Holographic Entanglement Entropy

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DERIVATIONS of

Holographic Entanglement Entropy

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Disclaimers!

- This talk is about derivations of holographic entanglement entropy in static situations.
- For time dependent situations, there are no derivations to date but some promising leads--wait for Gautam's talk.
- For connection between EE and thermodynamics see Mohsen's and Amir's

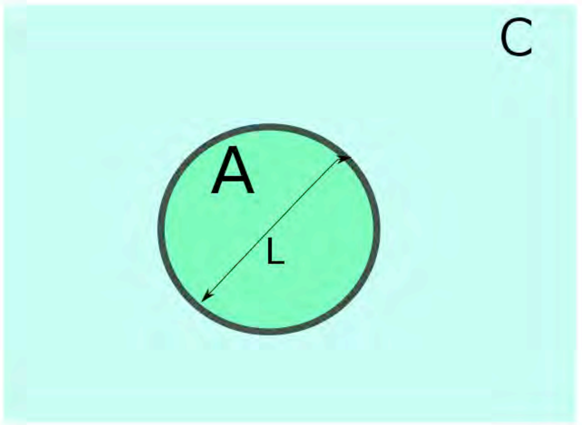
Entanglement entropy: Uses

- In classical information theory, the key concept is Shannon entropy. Consider a random variable X. The entropy of X is a measure of the amount of uncertainty in X before we learn the value.
- The quantum analog is von Neumann entropy (Entanglement entropy). It is a measure of the lack of quantum information.

- Experimental proposal to measure a close cousin Renyi entropy has been recently put forth by Abanin and Demler (2012).

- The area law followed by EE relates it to black hole entropy which was one of the original motivations to study it.

- EE is a non-local concept. In quantum many body systems it serves as an order parameter.



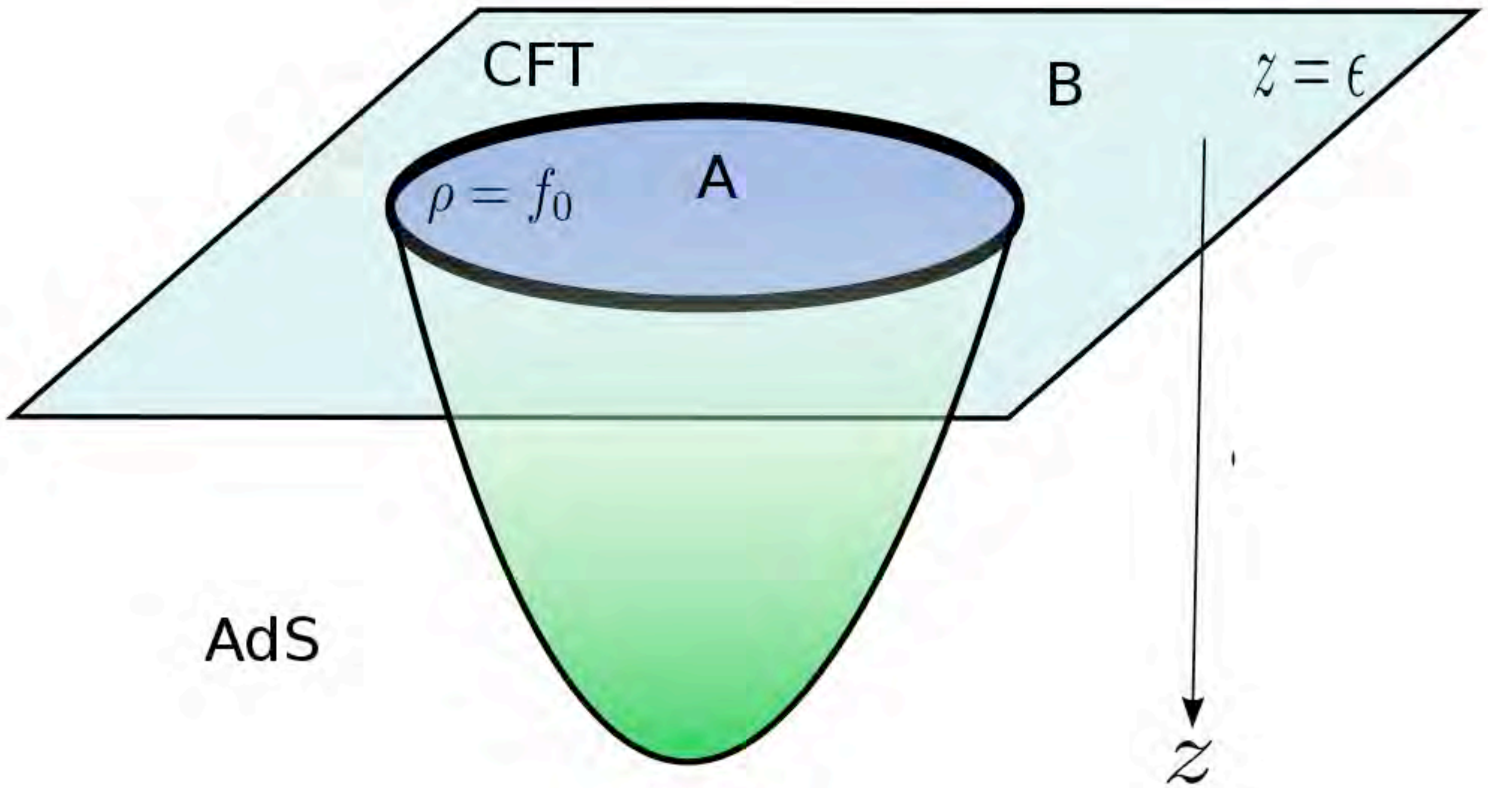
$L \to \infty$

- In gapped 2+1 d, topological order related to spectrum of anyons is given by the finite part of the entanglement entropy. [Kitaev, Preskill; Levin, Wen]

$$S = \alpha \frac{L}{\epsilon} - \gamma + \cdots$$

Universal

- EE can also be used to probe confinement. [Klebanov, Kutasov, Murugan]



- In the context of the AdS/CFT correspondence we wish to ask how much of the AdS space is needed to describe a finite region of the CFT.[Ryu, Takayanagi; Raamsdonk; Czech, Karczmarek, Nogueira, Raamsdonk; Bousso, Leichenauer, Rosenhaus; Hubeny, Rangamani]

Ryu-Takayanagi conjecture: Region enclosed by a minimal surface giving Entanglement Entropy to be $\frac{Min(\gamma_A)}{4G}$

- EE serves as a measure of the number of degrees of freedom in a CFT in even dimensions. [Ryu, Takayanagi]

- For odd dimensional CFTs it has been used to propose a generalization of the c-theorem in even dimensions. [Myers, AS]

- Using EE, Casini and Huerta have proved the c-theorem in 1+1 and 2+1 dimensions (with caveats). (However an EE proof of the Cardy-Komargodski-Schwimmer a-theorem in 3+1 is lacking).

- Consider a Hilbert space $H_{tot} = H_A \otimes H_B$
- A general state is written as

$$\psi = \sum c_{AB}|A\rangle|B\rangle \equiv \sum c_{AB}|AB\rangle$$

- A Bell state is $= \frac{1}{2}(|00\rangle + |11\rangle)$
- This is an example of an entangled state
- A useful way to check for entanglement is to compute the entanglement entropy S_{EE}

Some Basics

- For this we compute the reduced density matrix by tracing over B (or A).

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

- Then from this we calculate the von Neumann entropy $S_{EE} = -Tr(\rho_A \ln \rho_A)$

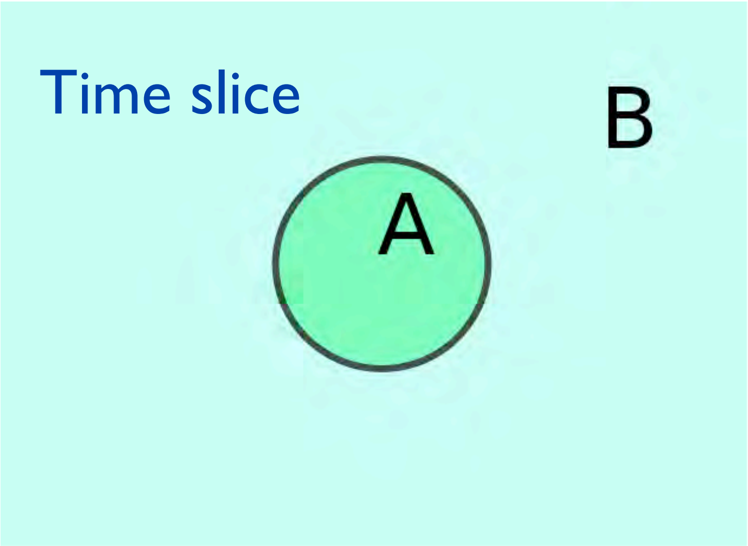
- If $S_{EE} > 0$ then the state is entangled. In this case $S_{EE} = \ln 2$

- This is an example of a maximally entangled state.

- Bipartite entanglement

$$\rho_A = Tr_B\, \rho_{tot}$$

$$S_A = -Tr_A\, \rho_A \ln \rho_A$$



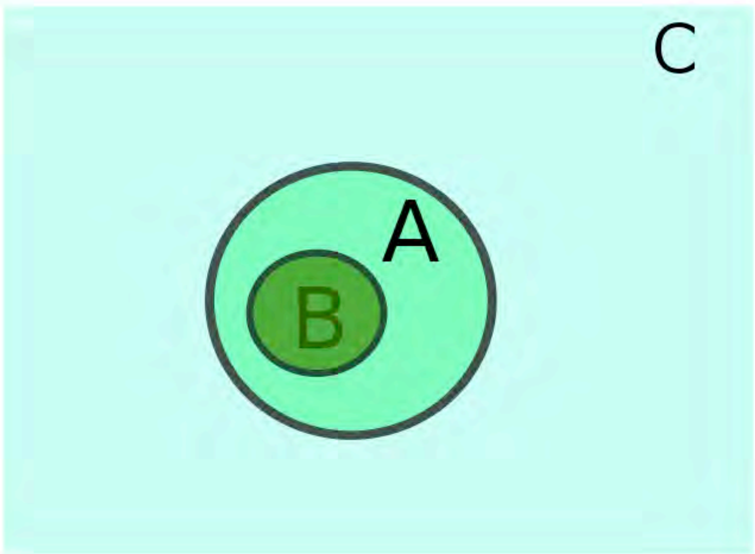
$$H_{tot} = H_A \otimes H_B$$

- If AB is pure: $S_A = S_B$

- Entanglement entropy is not extensive. The common property between A and B is the boundary. EE works out to be related to the area.

Strong subadditivity:

$$S_{B+A} + S_{A+C} \geq S_B + S_C$$



Static case

Entanglement entropy must satisfy strong subadditivity. This serves as a powerful check for any proposal .

Define:

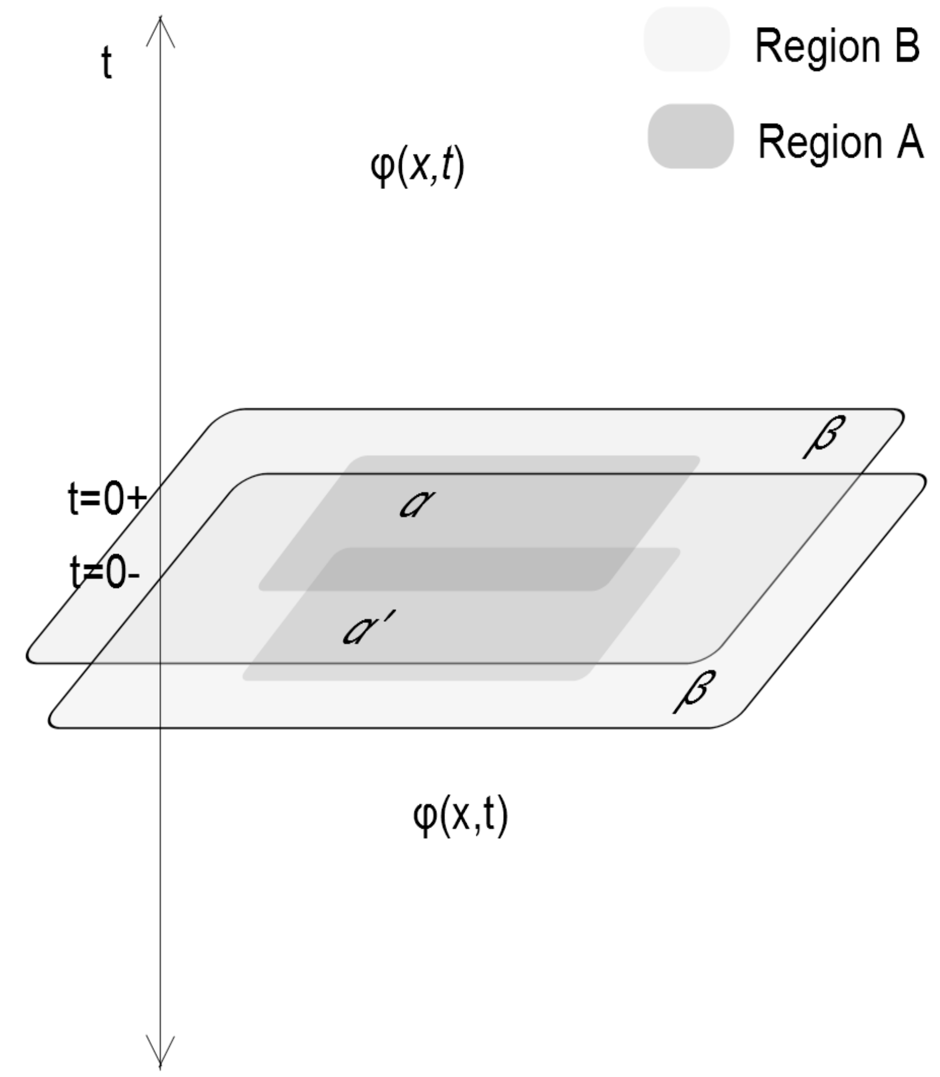
$$I_3(A,B,C) = S(ABC) - S(AB) - S(AC) - S(BA) + S(A) + S(B) + S(C)$$

In quantum field theory area law divergence has I_3=0.
This means purely pairwise correlations across entangling regions.
If $I_3 \leq 0$:monogamy.

Ways to compute EE

Even in free field theory, EE is quite hard to compute.

What is frequently used is the Replica Trick. [Callan, Wilczek; Calabrese, Cardy]



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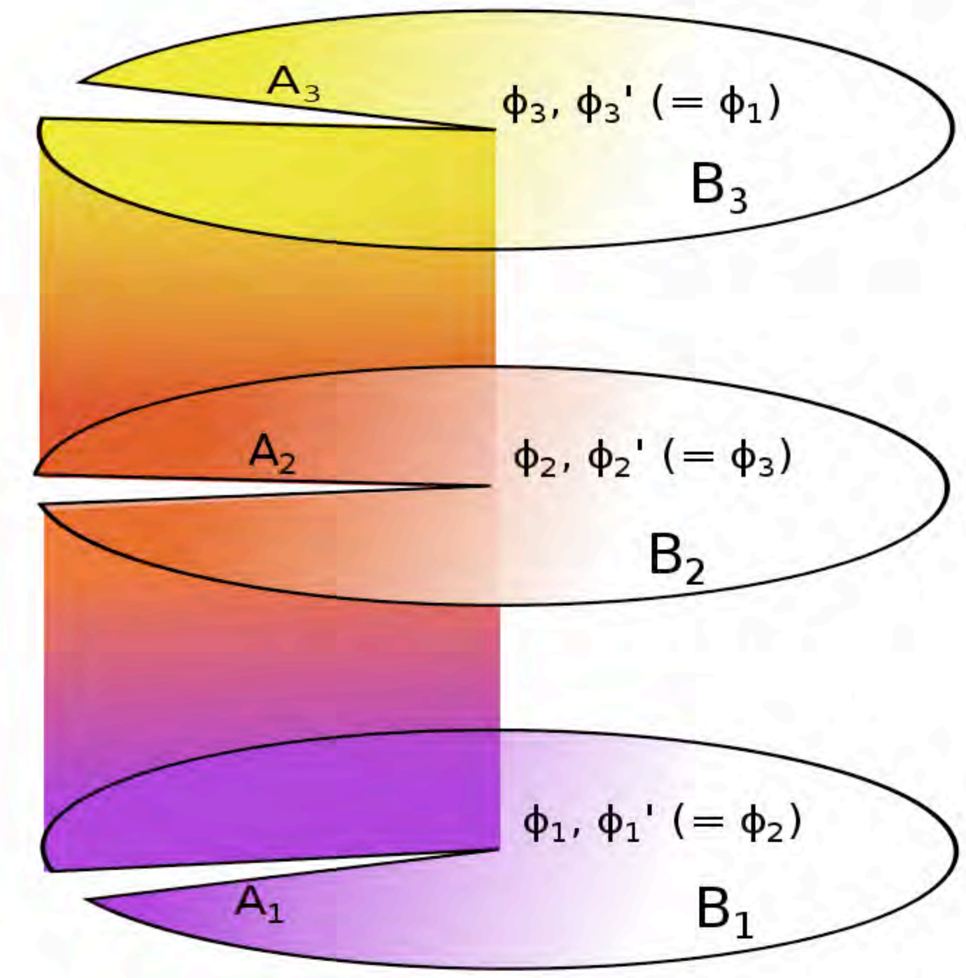

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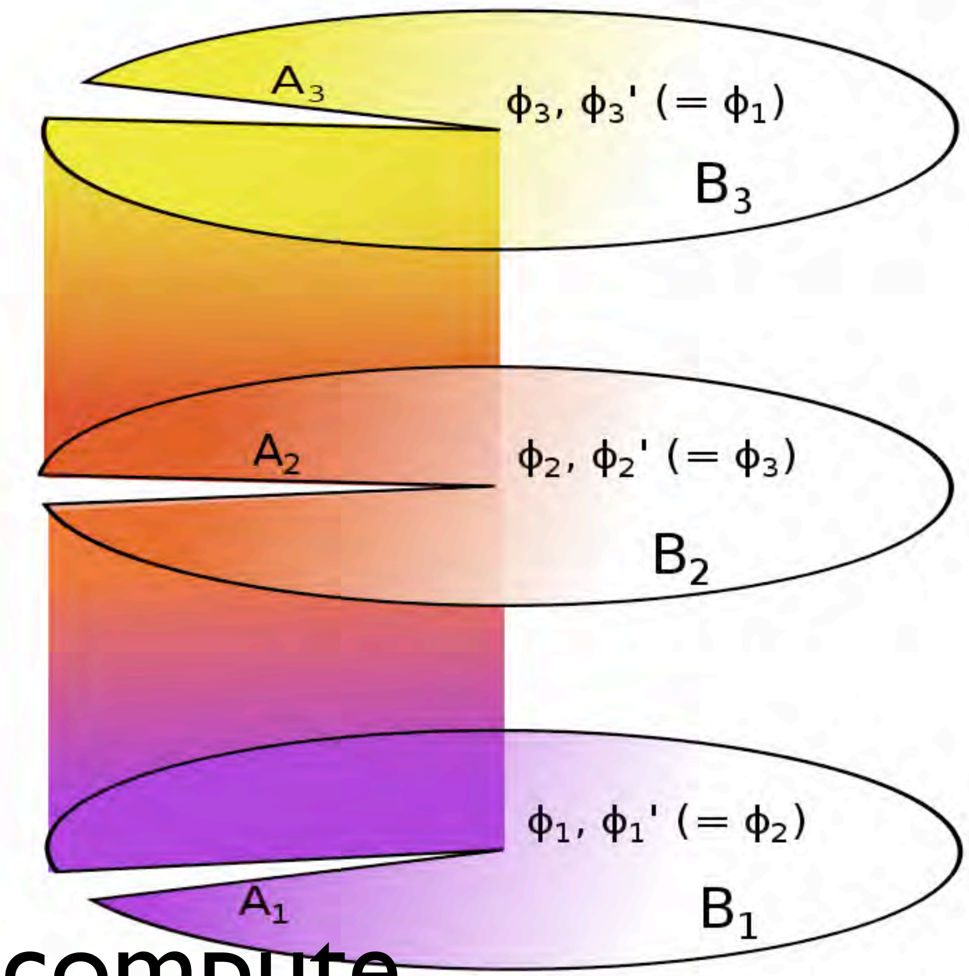
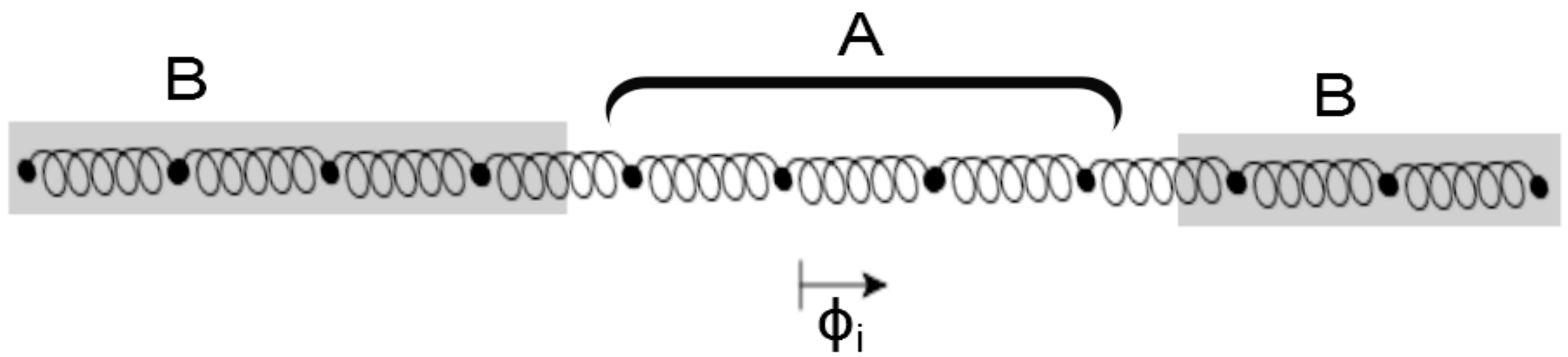


Rather than computing $Tr\rho\ln\rho$ we compute Renyi entropies $S_n = -\frac{1}{n-1}\ln Tr\rho^n$ with

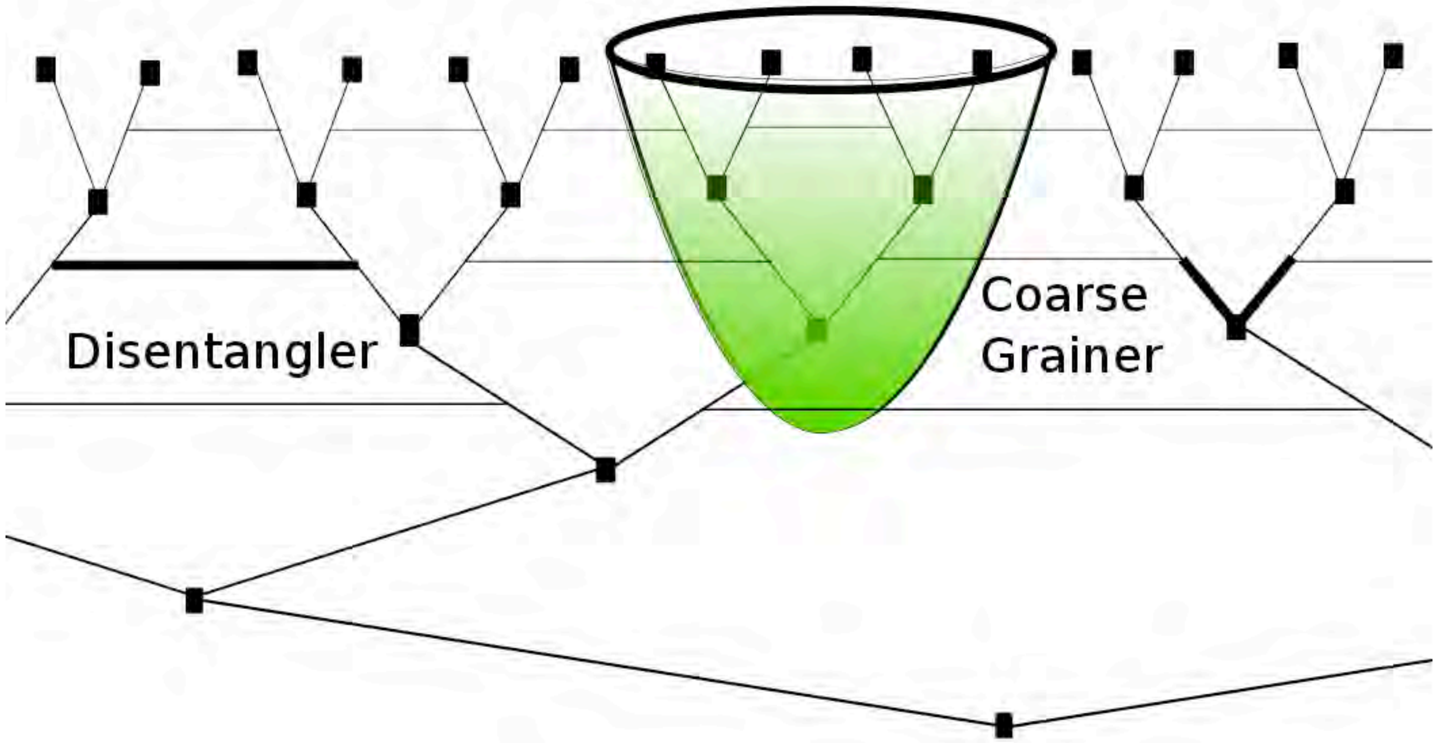
$$S_{EE} = \lim_{n\to 1} S_n$$



- Besides replica trick, there is also a more direct, but less developed real time approach (this does not introduce any conical singularity) which computes the reduced density matrix directly. [Bombelli, Koul, Lee, Sorkin]

- Better suited for calculations on the lattice.

- Can be more easily generalized; can include interactions perturbatively.



- HEE also has a bearing on cMERA (continuous Multiscale Entanglement Renormalization Ansatz)

[Vidal; Swingle]

- At each step in MERA one coarse grains using unitaries and disentanglers to make use of a computer possible for the ground state in many body systems.

Area law

[Bombelli-Koul-Lee-Sorkin 86; Srednicki 93]

- In d+1 dimensional quantum field theories we expect to have UV divergences. The leading divergent term in EE satisfies an area law. Namely

$$S_A \propto \frac{L^{d-1}}{\epsilon^{d-1}} + \cdots$$

- Due to cutoff dependence, the leading term is non-universal (except in 1+1d) while the subleading terms may be universal.

$$S_{EE} = \gamma_0 (\frac{L}{\epsilon})^{d-1} + \gamma_1 (\frac{L}{\epsilon})^{d-3} + \cdots$$

$$+c_{d-2} \log \frac{L}{\epsilon} + \cdots \quad \text{d+1:even}$$

$$+c_{d-2} + \cdots \quad \text{d+1:odd}$$

Derivation of HEE

[Casini, Huerta, Myers '11; Myers, AS '10]

- Fursaev had proposed a proof of HEE as proposed by Ryu-Takayanagi which had some problems since the conical singularities in the bulk were not accounted for properly.
- The only example where a concrete derivation exists of Holographic Entanglement Entropy is that for a spherical entangling surface.
- *This derivation does not use a minimal area prescription!*

- In holography, in a wide class of higher derivative theories, the null energy condition guarantees the existence of a c-function. [Myers, AS, '10]

- This c-function at the fixed points turns out to be related to the Wald entropy of topological black holes. This in turn was interpreted as an entanglement entropy of the dual CFT on R x S^{d-1} [Myers, AS '10] closely related to F-theorem [Jafferis, Klebanov, Pufu, Safdi '11]

- This observation was used to derive the HEE when the entangling surface is S^{d-2}. [Casini, Huerta, Myers, '11]

STEPS in derivation

- Consider CFT in d dimensions and entangling surface S^{d-2} radius R. Consider S_{EE}.
- Conformal mapping relates this to thermal entropy on R x H^{d-1} with curvature $1/R^2$ and $T=1/2\pi R$.
- Then $S_{EE} = S_{thermal}$
- Now can use holography to compute $S_{thermal}$

- Holography relates S_{thermal} to the Wald entropy for a topological black hole.

- This black hole corresponds to a hyperbolic foliation of empty AdS.

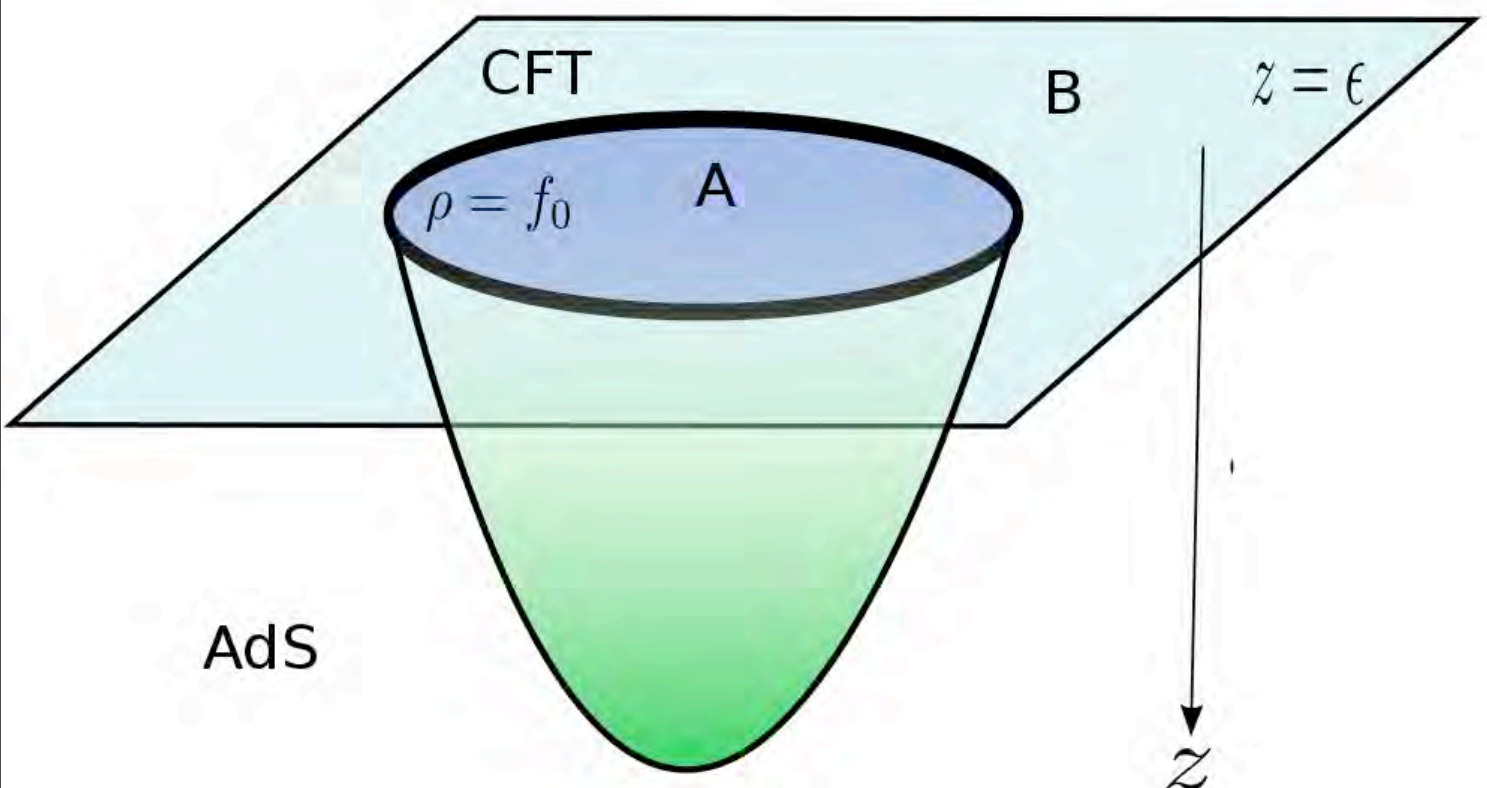
$$ds^2 = \frac{L^2}{\rho^2 - L^2} d\rho^2 - \frac{\rho^2 - L^2}{R^2} d\tau^2 + \rho^2 d\Sigma_2^{d-2}$$

- We can easily show that the resulting Wald entropy will pick out the Euler anomaly in odd dimensions [Imbimbo, Schwimmer, Thiesen, Yankelowicz; Myers, AS]

- In odd dimensions it leads to the identification of the finite part to play the role of the dofs.

- S_{EE} has UV divergences. These get mapped to IR divergences (integrating over infinite vol.) of $S_{thermal}$.
- Nowhere in this derivation have we resorted to a minimal area prescription.
- Gives the same result as Ryu-Takayanagi in Einstein gravity.
- Only works (for now) for spherical entangling surface.

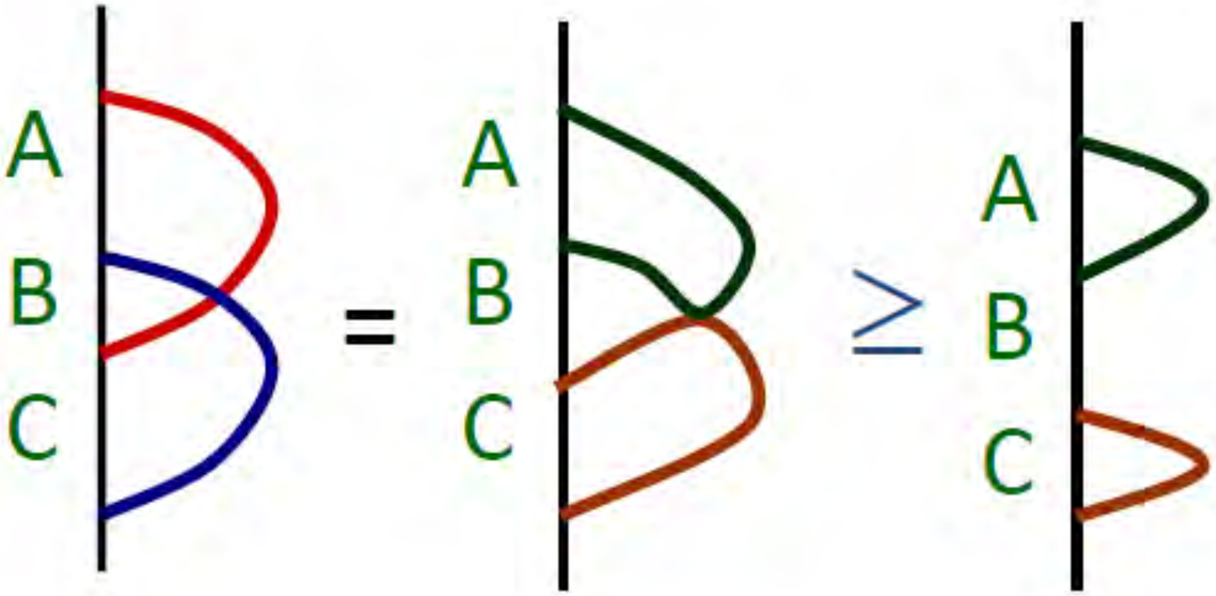
Holographic proposal



Ryu-Takayanagi '06 proposed:

$$S_A = \frac{Area(\gamma_A)}{4G_N}$$

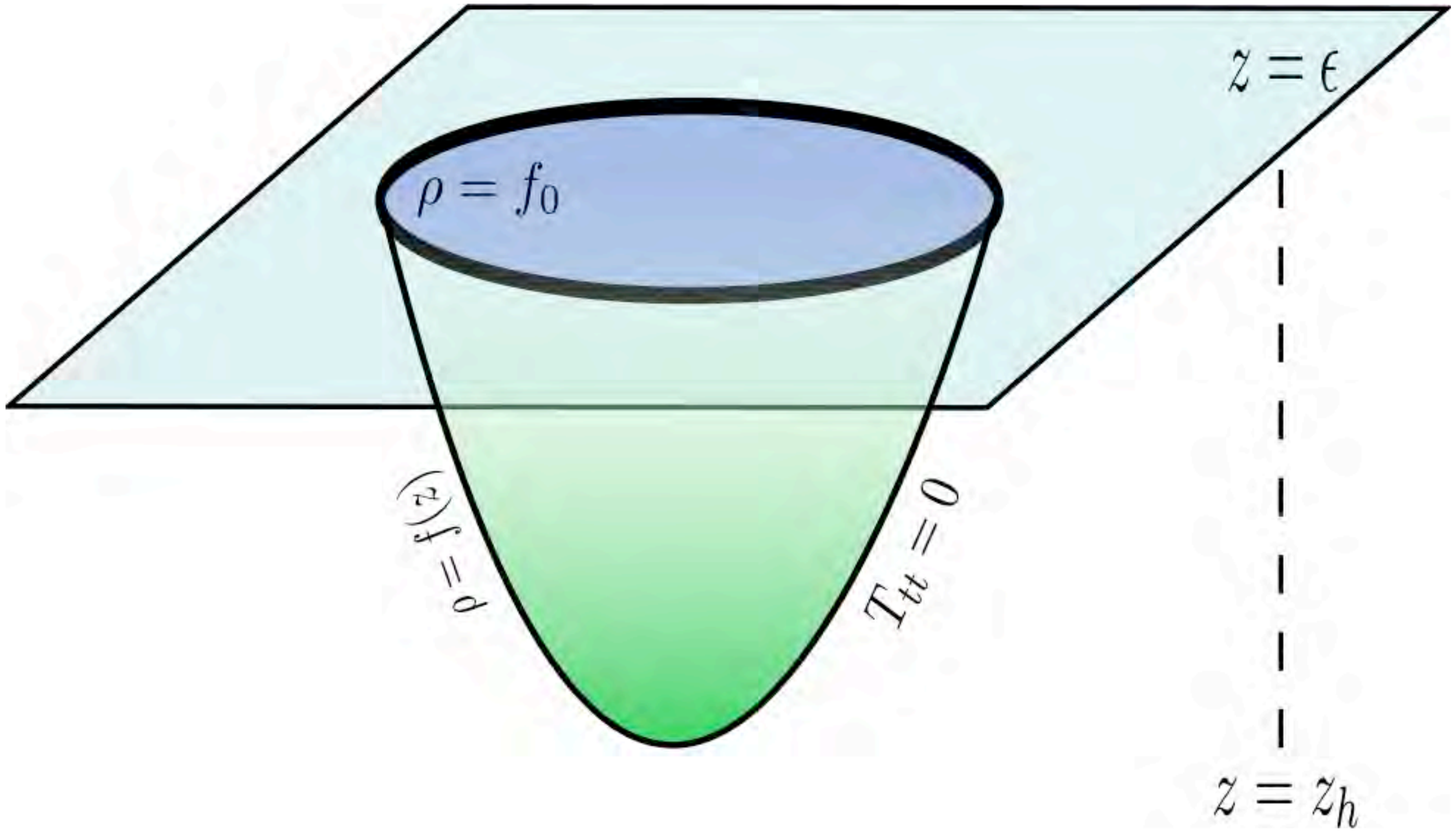
γ_A is the co-dim 2minimal surface sharing same boundary as A



$$\Rightarrow S_{A+B} + S_{B+C} \geq S_A + S_C$$

Takayanagi-Headrick proved the strong subadditivity (SSA) quite simply. The idea is that if the surface is a minimal surface then the above inequality trivially follows from the figure.

Thus the key ingredient for SSA seems to be an area functional whose minimization gives EE. It is unclear if SSA would follow otherwise. Monogamy also holds in holography. [Hayden, Headrick, Maloney]



- Let us illustrate the calculation of HEE using a spherical entangling surface. Start with AdS_5 (flat boundary)

$$ds^2 = \frac{L^2}{z^2}[dt^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) + dz^2]$$

- Now set $t = 0, \rho = f(z)$ so that we have a bulk co-dimension 2 surface.

- Plug this into the area functional

$$S_{RT} = \frac{8\pi^2 L^3}{\ell_P^3} \int dz \frac{f(z)^2 \sqrt{1 + f'(z)^2}}{z^3}$$

- Work out the eom for f(z) to find $f(z) = \sqrt{f_0^2 - z^2}$

- Plug this back into the area functional and expand around the boundary to get

universal

$$S_{RT} = \frac{4\pi^2 L^3}{\ell_P^3} \left[\frac{f_0^2}{\epsilon^2} - \ln \frac{f_0}{\epsilon} + \cdots \right]$$

- By setting $t = 0$ we lose covariance.
- The minimal area condition is the same as vanishing extrinsic curvature on the co-dimension-2 surface.
- We observe that on the co-dimension-1 surface set by $\rho = f(z)$, setting (tt component of Brown-York)

$$K^t_t - h^t_t(K^t_t +^{(3)} K) = -h^t_t{}^{(3)}K = 0$$

- gives precisely the RT minimal area condition for static surfaces. [Bhattacharrya, AS; Bhattacharrya, Kaviraj, AS]

HEE in higher derivative gravity

- The simplest guess for the area functional in a general theory of gravity would be the Wald formula [Myers, AS].

- What checks can we perform? For a 4d CFT

$$\langle T^\mu_\mu \rangle = -\frac{c}{8\pi} Weyl^2 + \frac{a}{2\pi}(Euler\ Density)$$

$$S_A = \gamma_1 \frac{L^2}{\epsilon^2} + \gamma_2 \ln \frac{L}{\epsilon} + \cdots$$

- Here [Ryu-Takayanagi; Schwimmer, Theisen; Solodukhin; Hung, Myers, Smolkin]

$$\gamma_2 = \frac{c}{2\pi}\int d^2x\,[C^{abcd}h_{ac}h_{bd} - \mathrm{Tr}K^2 + \frac{1}{2}(\mathrm{Tr}K)^2] - \frac{a}{2\pi}\int d^2x R$$

- For a spherical entangling surface $\gamma_2 \propto a$ while for a cylindrical entangling surface $\gamma_2 \propto c$

- Wald formula picks out `a' for both. Hence cannot be the correct guess.

- For Lovelock gravity (in 5d Gauss-Bonnet), there is a related entropy formula proposed first by Jacobson and Myers (differs by extrinsic curvature terms from Wald--gives correct result for black holes): [Hung, Myers, Smolkin; de Boer, Kulaxizi, Parnachev]

$$S_A = \frac{2\pi}{\ell_P^3} \int d^3x \sqrt{h}(1 + \lambda L^2 \mathcal{R})$$

- Minimizing this gives the correct values for the sphere and cylindrical entangling surface.

$$K + \lambda L^2[\mathcal{R}K - 2\mathcal{R}_{ij}K^{ij}] = 0$$

- Unfortunately no extension is known to more general theories. What are the rules??

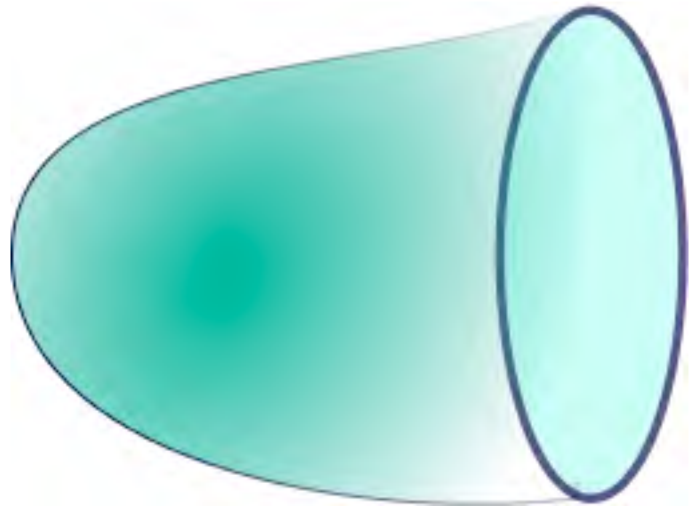
- Let us note here the tt component of the Brown-York tensor. It gives the same form for the surface near z=0. [Bhattacharyya, Kaviraj, AS]

$$K + \lambda L^2[\mathcal{R}K - 2\mathcal{R}_{ij}K^{ij} - \frac{1}{3}(K^3 - 3KK_{ij}K^{ij} - 2K_i^k K_k^l K_l^i)] = 0$$

*Absent in JM but present in Lewkowycz-Maldacena without 1/3

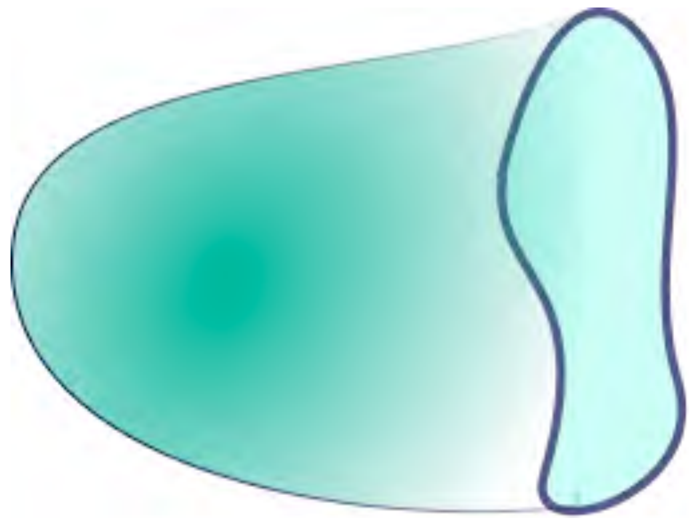
Generalized gravitational entropy

- Recently Lewkowycz and Maldacena have proposed a derivation of the Ryu-Takayanagi conjecture.
- They generalize the Gibbons-Hawking computation of entropy in Euclidean gravity.



U(1) symmetry

- Euclidean solution with U(1) symmetry is computing equilibrium thermodynamic partition function



U(1) symmetry

- The solution with no U(1) is interpreted as computing Tr ρ for an unnormalized density matrix

- Using the replica trick we can write

$$S_{EE} = -n\partial_n \left[Tr\rho^n - nTr\rho\right]|_{n=1}$$

- The idea is to interpret each term in holography.

- The last term corresponds to a solution where the boundary has a compact direction τ with periodicity 2π

- The 1st term corresponds to a solution where

$$\tau \sim \tau + 2\pi n$$

$$ds^2 = \frac{dr^2}{1+r^2} + r^2 d\tau^2 + (1+r^2)dx^2$$

- Consider the BTZ metric with U(1) symmetry with $\tau \sim \tau + 2\pi$
- Put in a scalar field with boundary conditions.

$$\phi = \eta e^{i\tau}$$

- This is not U(1) invariant.

- The n-th solution is given by

$$ds^2 = \frac{dr^2}{n^{-2} + r^2} + r^2 d\tau^2 + (n^{-2} + r^2) dx^2$$

$$\tau \sim \tau + 2n\pi$$

- In this background near $r \to 0$ the field

$$\phi \sim r^n e^{i\tau}, \phi \sim r^{-n} e^{i\tau}$$

- This scalar field can be thought to arise from a KK reduction

$$ds^2 = e^{2\phi_1} dy_1^2 + e^{2\phi_2} dy_2^2 + e^{-2\phi_1} dy_3^2 + e^{-2\phi_2} dy_4^2$$

- Thus we find that there will be singularities in the Riemann tensor components.

- Euclidean action can be shown to be finite in spite of these singularities.
- The singularities are to be viewed as harmless ones.
- The essential idea is to ascertain that the equations of motion are satisfied near the singularity. This will only happen for special surfaces.

$$ds^2 = e^{-\epsilon \log z\bar{z}}(dzd\bar{z}) + g_{ij}(dy^i + b^i_\alpha dx^\alpha)(dy^j + b^j_\alpha dx^\alpha)$$

$$g_{ij} = h_{ij} + z\,{}^zK_{ij} + \bar{z}\,{}^{\bar{z}}K_{ij}$$

- If we start with the metric we find that the zz component of the equation of motion (which has the only singularity) takes the form

$$-\frac{\epsilon}{z}\,{}^zK = 0$$

- Together with the conjugate this immediately gives us the minimal area condition of Ryu-Takayanagi.

- *Demanding that the equations of motion are non-singular gives the minimal area prescription.*

- Now that we have a method to compute the entangling surface equation, we should check what happens in higher derivative gravity theories. [Bhattacharyya, Kaviraj, AS; see also Chen, Zhang]

- Consider Gauss-Bonnet gravity

$$I = \int d^5x\sqrt{g}\left[R + \frac{12}{L^2} + \frac{\lambda}{2}L^2(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2)\right]$$

- In this case the equations near the conical singularity take the form

$$zz : K + \lambda L^2[\mathcal{R}K - 2\mathcal{R}_{ij}K^{ij} - \underline{(K^3 - 3KK_{ij}K^{ij} - 2K_i^k K_k^l K_l^i)]}$$

not there in S_{JM}

- So what went wrong?

- Turns out that there are further singularities in the equations of motion which are absent in Einstein-Gravity. Schematically

$$iz : \lambda L^2 \frac{\epsilon}{z} K \nabla K$$

$$ij : \lambda L^2 [\frac{\epsilon}{z} O(K^3) + \frac{\epsilon^2}{z^2} O(K^2)]$$

- Thus we need to be in a limit where these terms can be dropped. This will lead to

$$O(K^3) \ll O(K)$$

- Thus we will recover the same result. *The LM method needs tweaking to make it work for arbitrary extrinsic curvatures!*

- Some comments: One can with more approximations work out the surface equation in Weyl-squared.

- This takes the form: $K + \lambda_1 L^2[K\mathcal{R} - K_{ij}\mathcal{R}^{ij} + \frac{1}{3}\nabla^2 K] = 0$

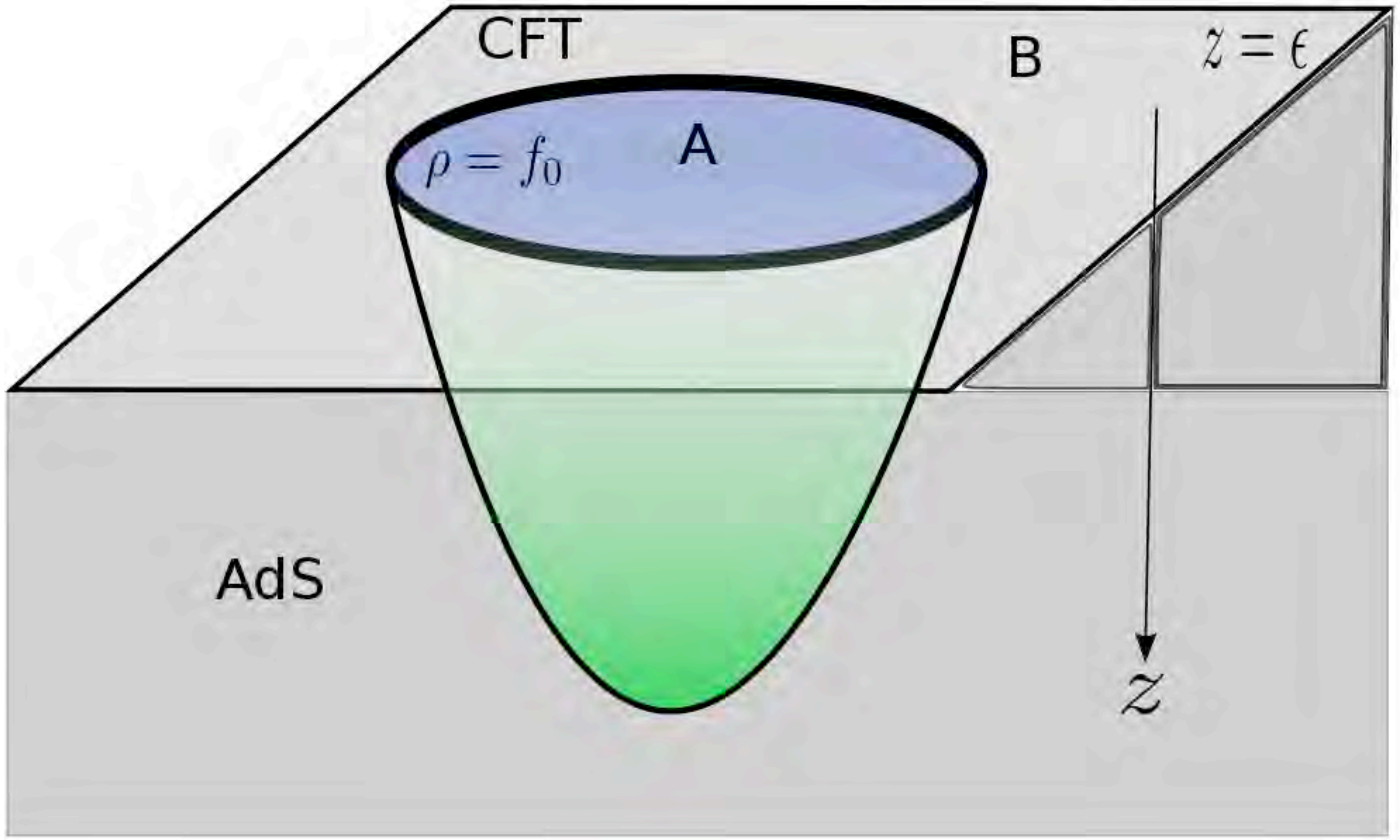
- We have not found an area functional giving this equation. It is quite likely that one does not exist.

- *It is not necessary for an area functional to exist for an arbitrary higher derivative theory of gravity. Contrast Wald formula.*

- Does the area functional have another role to play apart from giving the equation for the surface and the EE?

- ......I don't know. If we consider Euclidean path integral and integrate outside the entangling surface, then the result is divergent. But adding the area functional (also in Gauss-Bonnet) makes the result free of power law divergences and gives the universal part of EE.

- This suggests that it may be possible to derive the area functional using the Hamilton-Jacobi method.



- In AdS_5 the 3-dimensional counterterm (no time) can presumably thought of as arising from a D7-instantonic brane.

- Gibbons-Green-Perry had showed that the dual D-instanton in flat space can be interpreted as a worm-hole in string frame between 2 flat spaces.

Open problems

- Can one generalize the derivation due to Casini-Huerta-Myers to other entangling surfaces?
- Can one tweak the Maldacena-Lewkowycz method for arbitrary extrinsic curvatures?
- Afterwards generalize to time-dependent setups.
- Till then have fun keeping fingers crossed.

- Lots of online reviews: Ryu-Takayanagi; Takayanagi; Calabrese-Cardy; Casini-Huerta...
- Lots of online talks: Takayanagi; Myers; Cardy.....
- See also STRINGS 2013.

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Thank you
for listening