Generalised integrable λ, η-deformations

K. SIAMPOS

based on works with

G. Itsios, K. Sfetsos, D.C. Thompson and A. Torrieli

1404.3748, 1405.7803 and 1409.0554 and 1506.05784

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Integrable models – I

- Let a Hamiltonian system with

$$\frac{\mathrm{d}q_i}{\mathrm{d}\tau} = \frac{\partial H}{\partial p_i}\,, \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial H}{\partial q_i}\,, \quad H = H(q_i, p_i, \tau)\,, \quad i = 1, \ldots, N$$

- It is classically integrable if its eom can be recast to:

$$\frac{\mathrm{d}\mathcal{L}_1}{\mathrm{d}\tau} = [\mathcal{L}_0, \mathcal{L}_1]\,, \qquad I_i = \mathrm{Tr}\,\mathcal{L}_0^i\,, \qquad \{I_i, I_j\}_{\mathrm{PB}} = 0\,,$$

  where $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\tau;\mu)\,,\, \mu \in \mathbb{C}$; form a classical Lax pair.

- It is integrable in the Liouville sense; existence of action-angle variables.

Introduction and motivation

Integrable models – II

- Consider a 2d σ-model, whose eom can be written as:

$$\mathrm{d}\mathcal{L} = \mathcal{L} \wedge \mathcal{L} \quad \text{or} \quad \partial_0 \mathcal{L}_1 - \partial_1 \mathcal{L}_0 = [\mathcal{L}_0, \mathcal{L}_1] \,,$$

  where $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\tau, \sigma; \mu)\,, \quad \mu \in \mathbb{C}$.

- The Lax pair is defined up to gauge transformations: $\mathcal{L} \mapsto g^{-1}\mathcal{L}\, g - g^{-1}\mathrm{d}g$.

- There is a classically conserved *monodromy matrix*

$$M(\mu) := P \exp \int_{-\infty}^{\infty} \mathrm{d}\sigma\, \mathcal{L}_1(\tau, \sigma; \mu)\,, \quad \partial_0 M(\mu) = 0\,.$$

  Its expansion with respect to μ defines infinite conserved charges.

- Deforming the theory and keeping integrability is far from trivial.

- Difficulty increases with the number of deformation parameters.

SYNOPSIS

The λ-deformed isotropic Principal Chiral model

The λ-deformed Yang–Baxter σ-model

Discussion & Outlook

Plan of the talk

The λ-deformed isotropic PCM

The λ-deformed Yang–Baxter σ-model

Discussion & Outlook

THE λ-DEFORMED ISOTROPIC PCM

Consider the isotropic PCM, whose action reads:

$$S_{\rm PCM} = \frac{1}{2\pi t}\int \mathrm{d}^2\sigma\, R_+^T R_- \,.$$

Gauging procedure with WZW action reveals the action:

$$S_{k,\lambda}(g) = S_{\rm WZW,k}(\mathfrak{g}) + \frac{k\lambda}{2\pi}\int \mathrm{d}^2\sigma\, L_+^T(\mathbb{I}-\lambda D)^{-1}R_- \,, \quad \lambda = \frac{k\,t}{1+k\,t}\,, \quad 0 \leqslant \lambda < 1\,,$$

K. Sfetsos, Nucl. Phys. **B880** (2014) 225

Its equations of motion read:

$$\partial_+ I_- + \partial_- I_+ = 0\,, \qquad \partial_+ I_- - \partial_- I_+ + [I_+, I_-] = 0\,,$$
$$I_+ = -\frac{2\lambda}{1+\lambda}\left(\mathbb{I} - \lambda\, D^T\right)^{-1} L_+\,, \qquad I_- = \frac{2\lambda}{1+\lambda}\left(\mathbb{I} - \lambda\, D\right)^{-1} R_-\,.$$

The $su(2)$ case was tackled through a brut force computation.
J. Balog, P. Forgacs, Z. Horvath and L. Palla, Phys. Lett. **B324** (1994) 403
Glossary:

$$L^a_\pm = L^a_\mu \partial_\pm X^\mu = -i\,\mathrm{Tr}(T_a g^{-1}\partial_\pm g)\,, \quad R^a_\pm = R^a_\mu \partial_\pm X^\mu = -i\,\mathrm{Tr}(T_a \partial_\pm g g^{-1})\,,$$
$$R^a_\mu = D^a{}_b L^b_\mu\,, \quad D_{ab}(g) = \mathrm{Tr}(T_a g T_b g^{-1})\,, \quad g \in G\,.$$

THE λ-DEFORMED ISOTROPIC PCM

In a GWZW, there are two commuting copies of current algebras

$$S_+ = D_+ g g^{-1} + A_+ - A_- \,, \quad S_- = -g^{-1} D_- g + A_- - A_+ \,,$$
$$\{S_\pm^A, S_\pm^B\} = f_{ABC} S_\pm^C \delta_{\sigma\sigma'} \pm \frac{k}{2} \delta_{AB} \delta'_{\sigma\sigma'} \,, \quad \delta_{\sigma\sigma'} = \delta(\sigma - \sigma') \,,$$

P. Bowcock, Nucl. Phys. **B316** (1989) 80

The final action does not depend on derivatives of $A_\pm$

$$S_\pm + \frac{k}{2}\left(\lambda^{-1} A_\pm - A_\mp\right) \approx 0 \,, \quad \text{where} \quad A_\pm = -\frac{1}{2}(1+\lambda) I_\pm \,,$$

Rotating the base we find:

$$\{I_\pm^a, I_\pm^b\}_{\text{PB}} = e^2 f^{abc} (I_\mp^c - (1+2x) I_\pm^c) \delta_{\sigma\sigma'} \pm 2e^2 \delta^{ab} \delta'_{\sigma\sigma'} \,,$$
$$\{I_+^a, I_-^b\}_{\text{PB}} = -e^2 f^{abc} (I_+^c + I_-^c) \delta_{\sigma\sigma'} \,,$$

S. G. Rajeev, Phys. Lett. **B217** (1989) 123

T. J. Hollowood, J. L. Miramontes and D. M. Schmidtt, JHEP **1411** (2014) 009

where:

$$x = \frac{1+\lambda^2}{2\lambda} \geqslant 1 \,, \qquad e = \frac{2\lambda}{\sqrt{k(1-\lambda^2)}(1+\lambda)} \,.$$

The λ-deformed isotropic PCM

The eom can be written in terms of a classical Lax pair:

$$\mathcal{L}_0 = \mu \frac{I_1 + \mu I_0}{1-\mu^2}\,, \qquad \mathcal{L}_1 = \mu \frac{I_0 + \mu I_1}{1-\mu^2}\,, \quad I_{\left\{ {0 \atop 1} \right\}} = I_+ \pm I_-\,, \quad \mu \in \mathbb{C}$$

By expanding $M(\mu)$ in powers of μ we generate an infinite set of conserved charges:

$$M(\mu) = 1 + \mu\, Q_0 + \mu^2\, \widehat{Q} + \mathcal{O}(\mu^3)$$

The first two conserved charges read:

$$Q_0 = \int_{-\infty}^{\infty} d\sigma\, I_0(\sigma)\,, \quad Q_1 = \widehat{Q} - \frac{1}{2} Q_0^2 = \int_{-\infty}^{\infty} d\sigma\, I_1(\sigma) + \frac{1}{2} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\sigma} d\sigma' \left[I_0(\sigma), I_0(\sigma')\right]$$

M. Luscher and K. Pohlmeyer, Nucl. Phys. **B137** (1978) 46,
E. Brezin, C. Itzykson, J. Zinn-Justin and J. B. Zuber, Phys. Lett. **B82** (1979) 442,
D. Bernard, Commun. Math. Phys. 137 (1991) 191.

These charges turn out to satisfy a classical Yangian algebra $Y_C(\mathfrak{g})$.

THE λ-DEFORMED ISOTROPIC PCM

The Poisson brackets for Q_0

$$\{Q_0^a, Q_0^b\} = -2e^2(1+x) f_{abc}\, Q_0^c\,, \qquad \{Q_0^a, Q_1^b\} = -2e^2(1+x) f_{abc}\, Q_1^c\,,$$

and the *first Serre* relation for Q_1:

$$f_{dab}\{Q_1^c, Q_1^d\} + f_{dca}\{Q_1^b, Q_1^d\} + f_{dbc}\{Q_1^a, Q_1^d\} = 2e^2(1+x) \times\ F_{pqr}\, Q_0^p Q_0^q Q_0^r\,,$$

where: $F_{pqr} = \frac{1}{4} f_{aip} f_{bjq} f_{ckr} f_{ijk}$. They could be derived through the PB of the monodromy matrix.
J.M. Maillet Nucl. Phys. **B269** (1986) 54, Phys. Lett. **B281** (1992) 90

For the $\mathfrak{su}(2)$ case it vanishes and the *second Serre* relation reads

$$\{\{Q_1^a, Q_1^b\}, \{Q_0^c, Q_1^d\}\} + \{\{Q_1^c, Q_1^d\}, \{Q_0^a, Q_1^b\}\} = 2e^6\,(1+x)^3\, Q_0^e \left(\varepsilon_{abe}\, Q_0^{[c} Q_1^{d]} + \varepsilon_{cde}\, Q_0^{[a} Q_1^{b]}\right)$$

G. Itsios, K. Sfetsos, KS and A.Torrieli, Nucl.Phys. **B889** (2014) 64

Generalises MacKay's results for:
Current algebras of the PCM ($x = 1$) & Gross–Neveu ($e \to 0, x \to \infty, e^2 x = \text{finite}$)
N.J. MacKay, Phys. Lett. **B281** (1992) 90

RENORMALISABILITY

Consider a 1+1-dimensional non-linear σ-model with action

$$S = \frac{1}{2\pi\alpha'} \int (G_{\mu\nu} + B_{\mu\nu}) \partial_+ X^\mu \partial_- X^\nu \, .$$

The one-loop β-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ read:
Ecker–Honerkamp 71, Friedan 80, Braaten–Curtright–Zachos 85

$$\mu \frac{\mathrm{d}G_{\mu\nu}}{\mathrm{d}\mu} + \mu \frac{\mathrm{d}B_{\mu\nu}}{\mathrm{d}\mu} = R^-_{\mu\nu} + \nabla^-_\nu \xi_\mu \, ,$$

where the last term corresponds to field redefinitions (diffeomorphisms).

Generalities:

- The Ricci tensor and the covariant derivative includes torsion, i.e. $H = \mathrm{d}B$.
- The σ-model is renormalizable within the zoo of metrics and 2-forms.
- It is not given that the RG flows will retain the form at hand of $G_{\mu\nu}$ and $B_{\mu\nu}$.

RENORMALISABILITY

It turns out that the RG flow keeps the form of the σ-model, the coupling λ is flowing.

The β-function reads: G. Itsios, K. Sfetsos–KS, Phys.Lett. **B733** (2014) 265

$$\beta_\lambda = \mu \frac{d\lambda}{d\mu} = -\frac{c_G \lambda^2}{2k(1+\lambda)^2}\,, \qquad 0 \leqslant \lambda \leqslant 1\,, \qquad \text{and } k \text{ does not flow}$$

D. Kutasov, Phys. Lett. **B227** (1989) 68
The integrability is valid at one-loop in $1/k$.

Properties of the flow

- It behaves accordingly around $\lambda \ll 1 \Longrightarrow \beta_\lambda \simeq -\frac{c_G \lambda^2}{2k} + \mathcal{O}(\lambda^3)\,.$
- The β-function can be solved explicitly:

$$\lambda - \lambda^{-1} + 2\ln\lambda = -\frac{c_G}{2k}\,(t - t_0)\,,$$

  where UV at $\lambda \to 0$ and towards the IR at $\lambda \to 1^-$.
- It is invariant under the weak–strong duality, i.e. $\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k$
  D. Kutasov, Phys. Lett. **B233** (1989) 369
- The effective action shares the same invariance.

Plan of the talk

The λ-deformed isotropic PCM

The λ-deformed Yang–Baxter σ-model

Discussion & Outlook

The λ-deformed Yang–Baxter σ-model

Consider the Yang–Baxter σ-model:

$$S_{\rm YB} = \frac{1}{2\pi t}\int d^2\sigma\, R_+^T\, (\mathbb{I} - \eta\mathcal{R})^{-1}\, R_-$$

C. Klimčík and P. Ševera, Phys. Lett. **B351** (1995) 455
C. Klimčík, JHEP **0212** (2002) 051 and J. Math. Phys. **50** (2009) 043508

$\mathcal{R}$ is an anti-symmetric operator defining the bracket:

$$[A,B]_{\mathcal{R}} := [\mathcal{R}A, B] + [A, \mathcal{R}B]\,, \qquad \forall A, B \in \mathfrak{g}\,,$$

A sufficient condition for its Jacobi identity, leads to:

$$[\mathcal{R}A, \mathcal{R}B] - \mathcal{R}[A,B]_{\mathcal{R}} = -c^2[A,B]\,, \qquad \forall A, B \in \mathfrak{g}\,, \qquad c \in \mathbb{C}$$

Known as modified Yang–Baxter (mYB) equation.

- $[A,B]_{\mathcal{R}}$ defines a second Lie algebra $\mathfrak{g}_{\mathcal{R}}$, i.e. $\tilde{f}_{abc} = \mathcal{R}_{ad}f_{bdc} - \mathcal{R}_{bd}f_{adc}$.
- $\mathfrak{d} = g \oplus \mathfrak{g}_{\mathcal{R}}$ defines a Drinfeld double; f_{abc} and $\tilde{f}_{abc}$ satisfying the Jacobi identities.
- There is an underlying Poisson algebra

$$\{I^a_\pm, I^b_\pm\}_{\rm PB} = e^2 f^{abc}(I^c_\mp - (1+2x)I^c_\pm)\delta_{\sigma\sigma'} \pm 2e^2\delta^{ab}\delta'_{\sigma\sigma'}\,,$$
$$\{I^a_+, I^b_-\}_{\rm PB} = -e^2 f^{abc}(I^c_+ + I^c_-)\delta_{\sigma\sigma'}\,, \quad \text{where} \quad x < 1\,.$$

The λ-deformed Yang–Baxter σ-model

Gauging procedure with WZW action reveals the action:

$$S_{k,\lambda}(g) = S_{\mathrm{WZW,k}}(\mathfrak{g}) + \frac{k}{2\pi}\int \mathrm{d}^2\sigma\, L_+^T(\lambda^{-1} - D)^{-1} R_- \,,$$

where:

$$\lambda^{-1} = k^{-1}(E + k\,\mathbb{I})\,.$$

The equations of motion read ($\lambda \neq \mathbb{I}$):

$$\begin{aligned}
\partial_+ A_- - \partial_-\left(\lambda^{-T} A_+\right) &= [\lambda^{-T} A_+, A_-]\,,\\
\partial_+\left(\lambda^{-1} A_-\right) - \partial_- A_+ &= [A_+, \lambda^{-1} A_-]\,.
\end{aligned}$$

Goal: Try to rewrite the eom of $A_\pm$ in terms of a spectral dependent Lax pair.

The λ-deformed Yang–Baxter σ-model

Group case

$$E = \frac{1}{t}\ (\mathbb{I} - \eta\mathcal{R})^{-1}$$

Using the expression for E and the mYB we find that:

$$\pm\,\partial_\pm \widetilde{A}_\mp = \eta[\mathcal{R}\widetilde{A}_\pm, \widetilde{A}_\mp] + a\,[\widetilde{A}_+, \widetilde{A}_-]\,,$$
$$\widetilde{A}_\pm = (\mathbb{I} \pm \eta\,\mathcal{R})^{-1} A_\pm\,, \quad a = \frac{1 + c^2\eta^2\lambda_0}{1+\lambda_0}\,, \quad \lambda_0 = \frac{k\,t}{1 + k\,t}\,.$$

The above can be written in terms of a Lax connection:

$$\mathcal{L}_\pm = (\alpha_\pm \mathbb{I} \pm \eta\,\mathcal{R})(\mathbb{I} \pm \eta\,\mathcal{R})^{-1} A_\pm\,,$$
$$\alpha_\pm = \alpha_1 + \alpha_2\,\frac{\mu}{\mu \mp 1}\,, \quad \mu \in \mathbb{C}\,,$$
$$\alpha_1 = a - \sqrt{a^2 - c^2\eta^2}\,, \quad \alpha_2 = 2\sqrt{a^2 - c^2\eta^2}\,.$$

K. Sfetsos, KS, D.C. Thompson, 1506.05784

THE λ-DEFORMED YANG–BAXTER σ-MODEL

Comments:

1. It provides a two-parameter integrable deformation of the PCM, labelled as η and λ_0.

2. Generalises for an arbitrary semisimple group, the λ-deformation of the isotropic PCM.

3. Conjectured to be the bi-Yang–Baxter σ-model under *PL T-duality* and Wick rotation.
   C. Klimčík, J. Math. Phys. **50** (2009) 043508 and Letters in Mathematical Physics **104** (2014) 1095.
   K. Sfetsos, KS, D.C. Thompson, 1506.05784

$$S_{\text{bi}-\text{YB}} = \frac{1}{2\pi t}\int \mathrm{d}^2\sigma\, R_+^T \left(\mathbb{I} - \eta\,\mathcal{R} - \zeta\, D\mathcal{R}D^T\right)^{-1} R_-$$

Plan of the talk

The λ-deformed isotropic PCM

The λ-deformed Yang–Baxter σ-model

Discussion & Outlook

Discussion & Outlook

1. We studied the integrability and renormalisability of the λ-deformed isotropic PCM.

2. Conjectured to be the Yang–Baxter σ-model under PL T-duality and Wick rotation.
   B. Hoare and A. A. Tseytlin, Nucl.Phys. **B897** (2015) 448-478

3. We constructed the Lax pair for the λ-deformed Yang–Baxter; "related to" bi-YB.

4. These models turn to be renormalisable, and so are integrable, at one-loop in $1/k$.

5. In the $su(2)$ case, the PCM with $E = \text{diag}(E_1, E_2, E_3)$ is integrable & renormalizable
   I.V. Cherednik, Theor. Math. Phys. **47** (1981) 422
   L. Hlavatý, Phys. Lett. **A271** (2000) 207

   and so does its λ-deformation:
   K. Sfetsos and K.S, Phys. Lett. **B743** (2015) 160