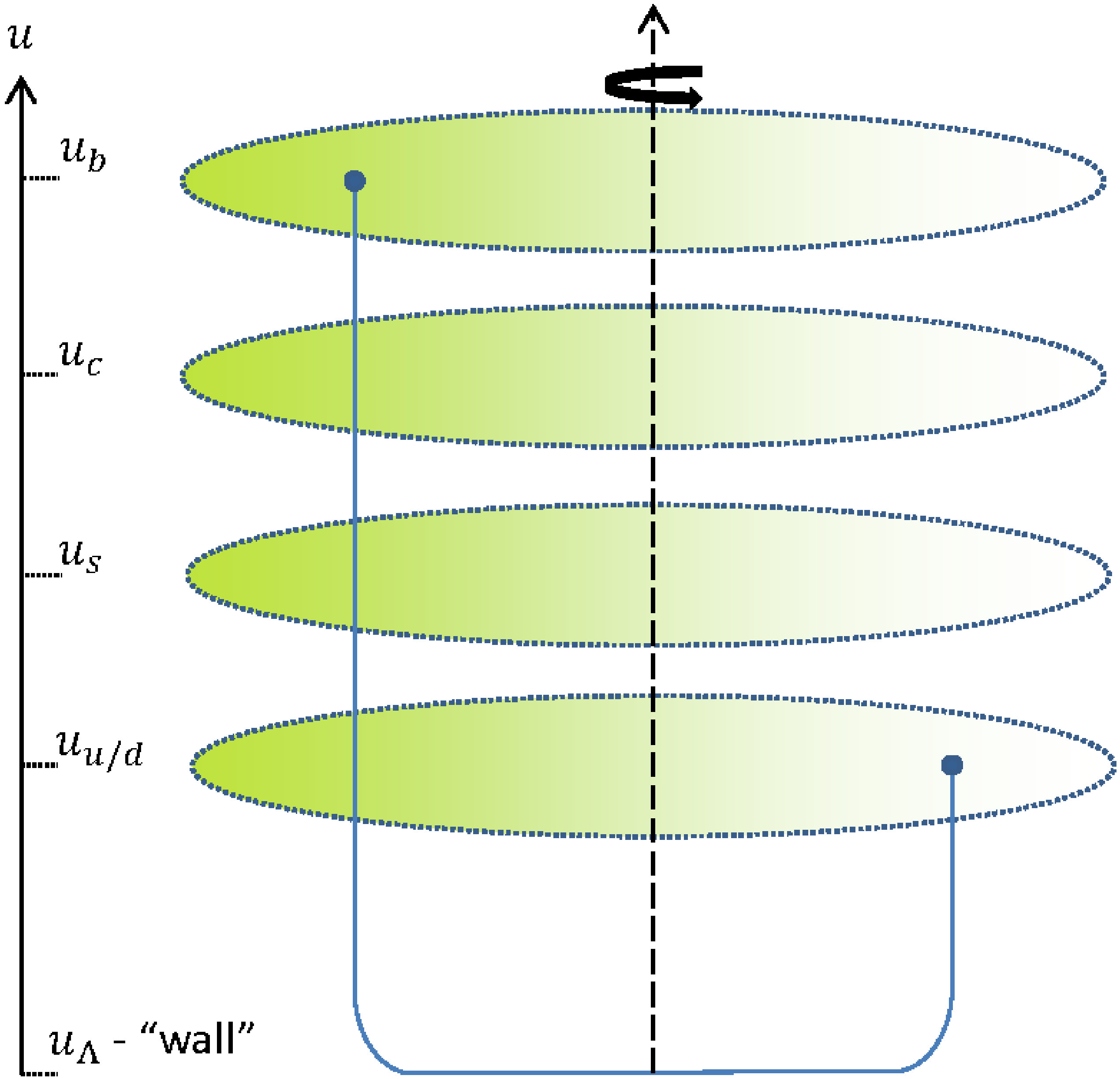
Stringy holographic Mesons

Stringy meson in U shape flavor brane setup

- The mesonic string is attached to flavor D branes in its two ends. For instance in the Sakai Sugimoto model



Example: The B meson



Rotating Strings ending on flavor branes

- Consider a general background of the form

$$ds^2 = G_{mm} dx^m dx^m = -G_{00} dt^2 + G_{xx} dx^2 + G_{uu} du^2 + G_{yy} dy$$

- G_{mm} (u) is a function of the radial direction u
- We look for rotating solutions of the eom

$$x^0 = e\tau, \theta = e\omega\tau, R = R(\sigma), u = u(\sigma), Y^i = Y^i(u_f)$$

- We assume that $u_f > u > u_\Lambda$

Strings ending on flavor branes

- Denote $f \equiv G_{00}$ with $G_{00} = G_{xx}$ and $g^2 = G_{00}G_{uu}$.
- The action in the σ=R gauge than reads

$$\boxed{S_{NG} = T\int d\tau dR[(fe^2 - f(e\omega R)^2)(f + G_{uu}\dot{u}^2)]^{\frac{1}{2}}}$$

- The equation of motion for u(R)

$$\boxed{\partial_R[g^2\frac{\epsilon\dot{u}}{G}] = \epsilon(\partial_u G))}$$

where $\epsilon \equiv \sqrt{1 - v^2}$

$G \equiv \sqrt{f^2 + g^2\dot{u}^2}$ $\quad v = \omega R$

Strings ending on flavor branes

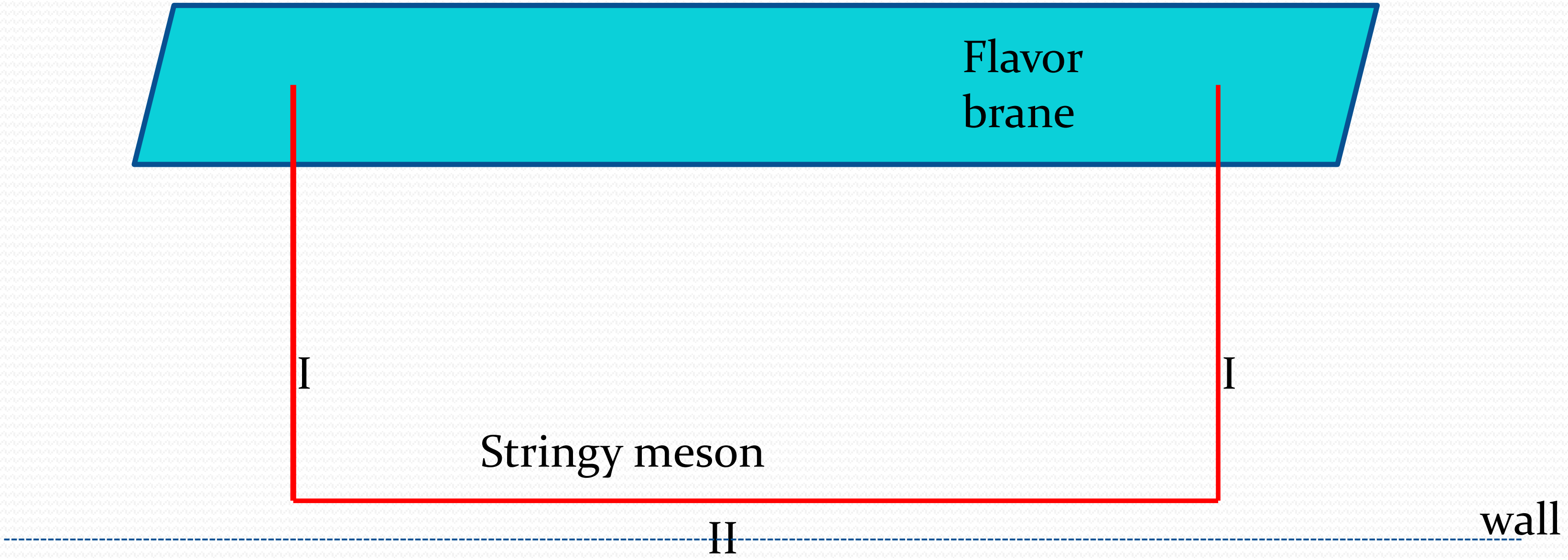
- We now separate the profile into two regions:

- Region (I) vertical $\dot{u} \longrightarrow \infty$

$$\sigma \in (-\pi/2, -\alpha), \sigma \in (\alpha, \pi/2)$$

- Region (II) horizontal $\dot{u} \longrightarrow 0,\ u = u_0$

$$\sigma \in (-\alpha, \alpha)$$



Flavor brane

I

I

Stringy meson

wall

II

String end-point mass

- We define the string end-point quark mass

$$\boxed{m_{sep} = T \int_{u_0}^{u_f} g(u) du = T \int_{u_0}^{u_f} \sqrt{G_{00} G_{uu}} du}$$

- For δ S=o the system has to obey the condition

$$T_{eff}(1 - v^2) = m_q \omega^2 R_0$$

$$T_{eff} = T f = T G_{00}$$

$$\boxed{\frac{T_{eff}}{\gamma} = m_{sep} \gamma \omega^2 R_0}$$

- The conditions to have mesons with Regge behavior in the limit of small msep are precisely the conditions to have a confining Wilson line

Energy and Angular momentum

- The Noether charges associated with the shift of t and θ

$$E = T \int d\sigma \frac{\sqrt{f^2 + g^2 (u')^2}}{\mathcal{E}} = T \int d\sigma \gamma \sqrt{f^2 + g^2 (u')^2}$$

$$J = T \int d\sigma \omega R^2 \frac{\sqrt{f^2 + g^2 (u')^2}}{\mathcal{E}} = T \int d\sigma \omega R^2 \gamma \sqrt{f^2 + g^2 (u')^2}$$

- The contribution of the vertical segments

$$E_I = = T \int_{u_\Lambda}^{u_f} du \gamma \sqrt{\frac{f^2}{(\dot{u})^2} + g^2} = 2\gamma_0 T \int_{u_\Lambda}^{u_f} du g(u) \equiv 2\gamma_0 m_{SEP}$$

$$J_I = = T \int d\sigma \omega R^2 \gamma \sqrt{f^2 + g^2 (u')^2} = 2\gamma_0 \omega R_0^2 T \int_{u_\Lambda}^{u_f} du g(u) \equiv 2\gamma_0 m_{SEP} \omega R_0^2$$

Energy and Angular momentum

- The horizontal segment contributes

$$E_{II} = = T \int_{-R_0}^{R_0} dR \gamma \sqrt{f^2 + g^2 (\dot{u})^2} = f(u_\Lambda) T \int_{-R_0}^{R_0} \frac{dR}{\sqrt{1 - \omega^2 R^2}} = 2 \frac{T_{eff}}{\omega} \arcsin(\omega R_0)$$

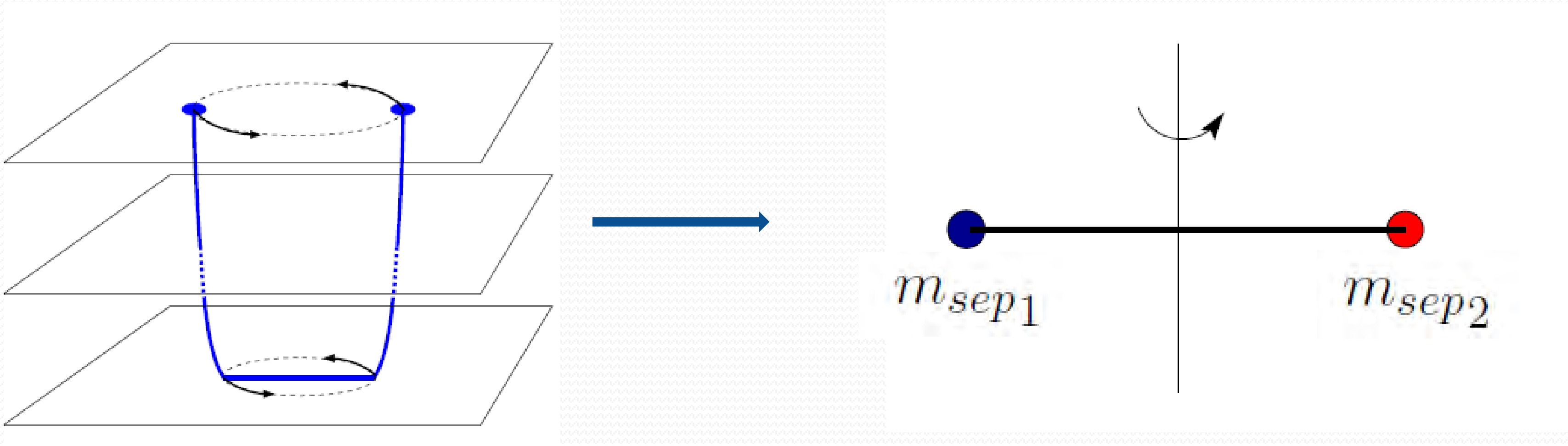
$$J_{II} = = T \int_{-R_0}^{R_0} dR \gamma \sqrt{f^2 + g^2 (\dot{u})^2} = f(u_\Lambda) T w \int_{-R_0}^{R_0} \frac{dR R^2}{\sqrt{1 - \omega^2 R^2}} = 2 T_{eff} [\frac{1}{\omega^2} \arcsin(\omega R_0) - \omega R_0 \sqrt{1 - \omega^2 R_0^2}$$

- Combining together all the segments we get which is precisely that of a rotating string with massive endpoints

$$\boxed{\begin{aligned} E &= 2 m_{sep} \gamma_0 + 2 \frac{T_{effe}}{\omega} \arcsin(\omega R_0) \\ J &= m_{sep} \gamma_0 \omega R_0^2 + \frac{T_{effe}}{\omega^2} \arcsin(\omega R_0) \end{aligned}}$$

From holographic string to string with massive endpoints

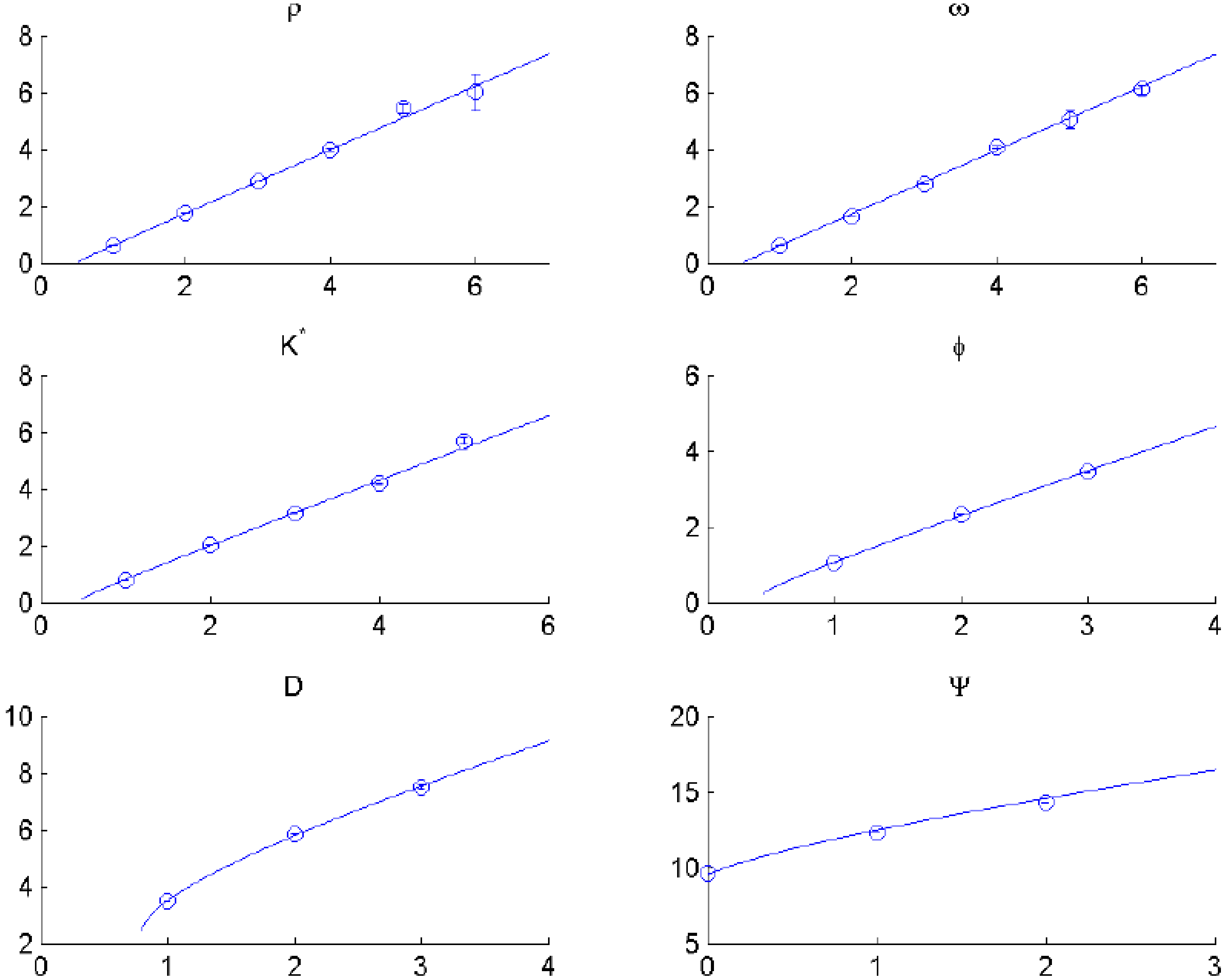
- One can map the energy and angular momentum of the holographic spinning string to those of a string in flat space time with massive endpoints. The masses are m_{sep_1} and m_{sep_2}



(M. Kruczenski, L. Pando Zayas D. Vaman J.S)

A brief review of fits of hadronic data

Toward a universal model



Toward a universal model

- The fit results for several trajectories simultaneously. The (J, M^2) trajectories of ρ, ω, K^*, ϕ D, and Ψ mesons
- We take the string endpoint masses

$$m_{u/d} = 60, m_s = 220, m_c = 1500$$

- Only the intercept was allowed to change. We got

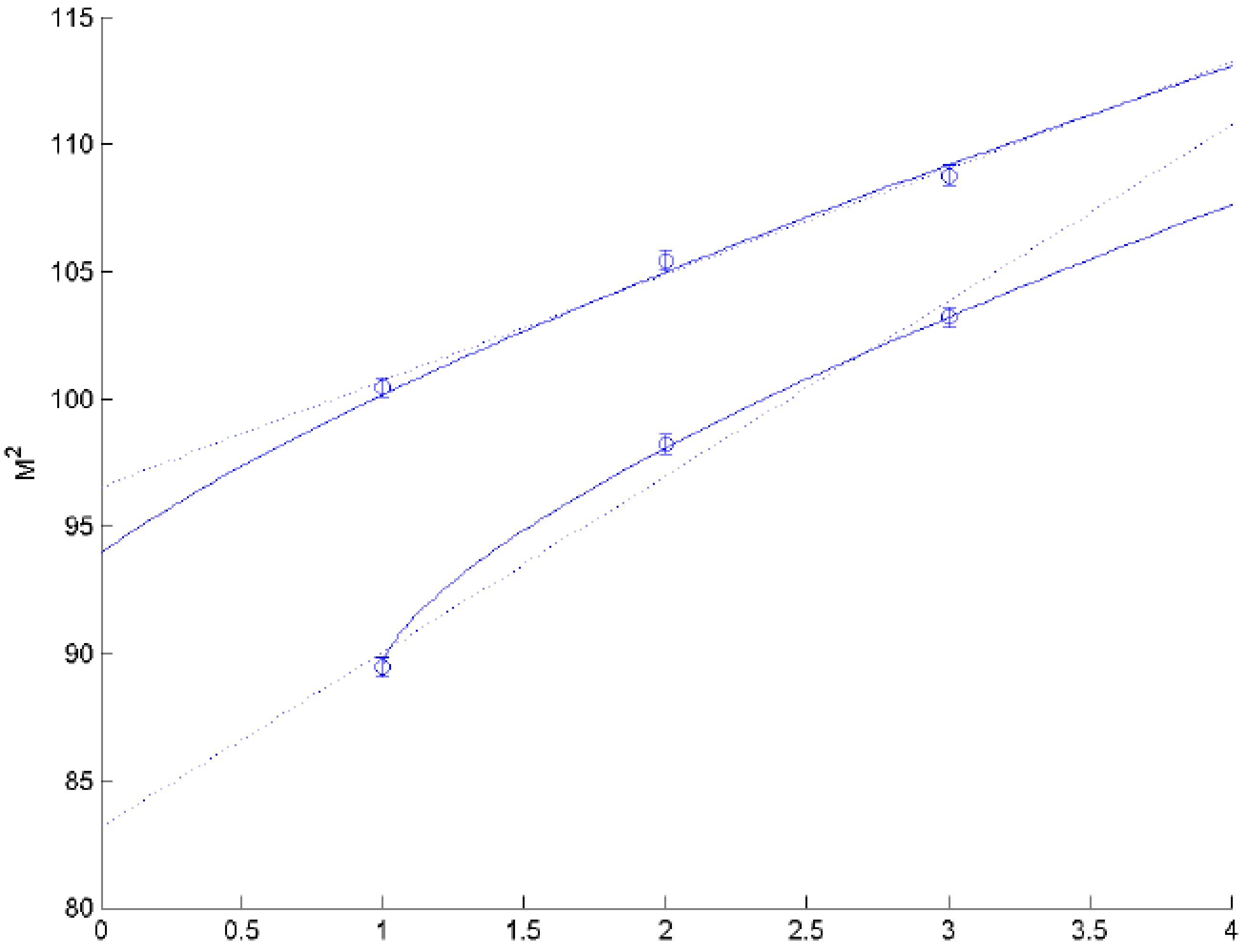
$$\alpha' = 0.899$$

$$a_\rho = 0.51, a_\omega = 0.52, a_{K^*} = 0.49$$

$$a_\phi = 0.44, a_D = 0.80, a_\Psi = 0.94$$

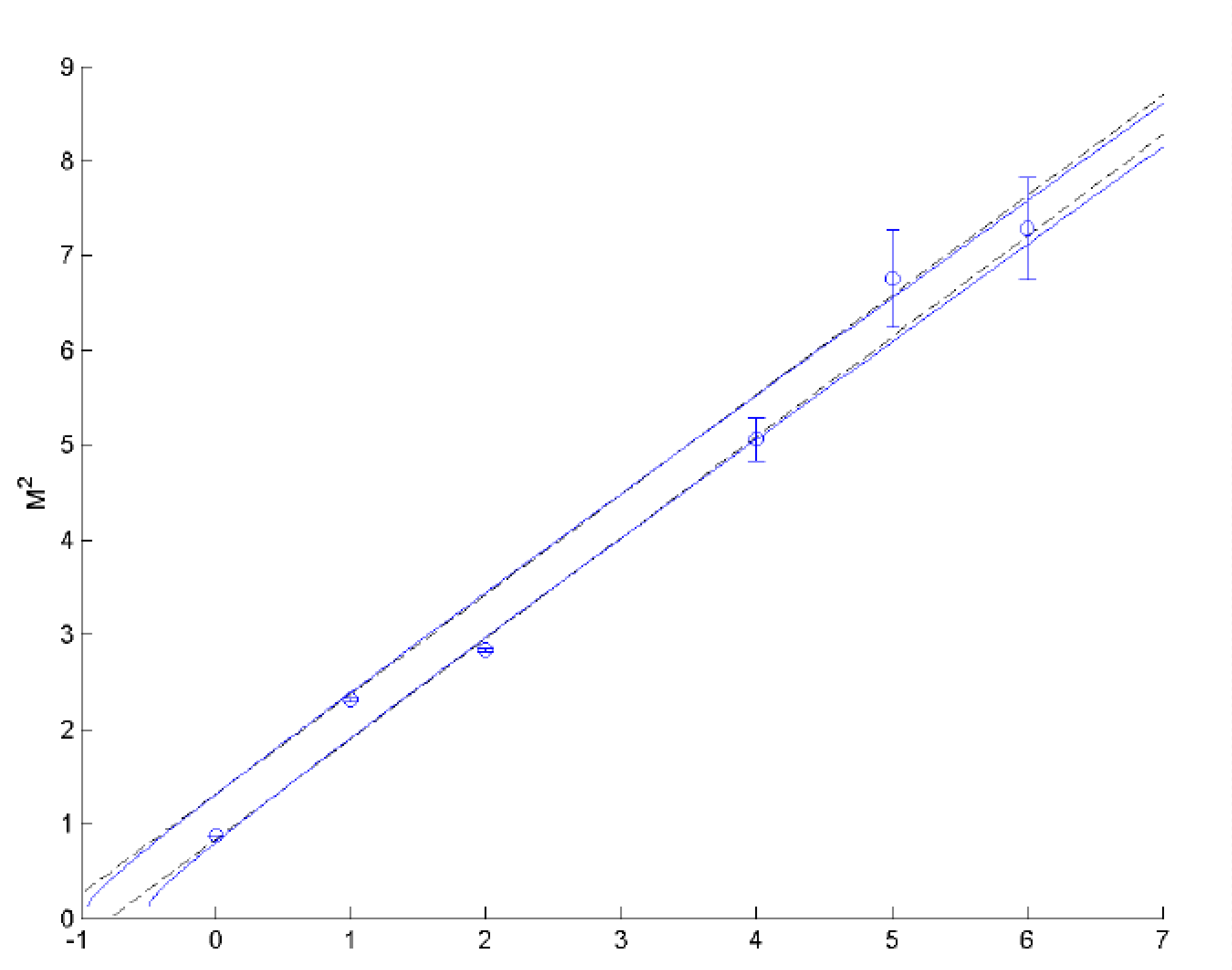
The botomonium trajectories

- To emphasize the deviation from the linearity we present the botomonium trajectories



Fit to Regge trajectories of Nucleons

- Fit of the Regge trajectories of the Nucleons



The trajectories of Ξ , Λc , Ξc

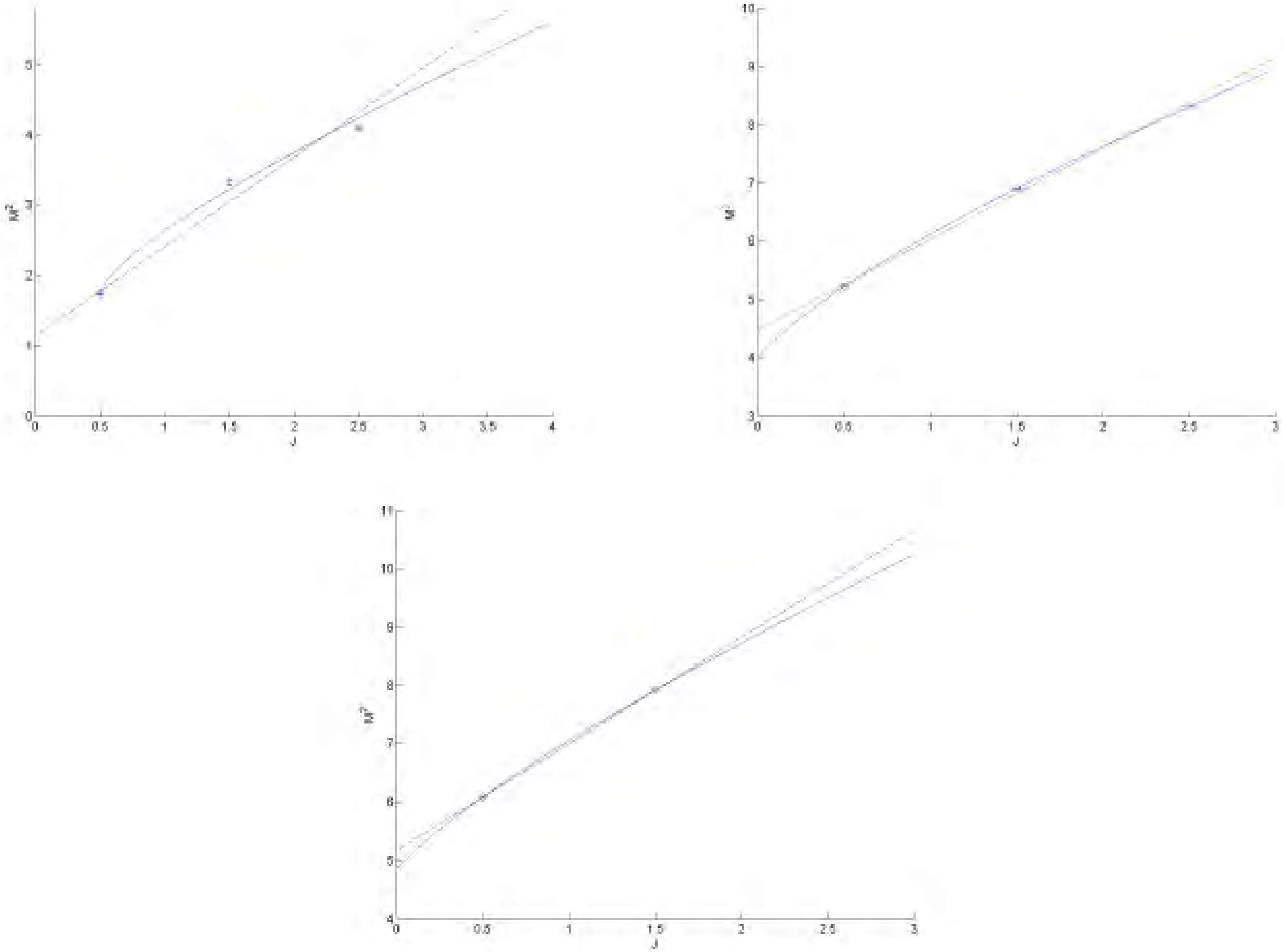


Figure 15. Left: The doubly strange Ξ baryon and its fit, $2m = 1320$. Right: The charmed Λ_c and its fit with $m_1 = 90$, $m_2 = 1720$. Bottom: The charmed-strange Ξ_c with its fit, $2m = 2060$.

The classical rotating bosonic string with massive particles on its ends

The classical action

- There are two ways to write the bulk string action

$$S_{NG} = -T \int d\tau \int_{-\delta}^{\delta} d\sigma \sqrt{-h} \qquad h_{\alpha\beta} \equiv \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$$

$$S_{Pol} = -\frac{T}{2} \int d\sigma d\tau \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$$

- There are two ways to write the endpoints action

$$S_{psq} = -m \int d\tau \sqrt{-\dot{X}^2} \qquad \dot{X}^\mu \equiv \partial_\tau X^\mu$$

$$S_{pa} = \frac{1}{2} \int d\tau \left[\frac{(\dot{X})^2}{\eta} - \eta m^2 \right]$$

Possible classical actions

- Thus there are 4 possible ways for the combined action

$$(i)\ S_{(NG,psq)} \qquad (ii)\ S_{(NG,pa)} \qquad (iii)\ S_{(Pol,psq)} \qquad (iv)\ S_{(Pol,pa)}$$

- In fact there is also another Weyl invariant-like action

$$)S_{Wi} = -\frac{T}{2}\int d\sigma d\tau \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + m\int d\tau \sqrt{\gamma_{\tau\tau}}\sqrt{\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu}|_{\sigma=-\delta,\sigma$$

- For (iv) we associate η with γ_{00} or to take it independent

$$\eta(\tau) = \frac{\sqrt{-\gamma_{\tau\tau}(\sigma,\tau)}|_{\sigma=-\delta,\sigma=\delta}}{m^2}$$

The equations of motion

- The variation of the bulk of the NG action yields

$$\boxed{\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu\right)=0}$$

- At the two boundaries we get

$$\boxed{T\sqrt{-h}\partial^\sigma X^\mu \pm m\partial_\tau\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}\right)=0}$$

- In (ii) the boundary equations and η equations are

$$\boxed{T\sqrt{-h}\partial^\sigma X^\mu \pm \partial_\tau\left(\frac{\dot{X}^\mu}{\eta(\tau)}\right)=0 \qquad \frac{(\dot{X})^2}{\eta(\tau))^2}+m^2=0}$$

The equations of motion

- In (iii) the bulk equation is

$$\boxed{\partial_a(\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0}$$

- The boundary equation is

$$\boxed{T\sqrt{-\gamma}\partial^\sigma X^\mu \pm m\partial_\tau\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}\right) = 0}$$

- The variation of the metric

$$\boxed{\partial^\alpha X^\mu \partial^\beta X_\mu - \frac{1}{2}\gamma^{\alpha\beta}\partial_\delta X^\mu \partial^\delta X_\mu = 0}$$

The solutions of the equations of motion

- A rotating classical solution (τ,σ) in $\mathcal{R}\times[-\delta,\delta]$

$$\boxed{X = L[\tau, \cos(w\tau)R(\sigma), \sin(w\tau)R(\sigma), 0]}$$

- Correspondingly the boundary condition

$$\boxed{\frac{m}{TL} = \frac{1-R^2(\delta)}{R(\delta)}}$$

- In particular

$$\boxed{X = L[\tau, \cos(\tau)\sin(\sigma), \sin(\tau)\sin(\sigma), 0] \qquad \frac{m}{TL} = \frac{\cos^2\delta}{\sin\delta}}$$

Classical Symmetries

- All 5 actions are invariant under certain global and local symmetries
- D dimensional Poincare invariance

$$X^\mu \longrightarrow \Lambda^\mu_\nu X^\nu + a^\mu \qquad \gamma^{\alpha\beta} \longrightarrow \gamma^{\alpha\beta} \qquad \eta \longrightarrow \eta$$

- World-line reparameterization of the world-line action

$$\begin{aligned} \tau &\longrightarrow \tau'(\tau) \\ X^\mu(\tau) &\longrightarrow X'^\mu(\tau') = X^\mu(\tau) \\ \eta(\tau) &\longrightarrow \eta'(\tau') = \frac{\partial\tau}{\partial\tau'}\eta(\tau) \end{aligned}$$

Classical symmetries

- The bulk actions are invariant under 2d diffeo

$$\tau \quad \rightarrow \quad \tau'(\tau\sigma), \qquad \sigma \rightarrow \sigma'(\tau,\sigma)$$

$$X^{\mu}(\tau,\sigma) \quad \rightarrow \quad X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma)$$

$$\gamma_{\alpha\beta}(\tau,\sigma) \quad \rightarrow \quad \gamma'_{\alpha\beta}(\tau'\sigma') = \frac{\partial\sigma^{\gamma}}{\partial\sigma'^{\alpha}}\frac{\partial\sigma^{\delta}}{\partial\sigma'^{\beta}}\gamma_{\alpha\beta}(\tau,\sigma)$$

- Weyl invariance of Polyakov’s action

$$\gamma_{\alpha\beta}(\tau,\sigma) \rightarrow e^{2w(\tau,\sigma)}\gamma_{\alpha\beta}(\tau,\sigma)$$

Classical symmeties

- For $S_{(Pol,pa)}$ one has tow options:

  (i) Equate $\eta(\tau) = \frac{\sqrt{-\gamma_{\tau\tau}(\sigma,\tau)}|_{\sigma=-\delta,\sigma=\delta}}{m^2}$

  (ii) Take $\eta(\tau)$ to be independent.

- In (i) the full action is not Weyl invariant
- In (ii) it is but the world line rep. cannot eliminate the auxiliary field.
- In both ways we cannot gauge fix to get a free action.

The Noether charges

- The Noether charge associated with the space-time translation reads

$$
\begin{aligned}
P^\mu \equiv Q^\mu &= T\int_{\delta}^{\delta} d\sigma \frac{1}{2\sqrt{det(h_{\alpha\beta})}}\partial_{\dot{X}^0} det(h_{\alpha\beta}) \\
&+ m_L \frac{\dot{X}^\mu}{2\sqrt{\dot{X}^\nu \dot{X}_\nu}}\Big|_{\sigma=-\delta} + m_R \frac{\dot{X}^\mu}{2\sqrt{\dot{X}^\nu \dot{X}_\nu}}\Big|_{\sigma=\delta}
\end{aligned}
$$

- The Lorentz transformation charges

$$
\begin{aligned}
J^{\mu\nu} &= T\int_{\delta}^{\delta} d\sigma \frac{1}{2\sqrt{det(h_{\alpha\beta})}}\left[\partial_{\dot{X}_\mu} det(h_{\alpha\beta})X^\nu - \partial_{\dot{X}_\nu} det(h_{\alpha\beta})X^\mu\right] \\
&+ m_L \frac{\dot{X}^\mu X^\nu - \dot{X}^\nu X^\mu}{2\sqrt{\dot{X}^\nu \dot{X}_\nu}}\Big|_{\sigma=-\delta} + m_R \frac{\dot{X}^\mu X^\nu - \dot{X}^\nu X^\mu}{2\sqrt{\dot{X}^\nu \dot{X}_\nu}}\Big|_{\sigma=\delta}
\end{aligned}
$$

The classical energy and angular momentum

- For the classical configuration

$$X^0 = l\tau \qquad \rho = L\sigma \qquad \phi = \omega\tau$$

- The energy and angular momentum are

$$E = \frac{2m}{\sqrt{1-\beta^2}} + 2Tl\frac{\arcsin(\beta)}{\beta}$$

$$J = 2ml\frac{\beta}{\sqrt{1-\beta^2}} + Tl^2\left(\frac{\arcsin(\beta) - q\sqrt{1-\beta^2}}{\beta^2}\right)$$

Small and large mass approximations

- Relations between E and J in the limits of
- Small mass

$$J = \alpha' E^2 - \alpha' \frac{4\pi^{1/2}}{3}(m_1^{3/2} + m_2^{3/2})\sqrt{E}$$

- Large mass

$$J_{lm} = \frac{2m^{1/2}}{T3\sqrt{3}}(E-2m)^{3/2} + \frac{7}{\sqrt{1083}m^{1/2}T}(E-2m)^{5/2}$$
$$-\frac{1003}{\sqrt{2332803}Tm^{3/2}}(E-2m)^{7/2}$$

Gauge fixing

- There are several ways to fix the 2d diffeomorphism

- The static gauge

$$\boxed{\tau = \frac{X^0}{L} \qquad \sigma = arcsin(\frac{\rho}{L})}$$

- The orthogonal gauge

$$\boxed{\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu = 0 \qquad \dot{X}^\mu X'_\mu = 0}$$

Or equivalently in ws light-cone coordinates

$$h_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0 \qquad h_{--} = \partial_- X^\mu \partial_- X_\mu = 0$$

Gauge fixing

- In this gauge

$$h_{\alpha\beta} = \cos^2(\delta)\eta_{\alpha\beta}$$

- Another way is to take the fluctuations to be orthogonal to the classical solution

$$\partial_\alpha X^\mu \eta_{\mu\nu} \delta X^\nu = 0$$

The quantization of static bosonic string with massive particles on its ends

The quantization of a static string

- The energy of the quantized static open string with no massive endpoints in the D dimension is (Arvis)

$$\boxed{E_n = \sqrt{(TL)^2 + 2\pi T\left(n - \frac{D-2}{24}\right)}}$$

- A naïve generalization of the static to a rotating string with no massive endpoints

$$\boxed{E_n = \sqrt{(2\pi TJ) + 2\pi T\left(n - \frac{D-2}{24}\right)}}$$

- Which translates to the Regge relation

$$\boxed{n + J = \alpha' E_n^2 + a \qquad a = \frac{D-2}{24}}$$

The quantization of the static string

- In solutions of the EQN are (in the TL/m<1 limit)

$$X^\mu = x^\mu + l^2 p^\mu \tau + il\sqrt{2}\sum_{n\neq 0}\frac{1}{w_n}\alpha_n^\mu \cos(w_n\sigma + \phi_n)e^{-iw_n\tau}$$

- The eigenfrequencies and phases are given by

$$\tan(\phi_n) = \frac{m^2 w_n}{T} \qquad \tan(w_n\pi) + \frac{2Tm^2 w_n}{T^2 - (w_n m^2)^2} = 0$$

- In the limit of massless and infinite mass we get

$$w_n = n.$$

The quantum energy of the static string

- The quantum energy is the sum of the classical energy and the energy from the quantum fluctuations

$$\boxed{\delta x^{\mu} = \frac{1}{\sqrt{2T}} \sum_{n \neq 0} e^{-i w_n t} \frac{\alpha_n^{\mu}}{w_n} u_n(\rho)}$$

- Subjected to the orthogonality conditions

$$\int_0^L d\rho u_n(\rho) u_m(\rho) \epsilon(\rho) = \delta_{nm}$$

$$\int_0^L dr u_n'(\rho) u_m'(\rho) = w_n^2 \delta_{nm}$$

$$\epsilon(\rho) = 1 + \tfrac{m}{T}[\delta(\rho + L/2) + \delta(\rho - L/2)]$$

The quantum energy of the static string

The usual quantization algebra

$$\boxed{[\alpha_n^i, \alpha_m^j] = \omega_n \delta^{ij}\, \delta_{n+m,0}\,,}$$

$$i, j = 1, \ldots, D-2\,, \quad n, m = \pm 1, \pm 2, \ldots.$$

When translated to Fock space operators

$$\alpha_n^j = \sqrt{\omega_n}\, a_n^j\,, \qquad \alpha_n^{j\,+} = \sqrt{\omega_n}\, a_n^{j\,+}\,,$$

$$[a_n^i, a_m^{j\,+}] = \delta^{ij}\, \delta_{nm}\,, \quad n, m = 1, 2, \ldots\,,$$

The quantum energy of the static string

- The quantum energy is

$$E = \frac{T}{2}\int_{-L/2}^{L/2} d\rho(\delta\dot{X}^{i})^{2}\epsilon(\rho) + (\delta X^{i'})^{2}$$

- Substituting the creation and annihilation operators

$$\begin{aligned} E &= \frac{1}{L}\sum_{n=1}^{\infty}\sum_{i=1}^{D-2}(\alpha_n^i \alpha_n^{i\,\dagger} + \alpha_n^{i\,\dagger}\alpha_n^i) \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{D-2}\alpha_n^{i\,\dagger}\alpha_n^i + \frac{D-2}{2}\frac{1}{L}\sum_{n=1}^{\infty} w_n \end{aligned}$$

- Thus the Casimir energy is

$$E_C(m) = \frac{1}{2}\sum_{n=1}^{\infty} w_n$$

The Casimir energy

- The Casimir energy (or the intercept) is given by

$$E_C(m) = \frac{1}{2}\sum_{n=1}^{\infty} w_n$$

- For the special cases

$$E_C(m=\infty) = E_C(m=0) = \frac{\pi}{2L}\sum_{n=1}^{\infty} n = -\frac{\pi}{24L}$$

- For finite mass

$$w_n = n + f(R)\frac{1}{n}$$

and we cannot use the zeta function regularization

The Casimir energy

- How can we sum over the eigenfrequencies for the massiv case?
- We use a contour integral to compute the sum using

(Lambiase Nesterenko)

$$\frac{1}{2\pi i}\oint_C dw\, w\frac{f'(w)}{f(w)} = \frac{1}{2\pi i}\oint_C dw\, w[Lanf(w)]' = \sum_k n_k w_k - \sum_l p_l \tilde{w}_l$$

we take zeros poles

$$f(w) = 2mTwcos(wL) - (m^2w^2 - T^2)sin(wL) = 0$$

So the Casimir energy is

$$E_C(m) = \frac{1}{4\pi i}\oint_C dw\, w[Lanf(w)]'$$

The Casimir energy

- Where C is a contour that includes the real semi-axis where all the roots of f(w) occur.
- Since f(w) does not have poles we deform the contour to a semi-circle of radius Λ and a segment along the imaginary axis $(-i\Lambda, i\Lambda)$.
- The Casimir energy thus reads

$$\begin{aligned} E_C^{(reg)}(m,L) &= \frac{1}{2\pi}\int_0^{\Lambda} dy Lan\left[2mTy\cosh(yL) + (m^2y^2 + T^2)\sinh(yL\right. \\ &+ \frac{1}{4\pi}\left[wLan[f(w)]\right]_{-\Lambda}^{\Lambda} + I_{sc}(\Lambda) \end{aligned}$$

- To regularize and renormalize the result we subtract

$$E_C^{(ren)}(m,L) = \lim_{\Lambda\to\infty}\left[E_C^{(erg)}(m,L) - E_C^{(reg)}(m, L\to\infty)\right]$$

The Casimir energy

- The subtracted energy is

$$E_C^{(erg)}(m, L \to \infty) = \frac{1}{2\pi}\int_0^{\Lambda} dy Lan\left[e^{(yL)}\frac{(my+T)^2}{2}\right]$$

- The renormalized Casimir energy is thus

$$E_C^{(ren)}(m, L) = \int_0^{\infty} dx Lan\left[1 - e^{-2x}\left(\frac{(x-a)}{x+a)}\right)^2 \cosh(xL) + (m^2x^2 + T^2)\sinh(xL)(\right]5.$$

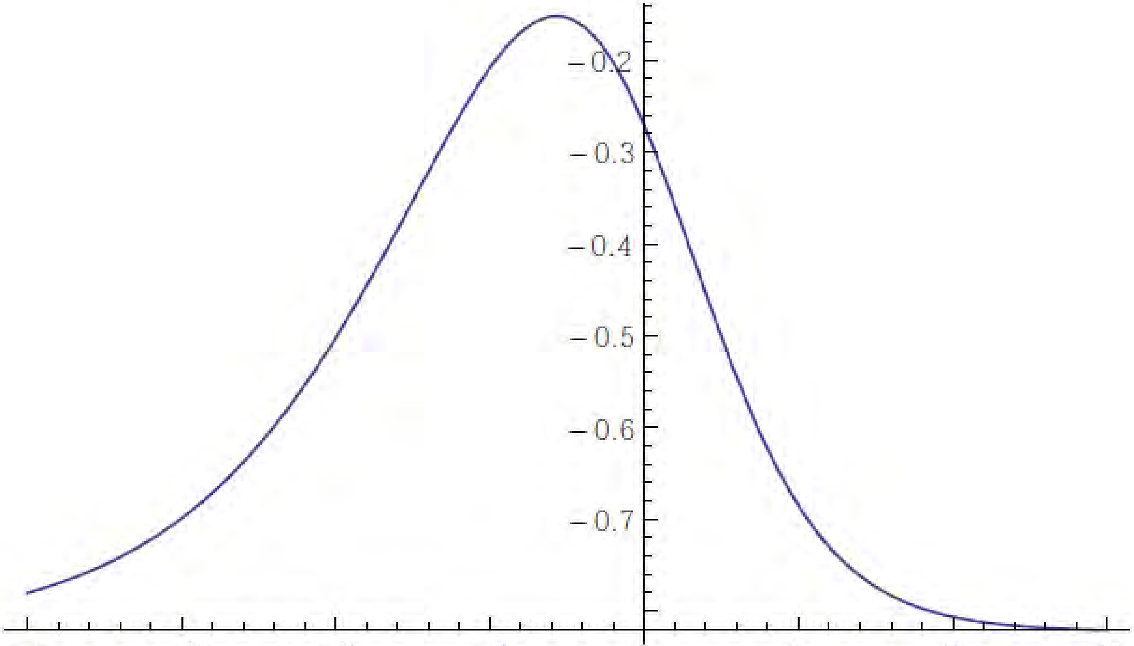
- For the massless and infinite mass cases

$$E_C^{(ren)}(m = 0, L) = E_C^{(ren)}(m = \infty, L) = \int_0^{\infty} dx Lan\left[1 - e^{-2xL}\right] = -\frac{\pi}{24L}($$

The Casimir energy

- Denoting a=m/TL we define the ratio

$$\eta(a) = \frac{E_c^{(ren)}(m,L)}{E_c^{(ren)}(m=\infty,L)} = -\frac{12}{\pi^2}\int_0^\infty dz Lan\left[1 - e^{-2z}\left(\frac{1-az}{1+az}\right)^2\right]$$



η as a function of $Lan_{10}(a)$

Back to the quantization of the rotating critical string

From the static string to the rotating one

- Before explicit analysis of the rotating string we can conjecture that (at least for the transverse modes) the result for the rotating string can be inferred from the static case
- For the rotating string we replace

$$\boxed{TL = \sqrt{\frac{2}{\pi}}\sqrt{TJ}\,f(\frac{m_{sep}}{M})}$$

- For the massless and small mass cases we have

$$f(\frac{m_{sep}}{M}) = 1 \quad for \quad a = 0 \qquad f(\frac{m_{sep}}{M}) = \sqrt{1 - \frac{8\sqrt{\pi}}{3}\left(\frac{m_{sep}}{M}\right)^{3/2}} \quad a << 1$$

- Large mass

$$\boxed{TL \sim \left(\frac{TJ}{\sqrt{m}}\right)^{2/3}}$$

The quantum trajecories

- The quantum energy and angular momentum is found by

(i) inserting into the expressions of the Noether charges

$$X^i = X^i_{cl}(\tau, \sigma) + \delta X^i(\tau, \sigma)$$

(ii) Expanding to quadratic order which means taking the leading order in TL/m

(iii) Imposing gauge fixing conditions

The quantum trajecoties

- The energy and angular momentum are given by

$$
\begin{aligned}
E &= T\int_{-\delta}^{\delta} d\sigma \frac{\left[\dot{X}^0\left(-(X'^0)^2+\rho'^2+\rho^2\phi'^2+\vec{z}'\cdot\vec{z}'\right)-X'^0\left(-(\dot{X}^0)X'^0+\dot{\rho}\rho'+\rho^2\dot{\phi}\phi'+\dot{\vec{z}}\cdot\vec{z}'\right)\right]}{\sqrt{det[h_{\alpha\beta}]}}\\
&+ m\left[\frac{\dot{(X^0)}}{\sqrt{-\dot{(X^0)}^2+\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{\vec{z}}\cdot\dot{\vec{z}}}}\right]_{\sigma=\pm\delta}\\
J &= T\int_{-\delta}^{\delta} d\sigma \frac{\left[\rho\dot{\phi}\left(-(X'^0)^2+\rho'^2+\rho^2\phi'^2+\vec{z}'\cdot\vec{z}'\right)-\rho^2\phi'\left(-(\dot{X}^0)X'^0+\dot{\rho}\rho'+\rho^2\dot{\phi}\phi'+\dot{\vec{z}}\cdot\vec{z}'\right)\right]}{\sqrt{det[h_{\alpha\beta}]}}\\
&+ m\left[\frac{\rho^2\dot{(\phi)}}{\sqrt{-\dot{(X^0)}^2+\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{\vec{z}}\cdot\dot{\vec{z}}}}\right]_{\sigma=\pm\delta}
\end{aligned}
$$

- The determinant is given by

$$
\begin{aligned}
det[h_{\alpha\beta}] &= \left(-(\dot{X}^0)^2+\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{\vec{z}}\cdot\dot{\vec{z}}\right)\left(-(X'^0)^2+\rho'^2+\rho^2\phi'^2+\vec{z}'\cdot\vec{z}'\right)\\
&- \left(-(\dot{X}^0)X'^0+\dot{\rho}\rho'+\rho^2\dot{\phi}\phi'+\dot{\vec{z}}\cdot\vec{z}'\right)
\end{aligned}
$$

The quantum trajectories of the massless case

- In the massless case we can follow either the static or the orthogonal gauges
- In the static gauge up to quadratic order

$$= TL\int_{-\pi/2}^{\pi/2} d\sigma\left[1+\frac{1}{2}\tan^2(\sigma)[(\dot{\phi})^2+(\phi')^2)+\frac{1}{2}\frac{1}{\cos^2(\sigma)}([\dot{\vec{z}}\cdot\dot{\vec{z}}+\vec{z}'\cdot\vec{z}']\right]$$

$$= TL^2\int_{-\pi/2}^{\pi/2} d\sigma\sin^2(\sigma)\left[1+\frac{1}{2}\tan^2(\sigma)[(\dot{\phi})^2+(\phi')^2)+\frac{1}{2}\frac{1}{\cos^2(\sigma)}([\dot{\vec{z}}\cdot\dot{\vec{z}}+\vec{z}'\right.$$

- We find that the following quantum deformation

$$\boxed{L\delta E = L(E-E_{cl}) = \alpha'\delta(E^2) = \int_{-\pi/2}^{\pi/2} d\sigma < |\mathcal{H}_{ws}| > \equiv a}$$

The quantum trajectories of the massless case

- In the orthogonal gauge

$$= T\int_{-\pi/2}^{\pi/2} d\sigma\partial_\tau(X^0_{cl}+\delta X^0) = \pi T L + T\int_{-\pi/2}^{\pi/2} d\sigma\dot{\delta}X^0$$

$$= T\int_{-\pi/2}^{\pi/2} d\sigma\rho^2\partial_\tau(\phi_{cl}+\delta\phi) = \frac{1}{2}\pi T L^2 + T L^2\int_{-\pi/2}^{\pi/2} d\sigma\left(\sin^2(\sigma)\delta\dot{\phi} + 2\sin(\sigma\right.$$

- We use the gauge fixing condition to rewrite the linear deformations with a quadratic form
- This leads again to

$$\boxed{\alpha'\delta E^2 \quad = \quad L\delta E = \int_{-\pi/2}^{\pi/2} d\sigma < |\mathcal{H}_{ws}(\delta X^\mu)| >}$$

The massive case- the transverse model

- For the string with massive endpoints we first switch off the fluctuations in the plane of rotation and consider the contribution to E and J from the transverse modes.
- Similar to the static string we get

$$\boxed{\alpha'\delta(E^2) =< |\mathcal{H}_{ws}(\delta X^\mu)| > +\mathcal{H}_{ws}|_{\sigma=\pm\delta}}$$

- This translates to

$$\boxed{a_{tramsverse} = \frac{D-3}{2}\sum_n w_n}$$

The contribution from the planar mode

- The contribution of δϕ is more subtle
- The NG action for it reads [Pando Zayas, Vaman,J.S]

$$\boxed{S_{NG} = T_s \left(\int \sqrt{e^4 \cos^4(\sigma)} + \frac{e^2 \tan^2(\sigma)}{2} (-\delta\dot{\phi}^2 + \delta\phi'^2) - \frac{1}{2} (\delta\dot{\vec{z}}\delta\dot{\vec{z}} - \delta\vec{z}'\delta\vec{z}') \right)}$$

- We can do a field redefinition $\delta\tilde{\phi} = \delta\phi \ \tan(\sigma)$
- This has to solve

$$\boxed{\left(\partial_\sigma^2 - \partial_\tau^2 - \frac{2}{\cos^2(\sigma)} \right) \delta\tilde{\phi}(\sigma) = 0.}$$

- The eigenmode expansion

$$\begin{aligned}
\delta\tilde{\phi}(\sigma,\tau) &= \tilde{\phi}_{nm}\chi_{nm}(\sigma,\tau) \\
&\equiv \sum_n \frac{\exp(in\tau)}{\sqrt{2\pi}} \left(\sum_{m=odd} \tilde{\phi}_{nm} \frac{1}{\sqrt{\pi(m^2-1)}} (m\sin(m\sigma) - \cos(m\sigma)\tan\sigma \right. \\
&+ \left. \sum_{m=even} \tilde{\phi}_{nm} \frac{1}{\sqrt{\pi(m^2-1)}} (m\cos(m\sigma) + \sin(m\sigma)\tan\sigma) \right)
\end{aligned}$$

The contribution from the planar mode

- With the normalization

$$(\partial_\tau^2 - \partial_\sigma^2 + \frac{2}{\cos^2\sigma})\chi_{nm}(\sigma,\tau) = (m^2 - n^2)\chi_{nm}(\sigma,\tau) \equiv \lambda_{nm}\chi_{nm}(\sigma,\tau),$$

$$\int_{-\pi}^{\pi} d\sigma\ \chi_{nm}\chi_{n'm'} = e^{i(n+n')\tau}\delta_{m-m'}.$$

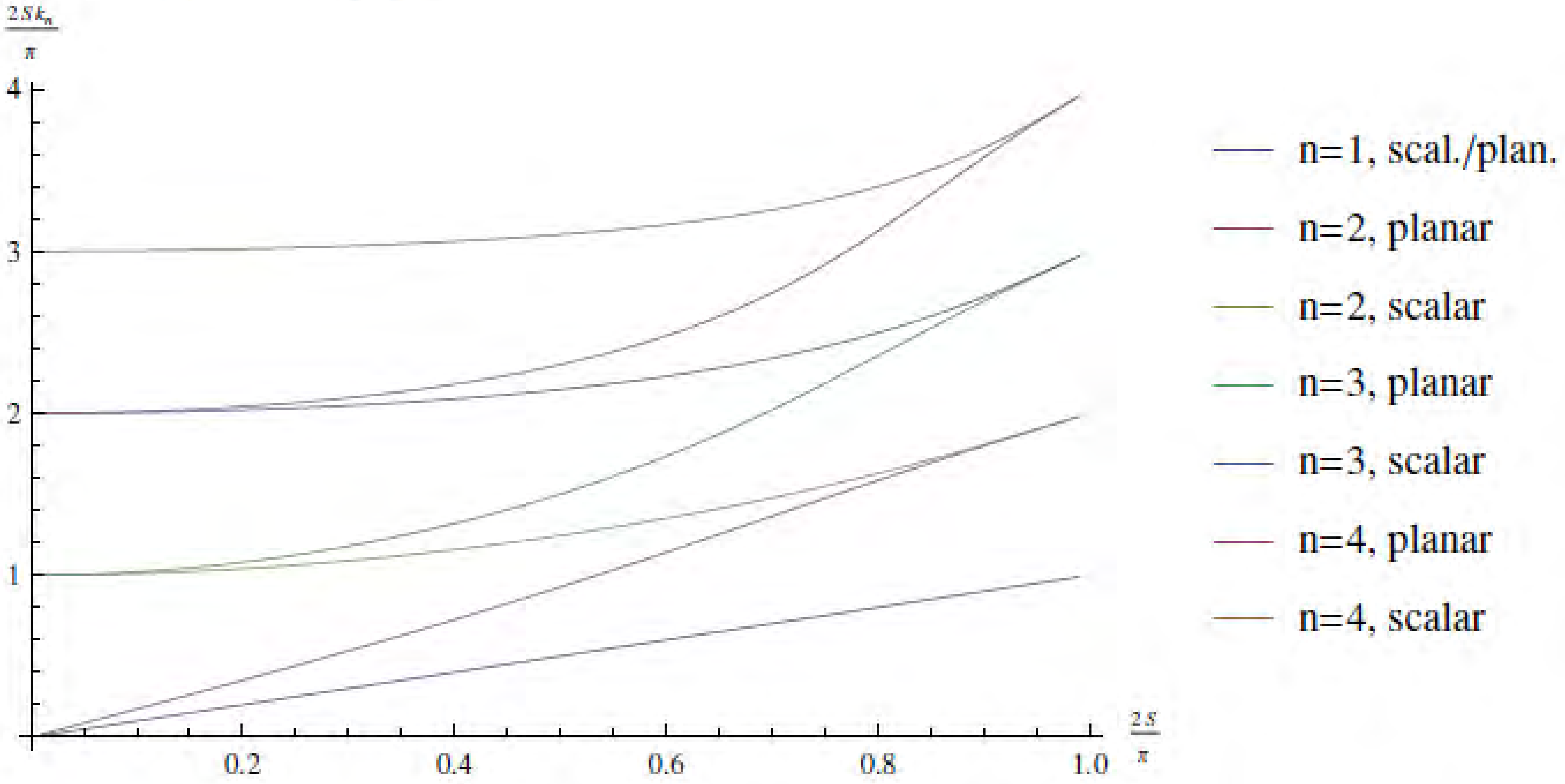
- The eigenmode with m=o vanishes and so we need a special solution for that case which is

$$\chi_{n\,m=1} = \frac{1}{\cos\sigma}.$$

- However this is not σ normalizable and hence cannot be counted.

The low lying spectrum

- The low lying spectrum as a function of δ where $\frac{m}{TL} = \frac{\cos^2\delta}{\sin\delta}$ was numerically derived by [Zahn]



On the quantization of the rotating string in non-critical dimensions

The non criticality term: Liuville term

- The quantum string action is inconsistent for a non-critical D dimensions.
- In the Polyakov formulation for quantum conformal invariance one has to add a Liouville term.
- It can be built from a ``composite Liouville field”

$$\boxed{\varphi \equiv -\frac{1}{2}\ln(g^{ab}\partial_a X^\mu \partial_b X_\mu)}$$

- The action then reads

$$S = S_{\text{Polyakov}} + S_{\substack{\text{composite}\\ \text{Liouville}}}$$

- The Liouville term is

$$\boxed{S_{\substack{\text{composite}\\ \text{Liouville}}} \equiv S_\varphi = \frac{\beta}{2\pi}\int d^2\sigma\,\sqrt{|g|}\,\left(g^{ab}\,\partial_a\varphi\,\partial_b\varphi - \varphi\,\mathcal{R}_{(2)}\right)}$$

where

$$\beta \equiv \frac{26-D}{12}$$

The non criticality term: The Polchinsky Strominger term

- In the Nambu-Goto formulation the anomaly is cancelled by adding a Polchinky Strominger term
- For a classical rotating string parameterized as

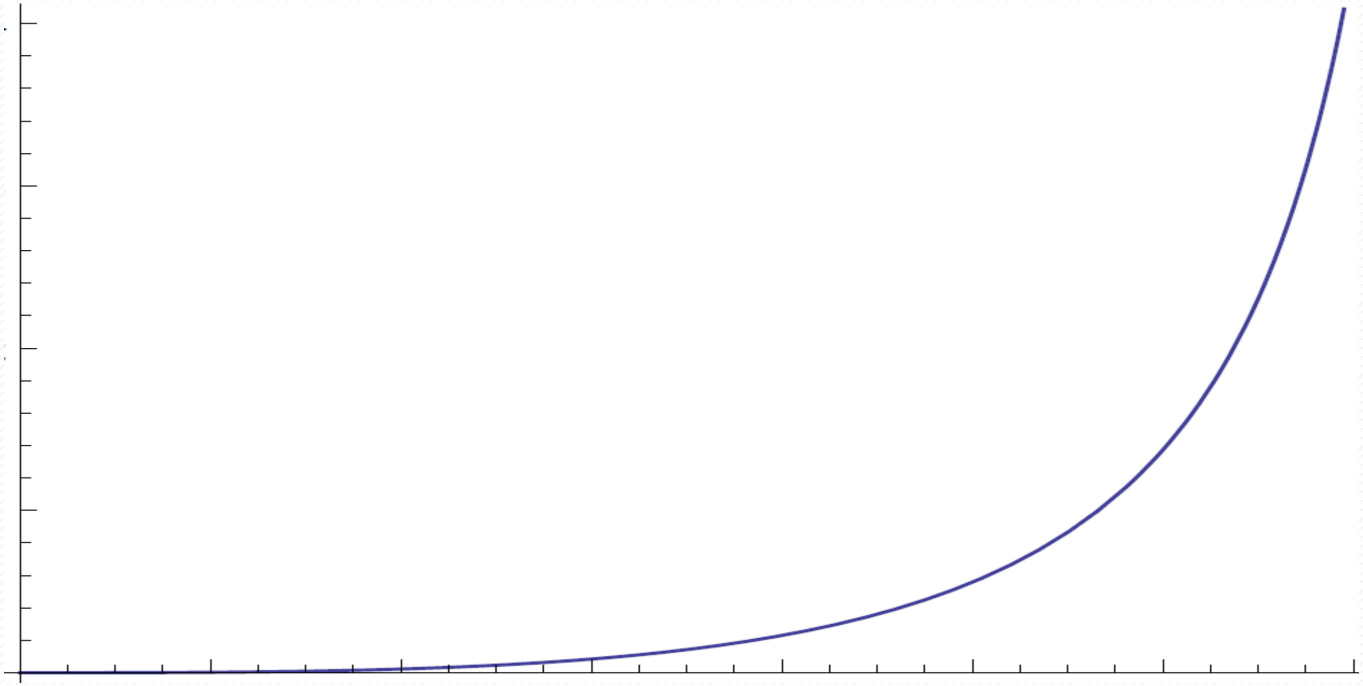
$$X = l(\tau, \cos(\tau)\sin(\sigma), \sin(\tau)\sin(\sigma), 0)$$

- The induced metric is $h_{\alpha\beta} = l^2 \cos^2(\sigma)\eta_{\alpha\beta}$
- For the range of (τ,σ) $\mathcal{R} \times [-\delta, \delta]$
- The boundary condition is $\boxed{\frac{m}{Tl} = \frac{\cos\delta}{\tan\delta}}$
- The PS term is

$$\boxed{\begin{aligned} S_{ps} &= \int_{-\delta}^{\delta} \frac{26-D}{24\pi} \frac{(\partial_+^2 X \cdot \partial_- X)(\partial_-^2 X \cdot \partial_+ X)}{(\partial_+ X \cdot \partial_- X)} = -\frac{26-D}{24\pi} \int_{-\delta}^{\delta} d\sigma \tan^2(\sigma) \\ &= -\frac{26-D}{12\pi}(\tan(\delta) - \delta) \end{aligned}}$$

The PS term

- The PS action as a function of δ



- The PS term diverges for the massless case. In that case one can use the renomalization procedure of Hellerman et al

The non-criticality term for the massless case

- Inserting the rotating classical string to the Liuville field one finds that
- The Liouville term= The Polchinsky Strominger term

- For the massless case δ=π/2 and the non-critical term diverges.
- Hellerman et al suggested a procedure to regularize and renormalize this divergence for the massless case.
- They found that the intercept dose not depend on D

$$\boxed{a = \frac{D-2}{24} + \frac{26-D}{24} = 1}$$

- The question is what happens in the massive case

Leading 1/m order quantum correction

- In the limit of large m/TL (v<<1) the boundary eom

$$\frac{TL}{\gamma} = m\gamma(wL)^2 \rightarrow \qquad (wL)^2 = \frac{TL}{m} << 1$$

- The classical trajectory

$$J \sim \frac{4\pi}{3\sqrt{3}}\alpha' \, m^{1/2}(E-2m)^{3/2} + O(E-2m)^{5/2}$$

- The quantum corrected trajectory involves

$$\alpha' E_{cl}^2 \rightarrow \alpha' E_{qm}^2 = \alpha' E_{cl}^2 + a = \alpha' E_{cl}^2 + (a_{Cas} + a_{PS})$$

Leading 1/m order quantum correction

- Thus the corrected trajectory reads

$$\boxed{J \sim \frac{4\pi}{3\sqrt{3}}\alpha' \, m^{1/2}\left(\sqrt{E^2+\frac{(a_{Cas}+a_{PS})}{\alpha'}}-2m\right)^{3/2}+O(E-2m)^{5/2}}$$

- The contribution of Sps to the intercept for D=4

$$a_{ps} = -\frac{26-D}{12\pi}(\tan(\delta)-\delta) = -\frac{11}{36\pi}\left(\frac{TL}{m}\right)^3$$

- We can replace the dependence on TL with

$$\frac{TL}{m} \simeq \left(\frac{3\sqrt{3}}{2}\frac{TJ}{m^2}\right)^{2/3} \qquad \boxed{a_{ps} \simeq -\frac{2}{\pi}\left(\frac{TJ}{m^2}\right)^2}$$

- We can approximate the a_{Cas}

$$\boxed{a_{Cas} \simeq \frac{3}{2\pi}[0.18\left(\frac{TL}{m}\right)-0.07] = 0.16\left(\frac{TJ}{m^2}\right)^{2/3}-0.06}$$

Summary and open questions

Summary and open questions

- Precise fits to hadronic spectra require the full quantization of a string with massive endpoints.
- In general this is not a free system
- We can solve it in the leading order in

$$\frac{TL}{m} \sim \left(\frac{TJ}{m^2}\right)^{2/3}$$

- We computed the Casimir energy due to the transverse modes. The planar modes are sbutle.
- The Polchinski Strominger action is singular for the massless case. It has to be amended.
- Will redo our fits to test the results of the quantization

Summary and open questions