Entanglement Entropy for Excited States

Based on

A.E.M (arXiv:1208.3204)
Amin Faraji, A.E.M. (JHEP03(2013)135,
arXiv:1301.1495)

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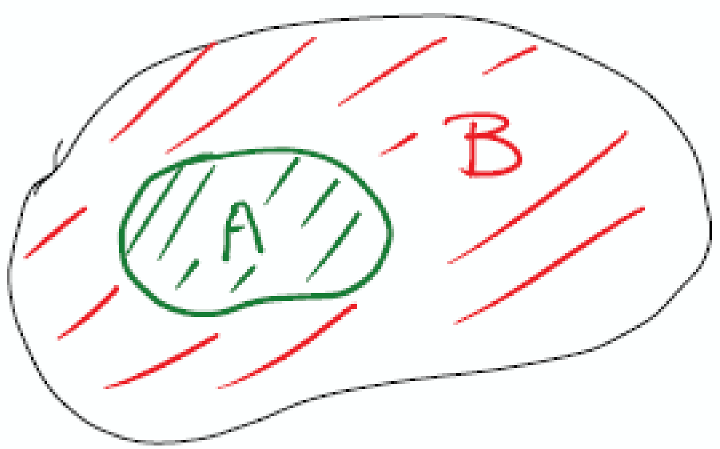
Basic Definitions

■ EE in QM

• Consider a QM'al system, divided into two complementary subsystems A and B



- The Hilbert space is divided into

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- A general state can be written as

$$|\psi\rangle = \sum_{i,a} \; C_{ia} \; |\phi_i\rangle \otimes |\chi_a\rangle$$

ϕ_i and χ_a are complete bases for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

- If the system prepared in a state $|\psi\rangle$, the density matrix will be

$$\rho = |\psi\rangle\langle\psi|$$

• The Reduced Density Operator for the subsystem A is defined as

$$\rho_A = tr_B\ \rho = \sum_{i,j,a} C_{ia} C^{\dagger}_{ja} |\phi_i\rangle\langle\phi_j|$$

• ρ_A is the density operator for an observer who has access only to the A degrees of freedom.

• Eventhough ρ may describe a <u>*Pure*</u> state, ρ_A will generically correspond to a <u>*Mixed*</u> state.

• The Entanglement Entropy of sybsystem A in the state $|\psi\rangle$ is defined as the *Von Neumann* Entropy of the reduced density operator ρ_A

$$S_A = -tr_A\{\rho_A \ \log \rho_A\}$$

• A useful quantity, *Renyi Entropy* is defined as

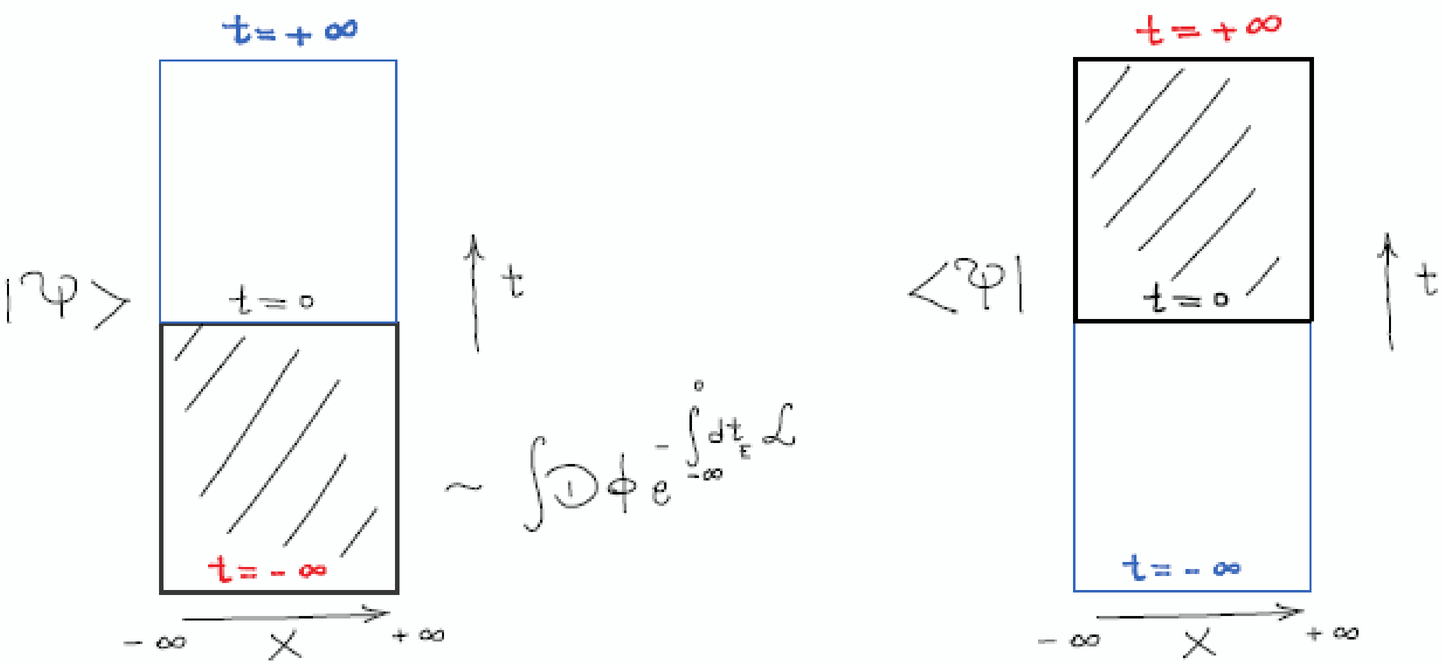
$$S_A^{(n)} = \frac{1}{1-n} \log tr_A(\rho_A)^n$$

and

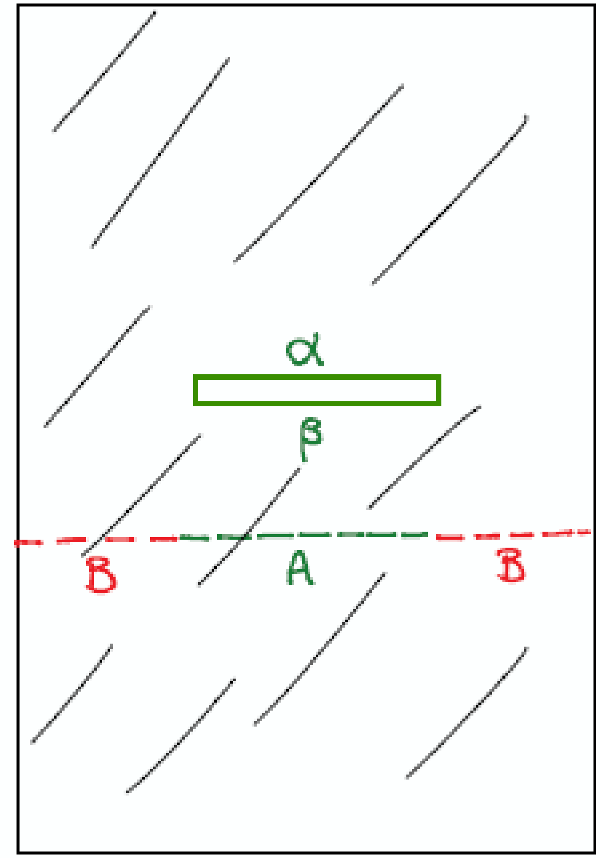
$$S_A = \lim_{n\to 1} S_A^{(n)}$$

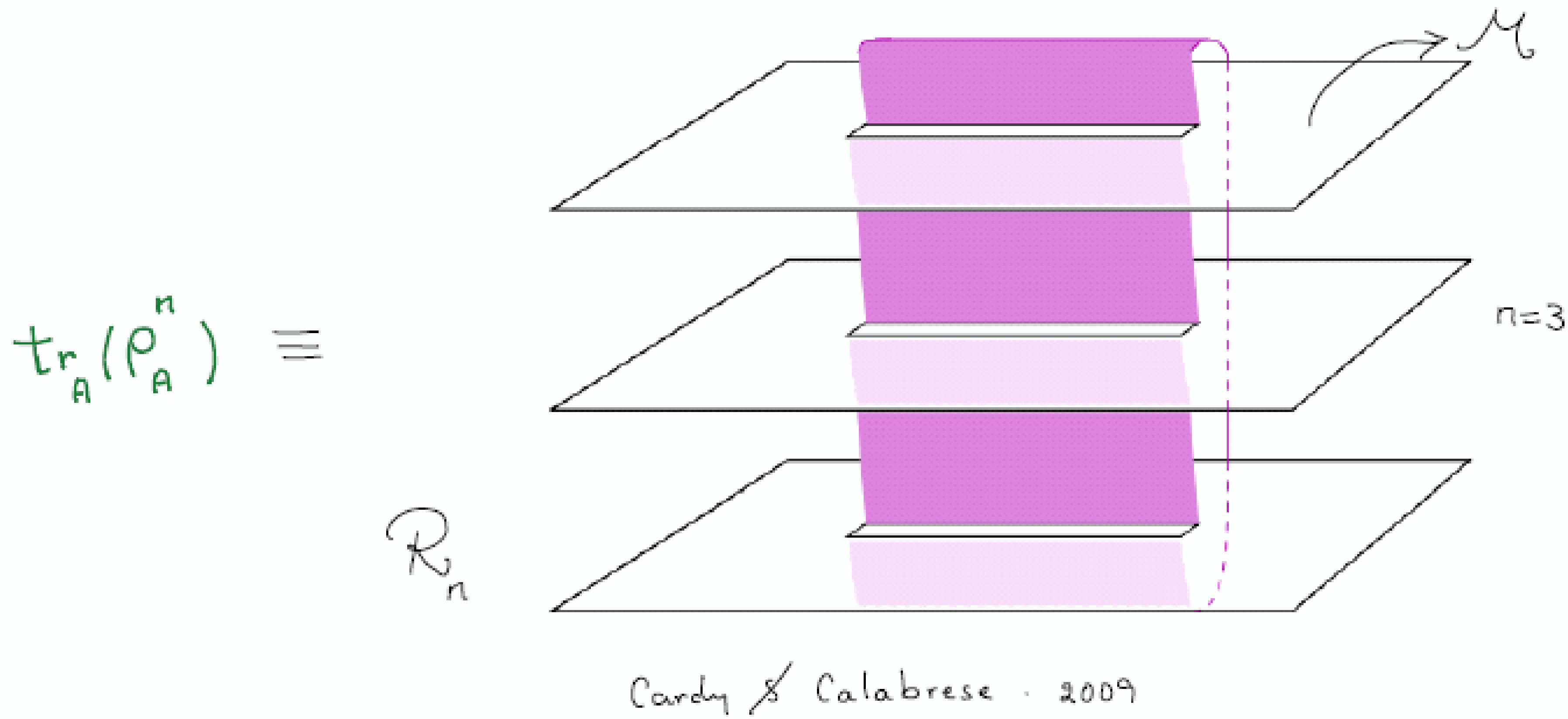
■ EE in QFT

Suppose that $|\psi\rangle$ is the Ground State of the theory. The Path Integral representation of ρ_A will be found as



$$\left[\rho_A = \mathrm{tr}_B\, |\psi\rangle\langle\psi|\right]_{\alpha\beta} \equiv$$





$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z_1^n}$$

$$Z_n \equiv Z_{\mathcal{R}_n} = \int [d\varphi(x)] \; e^{-S[\varphi]} \;, \qquad x \in \mathcal{R}_n$$

- $\mathcal{R}_n$ is a singular Riemann surface

- One can transfer the geometric complexities of $\mathcal{R}_n$ into the geometry of *Target Space* fields, $\varphi_i\ ,\ i = 1, 2, ..., n$

$$Z_{res} = \int_{res} [d^n \varphi(x)]\ e^{-S[\varphi_1, ..., \varphi_n]}\ , \quad x \in \mathcal{M}$$

where *res* stands for restrictions on the *replicated fields* and which replaces the nontrivial geometry of $\mathcal{R}_n$.

• One way of imposing the restrictions is to insert *Twist Operators* at the singular points

$$Z_{Twist} = \int [d^n\varphi(x)] \; e^{-S[\varphi_1,\dots,\varphi_n]} \; \prod \sigma_k \dots\dots \; , \quad x \in \mathcal{M}$$

• An alternative way is to move over to the *Covering Space* of the fields, $\mathcal{M}_C$ and calculate

$$Z_{\mathcal{M}_C} = \int [d\varphi(x)] \; e^{-S[\varphi]} \; , \quad x \in \mathcal{M}_C$$

The complexities of $\mathcal{R}_n$ are now encoded in the transformation

$$\mathcal{R}_n \to \mathcal{M}n$$

Entanglement Entropy for subsystems provide useful information specially when calculated for *Pure States*.

(For mixed states statistical entropy is also nonzero and may be subtracted by definition of other quantities such as mutual informayion...)

The *Pure States* can be the *Ground State* of the theory or an *Excited State*.

• We are interested in the Entanglement Entropy for Excited States, a much less studied case.

• Of particular interest is to find whether there exist any universal features in this case.

In the following we will be interested in the

Entanglement Entropy for Excited States in a Two Dimensional CFT on a circle (Line) with a Single Interval as the Entangling Surface

This problem was first studied *Analytically* in Alcaraz, Berganza, Sierra 2011, using the approach of Holzhey-Larsen-Wilczek ('94) and later by same authors using Calabrese-Cardy ('04) methods.

In the following we first derive the same results by using *Symmetric Orbifolds* (Lunin-Mathur (2000)), and then by *Holography*.

Outline of Symmetric Orbifolding

- We start with a theory on sphere, parametrized by $(z, \bar{z})$, with a flat metric and with two branch points of order n.

- By a coordinate transformation to $(w(z), \bar{w}(\bar{z}))$, which behaves as $w \approx z^{1/n}$ at branch points, one moves over to the covering sphere.

- By a Weyl transformation with a conformal factor $|\frac{dw}{dz}|^2$, one ends up with a third sphere with a fiducial metric $d\hat{s}^2$ which we have chosen to be flat.

• The partition functions on the first and third spheres are related by

$$Z = e^{S_L} \ \hat{Z}$$

where

$$S_L = \frac{c}{24\pi} \int dt^2 \sqrt{g}[\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + R\phi]$$

is the *Liouville* action and

$$e^{2\phi} = |\frac{dz}{dw}|^2$$

■ EE for Primary Excitations ■

• On z-sphere we put the branch points at

$$u = ae^{\frac{i}{2}(\pi+\theta)} \quad , \quad v = ae^{\frac{i}{2}(\pi-\theta)}$$

• To excite the theory to a highest weight state we create asymptotic *in* and *out* states by inserting the *primary* operator $\mathcal{O}$ at $z = \bar{z} = 0$ and

$$\tilde{\mathcal{O}}(\tilde{z}, \bar{\tilde{z}}) = \mathcal{O}(z, \bar{z}) z^{2h} \bar{z}^{2\bar{h}} \; \delta^{2(h+\bar{h})}$$

at $\tilde{z} = \bar{\tilde{z}} = 0$.

• The quantity of interest is the restricted path integral of the replicated theory on the z-sphere in presence of the insertions

$$tr\rho^n_{\mathcal{O}}(\theta) \equiv \frac{\int_{res}[d^n\varphi]e^{-S[\varphi_1,\ldots,\varphi_n]}\prod_{i=1}^n \mathcal{O}_i(0)\tilde{\mathcal{O}}_i(\infty)}{\left[\int[d\varphi]e^{-S[\varphi]}\mathcal{O}(0)\tilde{\mathcal{O}}(\infty)\right]^n}$$

• We now write everything in terms of quantities of the unreplicated theory on the z-sphere and the theory on the smooth, flat, w-sphere.

• w-sphere is found by the map

$$\frac{z-u}{z-v} = \frac{1}{1-(\frac{w-1}{w+1})^n}$$

• Putting everything together we find

$$tr\rho_{\mathcal{O}}^{n}(\theta) = e^{S_L}\frac{\widehat{Z}_w}{Z_z^n}\ \mathcal{T}\ \frac{\langle\prod_{k=0}^{n-1}\mathcal{O}(w_k)\tilde{\mathcal{O}}(w_k')\rangle_w}{\langle\mathcal{O}(0)\tilde{\mathcal{O}}(\infty)\rangle_z^n}$$

• The factor $\mathcal{T}$ comes from the transformation properties of the operators under the sequence

$$z \to w \text{ and } ds^2 \to d\hat{s}^2 = |\frac{dw}{dz}|^2 ds^2 \equiv e^{-2\phi}ds^2$$

• The sequence of transformations corresponds to a *conformal* transformation under which

$$\mathcal{O}(z,\bar{z}) = \left(\frac{dw}{dz}\right)^h \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}} \mathcal{O}(w,\bar{w})$$

- It turns out that excitations do not alter S_L. This is understood by a careful study of the Liouville field near the insertion points.

- The non-trivial effects come from the factor $\mathcal{T}$

$$\mathcal{T} = \left(\frac{d\tilde{z}}{d\tilde{w}}\bigg|_{w'_0} \times \prod_{k=1}^{n-1} \frac{d\tilde{z}}{dw}\bigg|_{w'_k} \times \prod_{k=0}^{n-1} \frac{dz}{dw}\bigg|_{w_k} \right)^{-h}$$

- One finds

$$\frac{tr\rho_{\mathcal{O}}^n(\theta)}{tr\rho^n(\theta)} \equiv \mathcal{F}_{\mathcal{O}}^{(n)}(\theta) = \frac{n^{-2n(h+\bar{h})} \langle \prod_{k=0}^{n-1} \mathcal{O}(\frac{\theta+2\pi k}{n}) \tilde{\mathcal{O}}(\frac{2\pi k}{n}) \rangle_{cy}}{\langle \mathcal{O}(\theta) \tilde{\mathcal{O}}(0) \rangle_{cy}^n}$$

- Recalling that

$$\mathcal{O}(w,\bar{w})\tilde{\mathcal{O}}(0,0) = \frac{1}{w^{2h}\bar{w}^{2\bar{h}}}[1 + \mathcal{Q}_{\Delta,\bar{\Delta}} w^{\Delta}\bar{w}^{\bar{\Delta}} + \ldots]$$

we find in the limit of $\theta \ll 2\pi$

$$\mathcal{F}^{(n)}_{\mathcal{O}}(\theta) = 1 + \frac{h+\bar{h}}{3}\Big(\frac{1}{n} - n\Big)(\frac{\theta}{2})^2 + O(\theta^{(\Delta+\bar{\Delta})})$$

- The *excess* of *Entanglement Entropy* will be

$$S_{\mathcal{O}}(\theta) - S_{GS}(\theta) = \frac{\partial}{\partial_n}\mathcal{F}^{(n)}_{\mathcal{O}}(\theta)|_{n=1}$$

■ EE for Excitations by Holography ■

• The objective is to find a gravitational (AdS_3) background that has the singular Riemann surface at the boundary.

• The gravitational on-shell action will give the partition function of the replicated theory by AdS/CFT.

• This can be done explicitly (Hung-Myers-Smolkin-Yale, '12).

• Alternatively, since in two dimensions all metrics are conformally flat, any non-trivial effect can be encoded in the conformal factor and hence in the shape of the regulator surface in the bulk.

• We take the latter route, i.e., assume a flat smooth boundary, but a non-trivial regulator surface.

• The regulator surface, when stated in the Fefferman-Graham coordinates, corresponds to the z-sphere.

• The regulator surface, when stated in the Poincare coordinates, corresponds to the w-sphere with ds^2.

• The regulator surface at a constant Poincare radius corresponds to the w-sphere after Weyl scaling and thus with $d\hat{s}^2 = e^{-2\phi} ds^2$.

• To account for the excitations, we turn on fields in the bulk with appropriate masses and boundary conditions.

• We use the Holographic Renormalization method to calculate the gravitational on-shell action and subsequently the field theory correlation functions.

- FG coordinates

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(\rho,z,\bar{z})}{\rho} dx^i dx^j \ , \quad i,j = 1,2$$

where

$$g_{ij}(\rho,z,\bar{z}) = g_{(0)_{ij}} + \rho g_{(2)_{ij}} + \cdots \ , \quad g_{(0)_{ij}} dx^i dx^j = dz d\bar{z}$$

- Make the following transformation (Krasnov, '03)

$$r = \frac{\rho^{e-\phi}}{1+\rho e^{-2\phi}|\partial_y \phi|^2} \quad , \quad w = y + \partial_{\bar{y}} \phi \frac{\rho e^{-2\phi}}{1+\rho e^{-2\phi}|\partial_y \phi|^2}$$

with

$$y \equiv \left(\frac{z-u}{z-v}\right)^{\frac{1}{n}}, \quad e^{\phi} = \frac{n}{l} \ |z-u|^{(1-1/n)} \ |z-v|^{(1+1/n)} = |\frac{dz}{dy}|^2$$

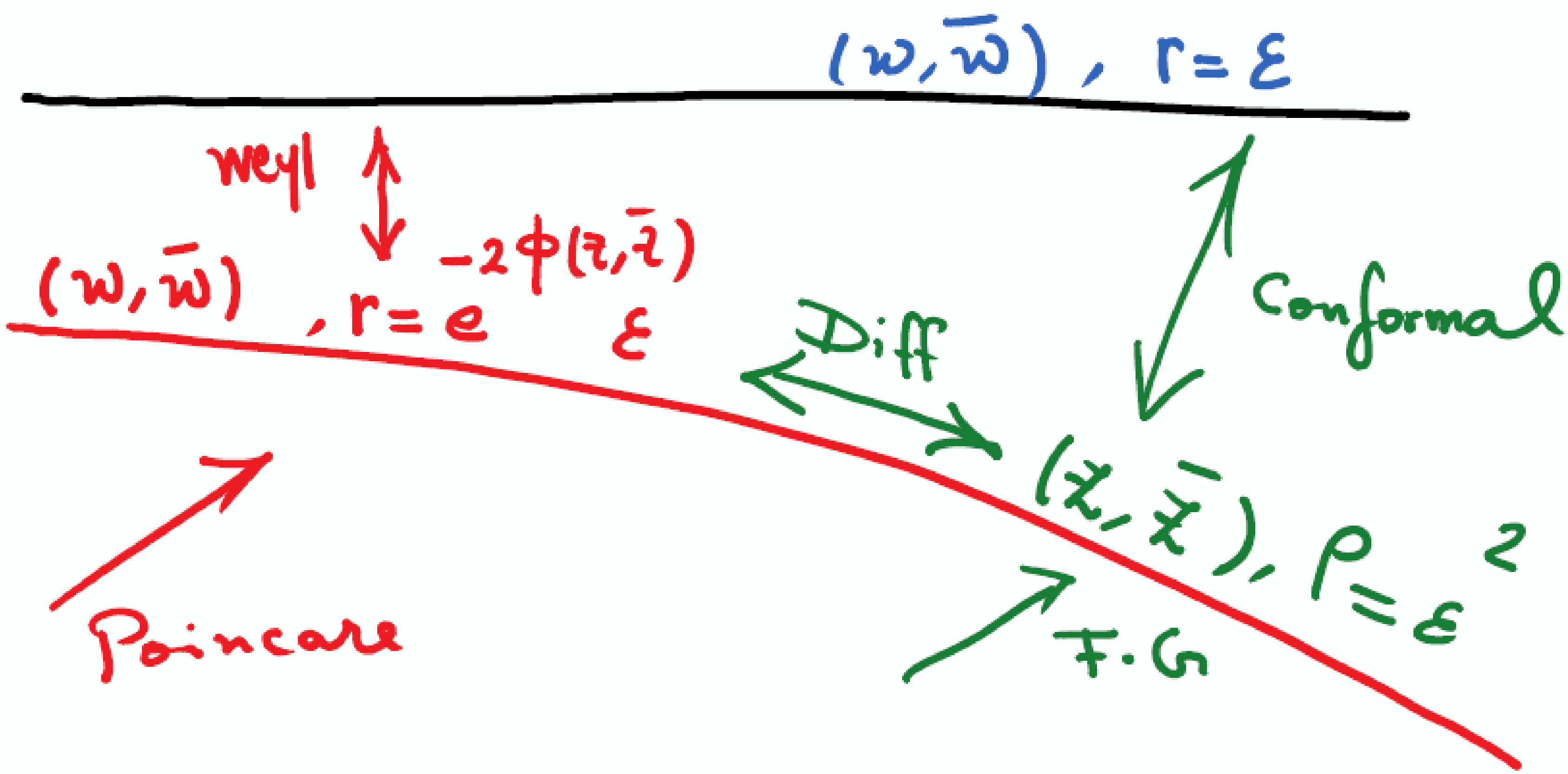
• This takes us to the Poinare coordinates

$$ds^2 = \frac{1}{r^2}(dr^2 + dw d\bar{w})$$

• The key point is that this transformation takes the surface $\rho = \epsilon^2 \ll 1$ in the FG coordinates to $r = \epsilon e^{-2\phi}$ in the Poincare coordinates with $e^{2\phi} = |dz/dw|^2$.

• The punch line is that the induced metric on the latter surface will be that on the w-sphere.

• The surface $r = \epsilon$, on the other hand, will have a flat induced metric and corresponds to the w-sphere with the Weyl rescaled metric, $d\hat{s}^2$.



(w, w̄) , r= ε
weyl
(w, w̄) , r= e^{-2φ(z, z̄)} ε
Diff
Conformal
(z, z̄), ρ= ε^2
F.G
Poincare

Outline of Calculations

- Action

$$S = S_g + S_m \ , \quad S_m = \int d^3x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2)$$

- We take Phi to be a scalar field

$$\Phi(r, w, \bar{w}) = \Phi'(\rho, z, \bar{z})$$

with the asymptotic expansion

$$\Phi(r, w, \bar{w}) = r^{2-\Delta} \phi(r, w, \bar{w}) \ ,$$
$$\phi(r, w, \bar{w}) = \phi_0(w, \bar{w}) + r^2 \phi_2(w, \bar{w}) + \cdots$$

with

$$m^2 = \Delta(2 - \Delta)$$

• One needs to first *regularize* the action and then add *counter terms* to it to find the *subtracted* action

$$S_{sub} = \int_{\mathcal{M}} \sqrt{\gamma} dw d\bar{w} \ \Phi[-\frac{r}{2}\partial_r \Phi + \frac{2-\Delta}{2}\Phi + \frac{1}{2(\Delta-2)}\Box_\gamma \Phi]$$

• The exact one point function is found as

$$\langle \mathcal{O}(w,\bar{w}) \rangle = \lim_{\epsilon\to 0} \left[\frac{1}{r^\Delta \sqrt{\gamma}} \frac{\delta S_{sub}}{\delta \Phi} \right]_{\mathcal{M}} = (2-2\Delta)\phi_{(2\Delta-2)}$$

Results

- We find the final result as

$$\langle \mathcal{O}'(z,\bar{z})\rangle = e^{-\phi\Delta}\langle \mathcal{O}(w,\bar{w})\rangle_{d\hat{s}^2}$$

which is equivalent to a scaling of the *external source* by

$$\phi_0 \to e^{\phi\Delta}\phi_0$$

• The n-point functions are thus obtained by

$$\prod_{i=1}^{n} \frac{\delta}{\delta\phi_0(w_i,\bar{w}_i)} \to \prod_{i=1}^{n} e^{-\phi(w_i,\bar{w}_i)\Delta} \frac{\delta}{\delta\phi_0(w_i,\bar{w}_i)}$$

• Plugging in the values of w_i for the insertion points and after several simplifications, we finally arrive at

$$\frac{tr\rho_{\mathcal{O}}^{n}(\theta)}{tr\rho^{n}(\theta)} \equiv \mathcal{F}_{\mathcal{O}}^{(n)}(\theta) = \frac{n^{-2n(h+\bar{h})}\langle\prod_{k=0}^{n-1}\mathcal{O}(\frac{\theta+2\pi k}{n})\tilde{\mathcal{O}}(\frac{2\pi k}{n})\rangle_{cy}}{\langle\mathcal{O}(\theta)\tilde{\mathcal{O}}(0)\rangle_{cy}^{n}}$$

■ Conclusions, Outlook

- EE for primary excitations are studied, by

symmetric orbifolding as well as by holography.

- Extensions for finite temperature and higher dimensions.

- Applications to thermodynamic properties of EE.

- ....