Gauge Theories

and

Non-Commutative Geometry

Eighth Crete Regional Meeting

in string theory

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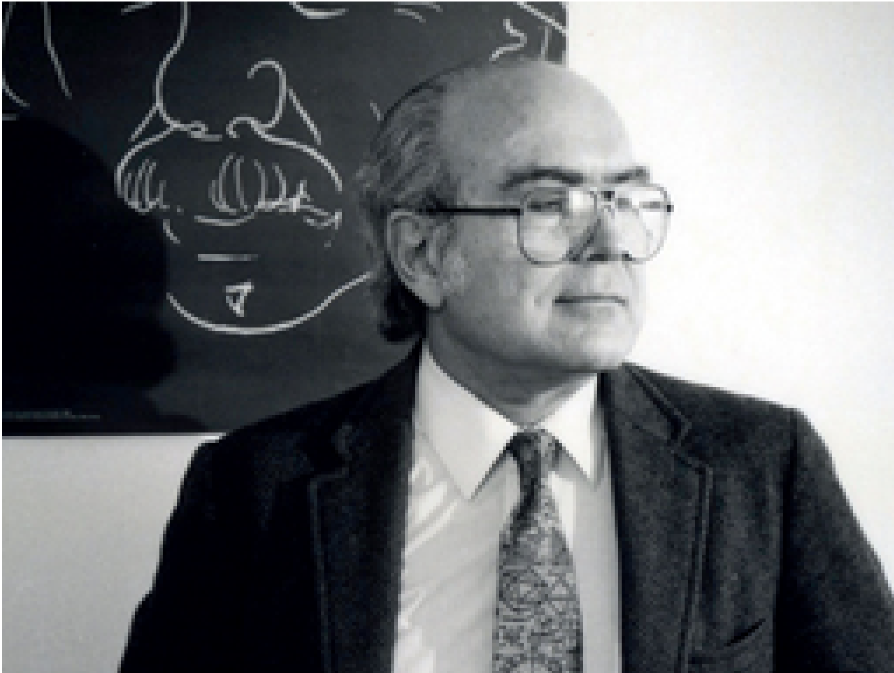
ENS Paris



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- 3) Seiberg-Witten map.
- 4) Large N gauge theories and matrix models.
- 5) The construction of gauge theories using the techniques of non-commutative geometry.

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- Define a * product
  $f * g = e^{\frac{i}{2}\frac{\partial}{x_\mu}\theta_{\mu\nu}\frac{\partial}{y_\nu}} f(x)g(y)|_{x=y}$

All computations can be viewed as expansions in θ
expansions in the external field

More efficient ways?

Quantum field theory in a space with non-commutative geometry?
BRS Symmetry?

Large N field theories

- $\phi^i(x)\ i = 1, ..., N\ ;\ N \to \infty$

  $\phi^i(x) \to \phi(\sigma, x)\ 0 \leq \sigma \leq 2\pi$

  $\sum_{i=1}^{\infty} \phi^i(x)\phi^i(x) \to \int_0^{2\pi} d\sigma(\phi(\sigma, x))^2$

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- For a Yang-Mills theory, the resulting expression is local

Gauge theories on surfaces

E.G. Floratos and J.I.

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- Given an $SU(N)$ Yang-Mills theory in a $d-$dimensional space

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- there exists a reformulation in $d{+}2$ dimensions

$$A_\mu(x) \to \mathcal{A}_\mu(x, z_1, z_2) \qquad F_{\mu\nu}(x) \to \mathcal{F}_{\mu\nu}(x, z_1, z_2)$$

with $\qquad [z_1, z_2] = \frac{2i}{N}$

$$[A_\mu(x), A_\nu(x)] \to \{\mathcal{A}_\mu(x, z_1, z_2), \mathcal{A}_\nu(x, z_1, z_2)\}_{Moyal}$$

$$[A_\mu(x), \Omega(x)] \to \{\mathcal{A}_\mu(x, z_1, z_2), \Omega(x, z_1, z_2)\}_{Moyal}$$

$$\int d^4x \ Tr\,(F_{\mu\nu}(x)F^{\mu\nu}(x)) \quad \to \quad \int d^4x dz_1 dz_2 \ \mathcal{F}_{\mu\nu}(x, z_1, z_2) * \mathcal{F}^{\mu\nu}(x, z_1, z_2)$$

These expressions are defined for *all* N!

Not necessarily integer ???

I. Large N

-A simple algebraic result:

At large N

The $SU(N)$ algebra $\rightarrow$ The algebra of the area preserving diffeomorphisms of a closed surface. (sphere or torus).

-The structure constants of $[SDiff(S^2)]$ are the limits for large N of those of $SU(N)$.

-Alternatively: For the sphere

$$x_1 = cos\phi\ sin\theta, \quad x_2 = sin\phi\ sin\theta, \quad x_3 = cos\theta$$

$$Y_{l,m}(\theta,\phi) = \sum_{\substack{i_k=1,2,3 \\ k=1,\dots,l}} \alpha^{(m)}_{i_1 \dots i_l}\ x_{i_1} \dots x_{i_l}$$

where $\alpha^{(m)}_{i_1 \dots i_l}$ is a symmetric and traceless tensor.
For fixed l there are $2l+1$ linearly independent tensors $\alpha^{(m)}_{i_1 \dots i_l}$, $m = -l, ..., l$.

Choose, inside $SU(N)$, an $SU(2)$ subgroup.

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

A basis for $SU(N)$:

$$S^{(N)}_{l,m} = \sum_{\substack{i_k=1,2,3 \\ k=1,\dots,l}} \alpha^{(m)}_{i_1 \dots i_l} \; S_{i_1} \dots S_{i_l}$$

$$[S^{(N)}_{l,m}, \; S^{(N)}_{l',m'}] = i f^{(N) l'', m''}_{l,m; \; l',m'} \; S^{(N)}_{l'',m''}$$

The three $SU(2)$ generators S_i, rescaled by a factor proportional to $1/N$, will have well-defined limits as N goes to infinity.

$$S_i \to T_i = \frac{2}{N} S_i$$
$$[T_i, T_j] = \frac{2i}{N} \epsilon_{ijk} T_k$$
$$T^2 = T_1^2 + T_2^2 + T_3^2 = 1 - \frac{1}{N^2}$$

In other words: under the norm $\|x\|^2 = Trx^2$, the limits as N goes to infinity of the generators T_i are three objects x_i which commute and are constrained by

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$\frac{N}{2i}\ [f,g] \to \quad \epsilon_{ijk}\ x_i\ \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k}$$

$$\frac{N}{2i}\ [T^{(N)}_{l,m}, T^{(N)}_{l',m'}] \to \ \{Y_{l,m}, Y_{l',m'}\}$$

$$N[A_\mu, A_\nu] \to \quad \{A_\mu(x,\theta,\phi), A_\nu(x,\theta,\phi)\}$$

II. To all orders

We can parametrise the T_i's in terms of two operators, z_1 and z_2.

$$T_+ = T_1 + iT_2 = e^{\frac{iz_1}{2}} \, (1 - z_2^2)^{\frac{1}{2}} \, e^{\frac{iz_1}{2}}$$
$$T_- = T_1 - iT_2 = e^{-\frac{iz_1}{2}} \, (1 - z_2^2)^{\frac{1}{2}} \, e^{-\frac{iz_1}{2}}$$
$$T_3 = z_2$$

If we assume that z_1 and z_2 satisfy:

$[z_1, z_2] = \frac{2i}{N}$

The T_i's satisfy the $SU(2)$ algebra.

If we assume that the T_i's satisfy the $SU(2)$ algebra, the z_i's satisfy the Heisenberg algebra

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- Question: Is there a space on which Internal symmetry transformations act as Diffeomorphisms?
- Answer: Yes, but it is a space with non-commutative geometry. A space defined by an algebra of matrix-valued functions

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- *New predictions for the B.E.H. mass?*

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- Related question: Is there a B.R.S. symmetry for this model?

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- Whether it will turn out to be convenient for us to use is still questionable.
- It will depend on our ability to simplify the mathematics sufficiently, or to master them deeply, in order to get new insights
- We need somebody with knowledge and imagination