Lifshitz Hydrodynamics

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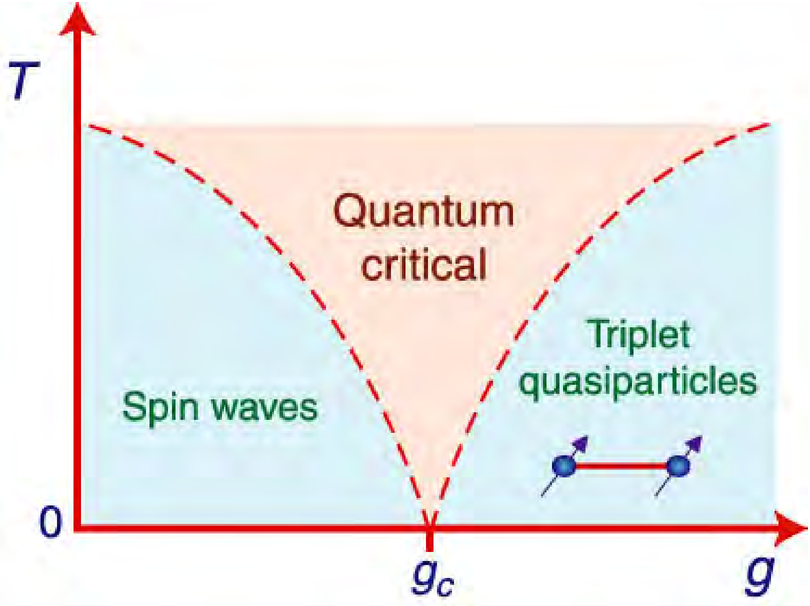
Outline

Strange Metals

- Heavy fermion compounds and other materials including high T_c superconductors have a metallic phase (dubbed as ‘strange metal’) whose properties cannot be explained within the ordinary Landau-Fermi liquid theory.
- In this phase some quantities exhibit universal behaviour such as the resistivity, which is linear in the temperature $\rho \sim T$.
- Such universal properties are believed to be the consequence of quantum criticality (Coleman:2005,Sachdev:2011).

Quantum Critical Points



Quantum Critical Points

- At the quantum critical point there is a Lifshitz scaling (Hornreich:1975,Grinstein:1981) symmetry

$$t \to \Omega^z t, \qquad x^i \to \Omega x^i, \quad i = 1, ...d \; . \tag{1}$$

- For $z = 1$ the spacetime symmetry can be enhanced to include the Lorentz group, and for $z = 2$ the Galilean group. For all other values of z, boost invariance will be explicitly broken.
- In the 'Galilean' case there is a conserved mass density and in the 'Lorentzian' case a maximal velocity.

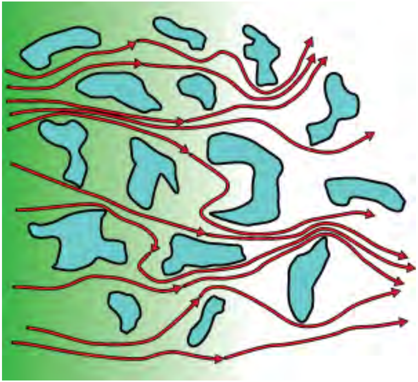
Quantum Critical Points

- Systems with ordinary critical points have a hydrodynamic description with transport coefficients whose temperature dependence is determined by the scaling at the critical point (Hohenberg:1977).
- Quantum critical systems also have a hydrodynamic description, e.g. conformal field theories at finite temperature, fermions at unitarity and graphene.
- At quantum critical regime the hydrodynamic description will be appropriate if the characteristic length of thermal fluctuations $\ell_T \sim 1/T$ is much smaller than the size of the system $L \gg \ell_T$ and both are smaller than the correlation length of quantum fluctuations $\xi \gg L \gg \ell_T$.

QCP Hydrodynamics

- Despite its obvious interest for the description of strange metals, the corresponding hydrodynamic description for quantum critical points with Lifshitz scaling has not been formulated yet.
- In contrast to the previous examples such a description should take into account the effects due to the lack of boost invariance.
- Our results are universal up to the value of the coefficients in the hydrodynamic expansion, which depend on the details of the critical point.
- The hydrodynamic expansion depends on whether the boost symmetry that is broken belongs to the Lorentz or the Galilean group. We study both cases.

QCP Hydrodynamics



New Transport Coefficient

- Our main new result is the discovery of a single new transport coefficient allowed by the absence of boost invariance. The effect of the new coefficient is a production of dissipation when the fluid is moving non-inertially.
- The result applies to any system with Lifshitz scaling, but also more generally to any system where boost invariance is explicitly broken. For instance, fluids moving through a porous medium or electrons in a dirty metal.

Conductivity

- We study the effects of the new coefficient on the conductivity of a strange metal using the Drude model and find a non-linear dependence on the electric field.
- Interestingly, we also find that scaling arguments fix the resistivity to be linear in the temperature, under the reasonable assumption that the dependence on the mass density is linear. This behaviour is universal: it is independent of the number of dimensions and the value of the dynamical exponent.

Lifshitz Symmetry

- We start considering the 'Lorentzian' case and will take the non-relativistic limit $c \to \infty$ later to obtain the 'Galilean' fluid.
- The generators of Lifshitz symmetry are time translation $P_0 = \partial_t$, spatial translations $P_i = \partial_i$, the scaling transformation $D = -zt\partial_t - x^i\partial_i$ and rotations.
- The subalgebra involving D, P_i and P_0 has commutation relations

$$[D, P_i] = P_i \ , \quad [D, P_0] = zP_0 \ . \tag{2}$$

Equation of State

- In a field theory the scaling symmetry is manifested as a Ward identity involving the components of the energy-momentum tensor

$$zT^0_0 + \delta^j{}_i T^i{}_j = 0 \ . \qquad (3)$$

- At finite temperature $T^0_0 = -\varepsilon$, $T^i{}_j = p\delta^i{}_j$, leading to the equation of state

$$z\varepsilon = dp \ . \qquad (4)$$

- This fixes the temperature dependence of energy and pressure. Taking the dimension of spatial momentum to be one, the scaling dimensions are

$$[T] = z \ , \quad [\varepsilon] = [p] = z + d \ . \qquad (5)$$

Equation of State

- The Lifshitz algebra can be generalized for constant velocities u^μ, $u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$ $(\mu, \nu = 0, 1, \cdots, d)$, with scaling dimension $[u^\mu] = 0$.
- We define the generators

$$P^{\parallel} = u^\mu \partial_\mu , \quad P^{\perp}_{\mu} = P_\mu^{\ \nu} \partial_\nu , \quad D = z x^\mu u_\mu P^{\parallel} - x^\mu P^{\perp}_{\mu} . \qquad (6)$$

Where $P_\mu^{\ \nu} = \delta_\mu^{\ \nu} + u_\mu u^\nu$. Then, the momentum operators commute among themselves and

$$[D, P^{\parallel}] = z P^{\parallel} \ , \quad [D, P^{\perp}_{\mu}] = P^{\perp}_{\mu} \ . \qquad (7)$$

- The Ward identity associated to D becomes

$$z T^{\mu}_{\nu} u_\mu u^\nu - T^{\mu}_{\nu} P_\mu^{\ \nu} = 0 \ . \qquad (8)$$

It coincides with (3) only when $z = 1$, but leads to the equation of state (4) for any velocity.

Energy-Momentum Tensor

- The conservation of the energy-momentum tensor determines the hydrodynamic equations $\partial_\mu T^{\mu\nu} = 0$.
- Lorentz symmetry forces the energy-momentum tensor to be symmetric. If boost or rotational symmetries are broken this condition can be relaxed.
- This allows many new terms in the hydrodynamic energy-momentum tensor, but as usual there are ambiguities in the definition of the hydrodynamic variables in the constitutive relations. In order to fix them, we impose the Landau frame condition

$$T^{\mu\nu} u_\nu = -\varepsilon u^\mu \; . \tag{9}$$

Energy-Momentum Tensor

- Then, the generalized form of the energy-momentum tensor is

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu + p\eta^{\mu\nu} + \pi_S^{(\mu\nu)} + \pi_A^{[\mu\nu]} + (u^\mu \pi_A^{[\nu\sigma]} + u^\nu \pi_A^{[\mu\sigma]}) u_\sigma \,. \qquad (10)$$

- The first line is the ideal part of the energy-momentum tensor, π_S contains symmetric dissipative contributions and must satisfy the constraint $\pi_S^{(\mu\nu)} u_\nu = 0$. π_A contains all possible antisymmetric terms.
- In a theory with rotational invariance $\pi_A^{[ij]} = 0$.

Entropy Current

- The new terms should be compatible with the laws of thermodynamics, in particular with the second law. Its local form in terms of the divergence of the entropy current is $\partial_\mu j_s^\mu \geq 0$.
- The divergence of the entropy current can be derived from the conservation equation

$$0 = \partial_\mu T^{\mu\nu} u_\nu = -T\partial_\mu (s u^\mu) - \pi_A^{[\mu\sigma]}(\partial_{[\mu} u_{\sigma]} - u_{[\mu} u^\alpha \partial_\alpha u_{\sigma]}) + \cdots . \tag{11}$$

- In the Landau frame we can define the entropy current as $j_s^\mu = s u^\mu$ to first dissipative order.

The Second Law

- The dots denote contributions originating in symmetric terms in the energy-momentum tensor. To first order in derivative corrections they will simply be the shear and bulk viscosity contributions, which are manifestly positive for positive values of the transport coefficients.
- The new terms are possible only if

$$\pi_A^{[\mu\nu]} = -\alpha^{\mu\nu\sigma\rho}(\partial_{[\sigma}u_{\rho]} - u_{[\sigma}u^{\alpha}\partial_{\alpha}u_{\rho]}) \, , \qquad (12)$$

where $\alpha^{\mu\nu\sigma\rho}$ contains all possible transport coefficients to first dissipative order and must satisfy the condition that, for an arbitrary real tensor $\tau_{\mu\nu}$,

$$\tau_{\mu\nu}\alpha^{\mu\nu\sigma\rho}\tau_{\sigma\rho} \geq 0 \, . \qquad (13)$$

Transport Coefficients

- If only boost invariance is broken, there is a single possible transport coefficient $\alpha \geq 0$

$$\pi_{A\,[0i]} = \alpha(\partial_{[0}u_{i]} - u_{[0}u^{\alpha}\partial_{\alpha}u_{i]})\ . \tag{14}$$

For a theory with Lifshitz symmetry the scaling dimension is $[\alpha] = d$, which determines the temperature dependence of the new transport coefficient to be

$$\alpha \sim T^{\frac{d}{z}}\ . \tag{15}$$

- There may be other new transport coefficients in a theory with more conserved charges, however for the example of a single conserved global current we did not find any to first order in derivatives.

Kubo Formula

- The first order transport coefficient is given by the Kubo formula

$$\alpha = -\lim_{\omega \to 0} \frac{1}{\omega} \frac{\delta_{ij}}{d} \mathrm{Im} \left\langle T^{[0i]} T^{0j} \right\rangle (\omega, \vec{q} = 0) \ . \tag{16}$$

- We have used that $\pi_A^{[0i]} = \langle T^{[0i]} \rangle$ and $\langle T^{[0i]} T^{jk} \rangle = 0$ by rotational invariance to derive this formula.
- It would be interesting to test this formula in a concrete model.

Non-relativistic Limit

- We now study fluids with broken Galilean boost invariance. In the relativistic fluid the maximal velocity c appears in $u^\mu = (1, \beta^i)/\sqrt{1-\beta^2}$, where $\beta^i = v^i/c$.
- In the non-relativistic limit $c \to \infty$, the pressure is not affected while the relativistic energy is expanded in terms of the mass density ρ and the internal energy U as

$$\varepsilon = c^2\rho - \frac{\rho v^2}{2} + U \,. \tag{17}$$

Non-relativistic Limit

- The relativistic hydrodynamic equations reduce to the non-relativistic form

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \ , \tag{18}$$

$$\begin{aligned}\partial_t U + \partial_i \left(U v^i \right) + p \partial_i v^i \\ = \frac{\eta}{2} \sigma^{ij} \sigma_{ij} + \frac{\zeta}{d} (\partial_i v^i)^2 + \frac{\alpha}{2} (V_A^i)^2 \ ,\end{aligned} \tag{19}$$

$$\begin{aligned}\partial_t(\rho v^i) + \partial_j(\rho v^j v^i) + \partial^i p \\ = \partial_j \left(\eta \sigma^{ij} + \frac{\zeta}{d} \delta^{ij} \partial_k v^k \right) \\ + \partial_t(\alpha V_A^i) + \partial_j \left(\frac{\alpha}{2} \left(v^j V_A^i + v^i V_A^j \right) \right) \ .\end{aligned} \tag{20}$$

- The shear tensor is $\sigma_{ij} = \partial_i v_j + \partial_j v_i - (2/d)\delta_{ij}\partial_k v^k$.

Non-relativistic Limit

- While taking the limit, we have absorbed factors of $1/c$ in the shear and bulk viscosities η and ζ and a factor $1/c^2$ in α.
- The vector V_A^i is defined as

$$V_A^i = 2D_t v^i + \omega^{ij} v_j \ , \tag{21}$$

where $D_t \equiv \partial_t + v^i \partial_i$, and $\omega_{ij} = 2\partial_{[i} v_{j]}$ is the tensor dual to the vorticity.
- The first term in V_A^i is proportional to the relative acceleration of the fluid, while the second term is proportional to an acceleration due to the Coriolis effect.
- Similarly to the viscosities, the coefficient α determines the dissipation that is produced in the fluid when the motion is not inertial.

Ward Identity

- In the non-relativistic limit with a non-zero mass density $\rho \neq 0$ the scaling symmetry needs to be modified.
- Consider a space-time diffeomorphism

$$t \to t + \xi^t \,, \quad x^i \to x^i + \xi^i \,, \tag{22}$$

  The Lifshitz equation of state is recovered if the theory has a symmetry

$$\xi^t = zt \,, \quad \xi^i = x^i + \frac{z-2}{2} v^i t \,. \tag{23}$$

- This is a combination of a scaling transformation (1) and a change of frame. When $z = 2$ the transformation is independent of the velocity and the symmetry group can be extended to include Galilean boosts and non-relativistic conformal transformations.
- The Ward identity becomes

$$-zU + dp = 0 \,. \tag{24}$$

Non-relativistic Lifshitz Scaling

- In a fluid with Lifshitz symmetry the scaling dimensions of the hydrodynamic variables are

$$[v^i] = z - 1, \quad [p] = [U] = z + d, \quad [\rho] = d + 2 - z, \quad (25)$$

while the temperature has scaling dimension $[T] = z$.

- We can determine the scaling dimensions of the transport coefficients by imposing that all the terms in the hydrodynamic equations have the same scaling. We find

$$[\eta] = [\zeta] = d \,, \quad [\alpha] = d - 2(z - 1) \,. \quad (26)$$

Drude Model of a Strange Metal

- We model the collective motion of electrons in the strange metal as a charged fluid moving through a static medium, that produces a drag on the fluid.
- We are interested in describing a steady state where the fluid has been accelerated by the electric field, increasing the current until the drag force is large enough to compensate for it.
- In order to simplify the calculation we will consider an incompressible fluid $\partial_i v^i = 0$.

Drude Model of a Strange Metal

- The fluid motion is described by the Navier-Stokes equations

$$\rho v^k \partial_k v^i + \partial^i p \tag{27}$$
$$= \rho E^i - \lambda \rho v^i + \eta \nabla^2 v^i + \frac{\alpha}{2} \partial_j \left(\left(v^j \sigma^{ik} + v^i \sigma^{jk} \right) v_k \right) \ .$$

- We have added two new terms: the force produced by the electric field E^i, and a drag term, whose coefficient λ has scaling dimension $[\lambda] = z$.
- We can solve this equation order by order in derivatives, keeping the pressure constant $\partial^i p = 0$. To leading order the current satisfies Ohm's law

$$J^i = \rho v^i \simeq \frac{\rho}{\lambda} E^i \ , \tag{28}$$

and the conductivity is simply $\boldsymbol{\sigma}_{ij} = \rho / \lambda \delta_{ij}$.

Conductivity

- At higher orders in derivatives we find the following corrections for a divergenceless electric field $E_x(y)$

$$\boldsymbol{\sigma}_{xx}(E_x) = \frac{\rho}{\lambda}\left[1 + \frac{1}{\rho\lambda E_x}\left(\eta\partial_y^2 E_x + \frac{\alpha}{6\lambda^2}\partial_y^2 E_x^3\right)\right] . \qquad (29)$$

- The conductivity depends on the electric field and its gradients. In the case where the electric field is linear $E_x = E_0 y/L$, the conductivity is simplified to

$$\boldsymbol{\sigma}_{xx} = \frac{\rho}{\lambda}\left[1 + \frac{\alpha E_0^2}{\rho L^2\lambda^3}\right] . \qquad (30)$$

- The contribution from the shear viscosity drops. This gives a way to identify the new transport coefficient α. It can be measured experimentally as an enhancement of the conductivity with the electric field.

Conductivity

- Another simple case is when the electric field takes the form $E_x = E_0 \cos(y/L)$. The contribution of α to the conductivity is y dependent

$$\boldsymbol{\sigma}_{xx}(y) = \frac{\rho}{\lambda}\left[1 - \frac{\eta}{\lambda\rho L^2} + \frac{\alpha E_0^2}{\lambda^3\rho L^2} - \frac{3\alpha E_x^2}{2\lambda^3\rho L^2}\right] . \tag{31}$$

- If we average on the y direction, we find again an enhancement of the conductivity with the electric field

$$\bar{\boldsymbol{\sigma}}_{xx} = \frac{\rho}{\lambda}\left[1 - \frac{\eta}{\lambda\rho L^2} + \frac{\alpha E_0^2}{4\lambda^3\rho L^2}\right] . \tag{32}$$

Lifshitz Scaling

- In contrast with a relativistic fluid, the density is approximately independent of the temperature. This introduces an additional scale, and in general the transport coefficients can be non-trivial functions of the ratio $\tau = T^{\frac{d+2-z}{z}}/\rho$.
- The conductivity will have the following temperature dependence

$$\sigma_{xx} = T^{\frac{d-2(z-1)}{z}}\hat{\sigma}(\tau) \simeq \frac{\rho}{T}\ , \tag{33}$$

  where we assumed a linear dependence on the density as obtained from the calculation with the drag term.
- This predicts a resistivity linear in the temperature and *independent* of the dynamical exponent and the number of dimensions.

Dissipative Effects

- Consider the effect of constant homogeneous forces on the heat production. Electric fields or temperature gradients will induce an acceleration

$$a^i = -\partial^i p/\rho + E^i = (s/\rho)\partial^i T + E^i \ . \qquad (34)$$

- We impose $\partial_t a^i = 0$, $\partial_j a^i = 0$. The Navier-Stokes equations for homogeneous configurations takes the form

$$\partial_t v^i - 2(\alpha/\rho)\partial_t^2 v^i + \lambda v^i = a^i \ . \qquad (35)$$

Dissipative Effects

- If the forces are suddenly switched on at $t = 0$, the evolution of the velocity is determined by this equation with the initial conditions $v^i(t = 0) = 0$,

$$\partial_t v^i(t = 0) = \frac{\rho a^i}{4\alpha\lambda}\left(\sqrt{\frac{8\alpha\lambda}{\rho} + 1} - 1\right) . \qquad (36)$$

- This choice is based on the physical requirement that at large times the velocity stays constant. When $\alpha \to 0$ it simply becomes $\partial_t v^i(t = 0) = a^i$.

Heat Production

- The heat production rate induced by the force is

$$\partial_t U = \lambda \rho v^2 + 2\alpha (\partial_t v)^2 \ . \tag{37}$$

- At late times the system evolves to a steady state configuration with constant velocity, so the heat production rate becomes constant $v^i = a^i/\lambda$. Subtracting this contribution for all times, the total heat produced is

$$\Delta Q = -\frac{\rho a^2}{2\lambda^2}\left(\sqrt{\frac{8\alpha\lambda}{\rho}+1}+2\right) \ . \tag{38}$$

Open Problems

- Experimental verification.
- Curvature effects and higher derivative effects.
- Holographic realization.
- Field theory calculation of the new transport coefficient.