Quantum Black Holes, Localization & Mock Modular Forms

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1. Quantum Entropy
2. Localization
3. Mock Modularity

References

- A. D., João Gomes, Sameer Murthy
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  arXiv:1012.0265
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  arXiv:1208.4078

- We give a new application of localization techniques in *gravitational* theories to reduce functional integral over string fields on $AdS_2 \times S^2$ to ordinary integrals.
- We uncover an unexpected connection with the fascinating mathematics of *mock modular forms*. It is closely related to the wall-crossing phenomenon.

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Two Related Motivations

Entropy of black holes remains one of the most important and precise clues about the microstructure of quantum gravity.

Can we compute exact quantum entropy of black holes including all corrections both microscopically and macroscopically?

Holography has emerged as one of the central concepts regarding the degrees of freedom of quantum gravity.

Can we find simple example of AdS/CFT holgraphy where we might be able to 'prove' it exactly?

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Black Hole Entropy

Bekenstein [72]; Hawking[75]

For a BPS black hole with electric charge vector Q and magnetic charge vector P, the leading Bekenestein- Hawking entropy precisely matches the logarithm of the degeneracy of the corresponding quantum microstates (*for large charges*) in accordance with the Boltzmann relation:

$$d(Q,P) \sim \exp\left[\frac{A(Q,P)}{4}\right] + \ldots \qquad (|Q|, |P| >> 1)$$

Strominger & Vafa [96]

This beautiful approximate agreement raises two important questions:

- What exact formula is this an approximation to?
- Can we systematically compute corrections to both sides of this formula, perturbatively and nonperturbatively in $1/|Q|$ and may be even exactly for arbitrary finite values of the charges?

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Why obsess with quantum black holes?

- We do not know which phase of string theory might correspond to the real world. For such a theory under construction, a useful strategy is to focus on *universal* properties that must hold in all phases. One universal requirement for a quantum theory of gravity is that in *any* phase of the theory that admits a black hole, it must be possible to interpret black hole as a statistical ensemble of quantum states.

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- Near horizon geometry of a BPS black hole is $AdS_2 \times S^2$.
- Using holography, a quantum generalization of Wald entropy is given in terms of a Wilson line expectation value

$$W(Q,P) = \left\langle \exp\left[- i\, Q_I \int_0^{2\pi} A^I\, d\theta \right] \right\rangle_{\mathrm{AdS}_2}^{\mathrm{finite}} \qquad I = 0, \ldots n_v\,.$$

This gives a precise quantum version of the equation we want to prove

$$\mathbf{d(Q,P) = W(Q,P)}$$

Our goal will be to compute both sides and compare.

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Computation of the quantum degeneracies $d(Q, P)$

In general $\mathcal{N} = 2$ compactifications this is a very hard dynamical problem and requires the computation of the number of bound states of some number of D-branes. Fortunately, for a class of states this problem has been completely solved and $d(Q, P)$ are computable explicitly:

- Half-BPS states in $\mathcal{N} = 4$ compactifications.
- One-eighth states in $\mathcal{N} = 8$ compactifications.
- Quarter-BPS states in $\mathcal{N} = 4$ compactifications.

In all cases, the degeneracies are given by Fourier coefficients of certain modular objects as we explain shortly.

- $\mathcal{N} = 4$ compactification is Heterotic on $T^6 \sim$ Type-II on $K3 \times T^2$.
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Half-BPS states in Heterotic on T^6

These are dual to purely electric states ($P = 0$). The degeneracy $d(Q)$ depends only on the duality invariant $n := Q^2/2$. And $d(n)$ are given in terms of Fourier coefficients of a modular form of weight -12:

$$
\begin{aligned}
F(\tau) &= \frac{1}{\eta^{24}(\tau)} = \sum_{n=-1}^{\infty} c(n) q^n \\
d(n) &:= c(n) = p_{24}(n+1)
\end{aligned}
$$

where $\eta(\tau)$ is the familiar Dedekind eta function

$$
\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \qquad \textit{with} \qquad q := e^{2\pi i \tau}
$$

Dabholkar & Harvey [89]

One-eighth BPS states in Type-II on T^6

Now, the degeneracy $d(Q,P)$ depends only on the U-duality invariant $\Delta = Q^2P^2 - (Q\cdot P)^2$, and $d(\Delta)$ is given in terms of the Fourier coefficients of :

$$
\begin{aligned}
A(\tau,z) &= \frac{\vartheta_1^2(\tau,z)}{\eta^6(\tau)} = \sum_{n,\ell} c(n,\ell) q^n y^\ell \\
\vartheta_1(\tau,z) &= q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)\,,
\end{aligned}
$$

with $y := e^{2\pi i z}$. The $c(n,\ell)$ depend only on $4n-\ell^2$ so we can talk about $C(4n-\ell^2)$ and the degeneracies are given by $d(\Delta) := (-1)^{\Delta+1}C(\Delta)$.

Moore, Maldacena, Strominger [99]

Jacobi Forms

The function $A(\tau, z)$ is a (weak) Jacobi form of weight -2 and index 1.
A Jacobi form is a holomorphic function $\varphi(\tau, z)$ from $\mathbb{H} \times \mathbb{C}$ to $\mathbb{C}$ which is "modular in τ and elliptic in z" in the sense that it transforms under the modular group (global diffeomorphisms) as

$$\varphi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) \;=\; (c\tau + d)^k \, e^{\frac{2\pi imcz^2}{c\tau+d}} \, \varphi(\tau, z)$$

and under the translations (global gauge transformations) of z by $\mathbb{Z}\tau + \mathbb{Z}$ as

$$\varphi(\tau, z + \lambda\tau + \mu) \;=\; e^{-2\pi im(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z) \qquad \forall \quad \lambda, \, \mu \in \mathbb{Z}\,,$$

where k is an integer and m is a positive integer.

Quarter-BPS states in Heterotic on T^6

Now, the degeneracy $d(Q, P)$ depends on the three T-duality invariants $m := P^2/2$, $n := Q^2/2$, and $\ell := Q \cdot P$, and $d(m, n, l)$ is given in terms of Fourier coefficients of a Siegel modular form of weight -10

$$\frac{1}{\Phi_{10}(\sigma, \tau, z)}$$

Now extracting the Fourier coefficients is more subtle because this function has a double pole. Understanding the consequences of this meromorphy has revealed very interesting physics and mathematics (wall-crossing, Borcherds algebras, mock modular forms) .
For our purposes, we will first expand in $p := e^{2\pi i \sigma}$ to get

$$\frac{1}{\Phi_{10}(\sigma, \tau, z)} = \sum_{-1}^{\infty} p^m \psi_m(\tau, z)$$

Quarter-BPS states in Heterotic on T^6

$$\psi_m(\tau, z) = \frac{1}{A(\tau, z)} \frac{1}{\eta^{24}(\tau)} \chi_{m+1}(\tau, z)$$

Transforms as Jacobi form of weight -10 and index m but has a pole in z.

Here $\chi_{m+1}(\tau, z)$ is the elliptic genus of symmetric product $m+1$ copies of $K3$. Its Fourier coefficients can be computed from Fourier coefficients of

$$\chi_1(\tau, z) = 2B(\tau, z) = 8\Big[\frac{\vartheta_2^2(\tau, z)}{\vartheta_2^2(0, z)} + \frac{\vartheta_3^2(\tau, z)}{\vartheta_3^2(0, z)} + \frac{\vartheta_4^2(\tau, z)}{\vartheta_4^2(0, z)}\Big]$$

Dijkgraaf, Verlinde, Verlinde [96]; Gaiotto, Yin, Strominger [06]; David, Sen[07]

We turn briefly to $W(Q, P)$ and then return to $d(Q, P)$ again.

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Computing $W(Q, P)$ and AdS_2 Functional Integral

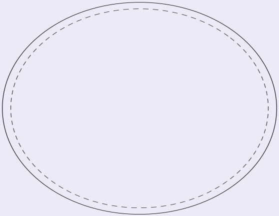

Figure: Wilson line inserted at the boundary with a cutoff at $r = r_0$.

$$ds^2 = (r^2 - 1)d\theta^2 + \frac{dr^2}{r^2 - 1} \qquad 1 \leq r < r_0$$

Localization in Supergravity

- A formal functional integral over spacetime string fields in AdS_2. One can integrate out massive fields to get a functional integral over supergravity fields. Even so, it seems almost impossible to tackle.
- *One of our main results is evaluation of a functional integral in supergravity by 'localizing' onto finite-dimensional manifold in field space. of instanton solutions*

$\mathcal{N} = 2$ supergravity coupled to n_v vector multiplets

Vector multiplet: vector field A_μ^I, complex scalar X^I, $SU(2)$ triplet of auxiliary fields Y_{ij}^I, fermions Ω_i^I. Here i in doublet.

$$\mathbf{X}^I = \left(X^I, \Omega_i^I, A_\mu^I, Y_{ij}^I\right) \qquad I = 0, \ldots, n_v\,.$$

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Localizing Instanton Solution

$$X^I = X^I_* + \frac{C^I}{r}\,, \qquad \bar{X}^I = \bar{X}^I_* + \frac{C^I}{r}$$

$$Y_1^{I1} = -Y_2^{I2} = \frac{2C^I}{r^2}\,, \qquad f^I_{\mu\nu} = 0\,.$$

Solves a major piece of the problem by identifying the off-shell field configurations onto which the functional integral localizes. This instanton is *universal* and does not depend on the physical action.

Scalar fields are very off-shell far away in field space from the classical attractor values X^I_* and *auxiliary fields* get nontrivial position dependence. Gravity multiplet not excited. Gupta & Murthy [12].

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Renormalized Action for Chiral Physical Actions

- A chiral physical action is described by a prepotential F which is a function of the scalar superfields. We substitute the above solution and can extract the finite piece.
- After a tedious algebra, one obtains a remarkably simple form for the renormalized action S_{ren} as a function of $\{C^I\}$.

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p)$$

with $\phi^I := e_*^I + 2C^I$ and $\mathcal{F}$ given by

$$\mathcal{F}(\phi, p) = -2\pi i \left[F\Big(\frac{\phi^I + ip^I}{2}\Big) - \bar{F}\Big(\frac{\phi^I - ip^I}{2}\Big) \right] ,$$

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Integration Measure

The line element on ϕ-space is

$$d\Sigma^2 = M_{IJ}\,\delta\phi^I\delta\phi^J$$

with the metric

$$M_{IJ} = K_{IJ} - \frac{1}{4}\frac{\partial K}{\partial\phi^I}\frac{\partial K}{\partial\phi^J}$$

given in terms of the Kähler potential

$$e^{-K} := -i(X^I\bar{F}_I - \bar{X}^I F_I)$$

The functional integral has collpased to an ordinary integral

$$\int \prod_{I=0}^{n_v} d\phi^I \sqrt{\det(M)}\; e^{S_{ren}(\phi)}\,.$$

For $N = 2$ chiral truncation of $N = 8$ the classical prepotential is quantum exact

$$F(X) = \frac{X^1 X^a X^b C_{ab}}{X^0} \qquad a = 1, \ldots 6 .$$

It turns out one can even evaluate the finite-dimensional integral to obtain

$$W_1(\Delta) \;=\; (-1)^{\Delta+1}\, 2\pi \Big(\frac{\pi}{\Delta}\Big)^{7/2} I_{\frac{7}{2}}\big(\pi\sqrt{\Delta}\big) \ .$$

where $\Delta = q^2 p^2 - (p \cdot q)^2$ is the U-duality invariant and

$$I_\rho(z) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\sigma}{\sigma^{\rho+1}} \exp[\sigma + \frac{z^2}{4\sigma}]$$

is the Bessel function of first kind of index ρ.

Note that the contour is parallel to imaginary axis and not real axis. Related to the analytic continuation of the conformal factor of the metric.

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Table: Comparison of the microscopic degeneracy $d(\Delta)$ with $W_1(\Delta)$ and the exponential of the Wald entropy. Note $d(\Delta)$ positive!)

Δ	3	4	7	8	11	12
$C(\Delta)$	8	-12	39	-56	152	-208
$W_1(\Delta)$	7.972	12.201	38.986	55.721	152.041	208.455
$\exp(\pi\sqrt{\Delta})$	230.765	535.492	4071.93	7228.35	33506	53252

The area of the horizon goes as $4\pi\sqrt{\Delta}$ in Planck units. Already for $\Delta = 12$ this area would be 50, and one might expect that the Wald entropy would be a good approximation. Not true! The discrepancy between the degeneracy and the exponential of the Wald entropy arises entirely from integration over massless fields.

Table: Comparison of the microscopic degeneracy $d(\Delta)$ with $W_1(\Delta)$ and the exponential of the Wald entropy. Note $d(\Delta)$ positive!)

Δ	3	4	7	8	11	12
$C(\Delta)$	8	-12	39	-56	152	-208
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The $d(\Delta)$ admits an **exact** expansion. Rademacher

$$d(\Delta) = \sum_{c=1}^{\infty} d_c$$

$$d_c(\Delta) = (-1)^{\Delta+1}\, 2\pi \Big(\frac{\pi}{\Delta}\Big)^{7/2} I_{\frac{7}{2}}\Big(\frac{\pi\sqrt{\Delta}}{c}\Big)\, \frac{1}{c^{9/2}} K_c(\Delta)\,.$$

The sum $K_c(\Delta)$ is a discrete version of the Bessel function

$$K_c(\Delta) := e^{5\pi i/4} \sum_{\substack{-c\leq d<0;\\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)}\; M(\gamma_{c,d})^{-1}_{\ell 1}\; e^{2\pi i \frac{a}{c}(-1/4)}$$

- *An* **exact** *expansion and not just an asymptotic expansion. Because of* **localization**, *it is meaningful to consider subleading exponentials.*
- It is **guaranteed** to add up to an integer but only after adding all terms and not at any finite order even though it converges very fast.

Nonperturbative contributions from orbifolds

- Consider Z_c orbifolds of the disk which implies $0 \leq \theta < 2\pi/c$. But by a coordinate trasnformation $\tilde{\theta} = c\theta$ and $\tilde{r} = r/c$ we get the same asymptotic metric
$$ds^2 \sim \tilde{r}^2 d\tilde{\theta}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}$$
Hence there are more localizing instantons but with an action reduced by a factor of c. This correctly reproduces the Bessel function with a reduced factor of c in the argument.
- If we accompany by a shift in a charge lattice then one also picks up a phase from the Wilson line exactly as in the sum K_c.

It seems possible therefore to reproduce the integer exactly.

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Caveats and Open Problems

- Show that D-terms do not contribute.
  Near horizon has enhanced supersymmetry, Murthy & Rey (10)
- Show that the orbifold phases reproduce the Kloosterman sum.
  Wilson lines on orbifolds give right structure of phases
- In general, there will be additional contributions from brane-instantons and one-loop determinants.

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On a philosophical note,

- The functional integral of quantum string theory near black hole horizons appears to have the ingredients to reproduce an integer — the bits of the branes. It appears to be an exact dual description with its own rules of computation rather than an emergent description.
- That the bulk can 'see' this integrality may be relevant for information retrieval because a necessary requirement for information retrieval is that gravity sees the 'discreteness' of quantum states.
- Conversely, we can use the data from microscopic bits to learn about the nonperturbative rules of the functional integral of quantum gravity. It is useful to have explicit answers to compare with.

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Ramanujan's example

In Ramanujan's famous last letter to Hardy in 1920, he gives 17 examples of mock theta functions, without giving any complete definition of this term. A typical example (Ramanujan's second mock theta function of "order 7" — a notion that he also does not define) is

$$\begin{aligned} \mathcal{F}_7(\tau) &= -q^{-25/168} \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q^n)\cdots(1-q^{2n-1})} \\ &= -q^{143/168}\left(1+q+q^2+2q^3+\cdots\right) . \end{aligned}$$

Hints of modularity such as Cardy behavior of Fourier coefficients but not quite modular! Despite much work, this fascinating 'hidden' modular symmetry remained mysterious until the thesis of Zwegers [2005].

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Mock Modular Form and its Shadow Zwegers[05], Zagier [07]

A mock modular form $h(\tau)$ of weight k is the first of the pair (h, g)

1. $g(\tau)$ is a modular form of weight $2-k$,
2. the sum $\widehat{h} = h + g^*$, of h is modular with weight k with

$$g^*(\tau,\bar{\tau}) = \left(\frac{i}{2\pi}\right)^{k-1}\int_{-\bar{\tau}}^{\infty}(z+\tau)^{-k}\overline{g(-\bar{z})}\,dz\,.$$

Then g is called the *shadow* of h and $\widehat{h}$ is called *modular completion* of h which obeys a 'holomorphic anomaly' equation

$$(4\pi\tau_2)^k\,\frac{\partial\widehat{h}(\tau)}{\partial\bar{\tau}} = -2\pi i\,\overline{g(\tau)}.$$

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"My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions . . . But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further."

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Physical Motivation

- The counting function for horizon degeneracies for a large class of black holes in string theory is expected to be modular from the perspective of holography. Boundary of AdS_3 is a conformal torus with an $SL(2,\mathbb{Z})$ symmetry as global diffeomorphism.
- With wall-crossing, there is an apparent loss of modularity! Disaster!!
- It is far from clear if and how such a counting function can be modular. Very hard to analyze this question in general $\mathcal{N}=2$ case.

The quarter-BPS black holes in $\mathcal{N}=4$ theories provide an ideal set-up to understand the essential conceptual issues. We develop the required analytic machinery that provides a complete answer to this question which leads naturally to the mathematics of mock modular forms.

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Mock Modular Forms and Quantum Black Holes

A. D., Sameer Murthy, Don Zagier [2012]

A summary of results

- Quantum degeneracies of single-centered black holes in $N = 4$ theories are given by Fourier coefficients of a mock modular form.
- Mock modularity is a consequence of *wall-crossing in moduli space of a spacetime index* and *noncompactness* of the microscopic SCFT.

This hidden modular symmetry is essential for two reasons

- Conceptually, for AdS_2 and AdS_3 holography.
- Practically, for developing a Rademacher type expansion.

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Meromorphic Jacobi Forms

Recall that the asymptotic counting function of quarter-BPS states in $N = 4$ theory is

$$\psi_m(\tau, z) = \frac{1}{A(\tau, z)} \, \frac{1}{\eta^{24}(\tau)} \, \chi_{m+1}(\tau, z)$$

for $m = P^2/2$ with τ and z as chemical potentials for $Q^2/2$ and $P \cdot Q$. Now A, which has a double zero is in the denominator. So we have to figure out how to extract Fourier coefficients taking into account a double pole. This innocent looking pole is physically relevant for wall-crossing and is at the heart of the connection to mock modularity.

Meromorphy, Moduli Dependence, and Wall-crossing

Meromorphy and Moduli Dependence

- This is meromorphic with a double pole at $z = 0$. Degeneracies depend on the contour.
- This is a problem because then the degeneracies are not uniquely defined. For any given contour they are not duality invariant. Moreover, they do not have any moduli dependence as is expected.
- These two problems solve each other and become features if the contour is chosen to depend on the moduli appropriately. Dabholkar, Gaiotto, Nampuri [07], Sen[07], Cheng, Verlinde [07]

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Wall-crossing and Multi-centered black holes

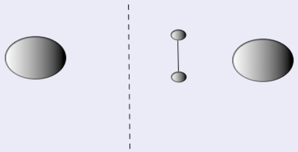

Figure: On the left of the wall there are only single-centered black holes but on the right of the wall there are both single-centered and multi-centered black holes.

Contours, Poles, and Walls

- Contour depends upon moduli.
- Pole-crossing corresponds to wall-crossing.
- Residue at the pole gives the jump in degeneracy upon wall-crossing.

How to isolate the degeneracies of single-centered black holes?

Under modular transformation $z \to z/c\tau + d$, the contour shifts. As a result, Fourier coefficients no longer have nice modular properties.

Modular symmetry is lost. How to restore modular symmetry?

A modular form is completely specified by a few invariants—its weight and first few Fourier coefficients. Without modular symmetry, life is difficult!

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Decomposition Theorem DMZ [12]

There is a unique decomposition of the counting function:

$$\psi_m(\tau, z) = \psi_m^F(\tau, z) + \psi_m^P(\tau, z)\,,$$

such that

- $\psi_m^P(\tau, z)$ has the same pole structure in z as $\psi_m(\tau, z)$:

$$\psi_m^{\mathrm{P}} \;:=\; \frac{p_{24}(m+1)}{\eta^{24}(\tau)} \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}\,y^{2ms+1}}{(1-q^s y)^2}\,,$$

- $\psi_m^F(\tau, z)$ has no poles.

The nontrivial part of the theorem is that both ψ_m^F and ψ_m^P admit modular completions $\widehat{\psi}_m^F$ and $\widehat{\psi}_m^P$ respectively.

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Mock Jacobi Form

Mock modularity and holomorphic anomaly

The completion is a mock Jacobi form ('mock modular in τ and elliptic in z'). It satisfies the 'anomaly' equation

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Mock modularity and noncompactness

For a compact SCFT, the ellptic genus counts right-moving ground states and left-moving excitations. Hence is holomorphic. *This can fail for a noncompact SCFT because of a continuum of states.*

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Physical Interpretation and AdS_2 Holography

Both pieces in the decomposition have a natural physical interpretation.

- ψ_m^{P} is the counting function of multi-centered black holes
- ψ_m^{F} is the counting function of single-centered black holes

This enables us to cleanly isolate the contribution of single-centered black holes at the *microscopic* level.

- Fourier coefficients of ψ_m^{F} are the degeneracies $d(Q, P)$ of single-centered black holes that we require for AdS_2 holography.
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Mock Modularity and AdS_3 Holography

- Modular transformations are global diffeomorphisms of boundary torus. Hence, restoring modular symmetry is essential for holography.
- Elliptic transformations are large gauge transformations of the three-form field which correspond to integer shifts of the axion field $a \to a + b$ and $Q \to Q + bP$ and hence is responsible for spectral flow symmetry or the elliptic symmetry.

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Physical Interpretation of the Modular Completion

What object can correspond to the indexed partition function of the CFT?

- It cannot possibly be ψ_m or ψ_m^P because both have wall-crossings.
- It cannot possibly be ψ_m^F because it is not modular.

We propose that it is the completion $\widehat{\psi}_m^F$ which is naturally identified with the generalized elliptic genus of the SCFT dual to a single-centered AdS_3 geometry. This microscopic picture is consistent with all symmetries, and with the macroscopic supergravity analysis of de Boer, Denef, El-Showk, Messamah, Van den Bleeken [10]. More work is needed to fully understand its implications, in particular of the holomorphic anomaly.

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Connection to Number Theory

- Our decomposition theorem was partially motivated by the notion of 'attractor contour' in black hole physics.
- Using our results, the infinite family of meromorphic black hole counting functions $\{\psi_m\}$ and another related family furnish an infinite list of examples of mock modular forms.
- This list contains most known mock modular forms including the mock theta functions of Ramanujan, the generating function of Hurwitz-Kronecker class numbers, the mock modular form conjecturally related to the Mathieu group M_{24}, *as well as an infinite number of new examples.*

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