Holographic entanglement entropy and Entanglement thermodynamics

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Based on

D. Allahbakhshi, M. A. and A. Naseh, "Entanglement Thermodynamics," arXiv:1305.2728 .

D. Allahbakhshi and M. A, "Probing Fractionalized Charges," arXiv:1301.4815.

see also

J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, "Thermodynamical Property of Entanglement Entropy for Excited States," arXiv:1212.1164

D. D. Blanco, H. Casini, L. -Y. Hung and R. C. Myers, "Relative Entropy and Holography," arXiv:1305.3182

Plan of the talk

- Review of (holographic) entanglement entropy
- First law of entanglement thermodynamics
- Other laws
- Charged system
- Summary

Entanglement entropy

Consider a state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, which evolves in time by its Hamiltonian H

Physical quantities are computed as expectation values of operators as follows

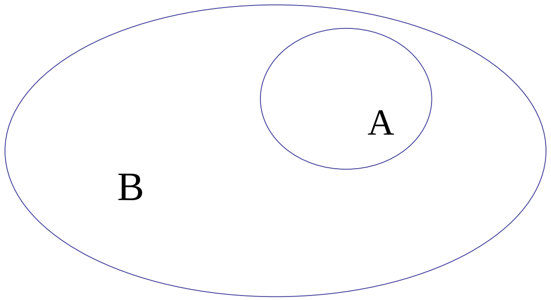
$$\langle O\rangle = \langle\psi|O|\psi\rangle = \mathrm{Tr}(\rho O)$$

where we defined the density matrix $\rho = |\psi\rangle\langle\psi|$. This system is called a pure state as it is described by a unique wave function $|\psi\rangle$.

In mixed states, the system is described by a density matrix ρ. An example of a mixed state is the canonical distribution

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

Assume that the quantum system has multiple degrees of freedom and so one can decompose the total system into two subsystems A and B



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

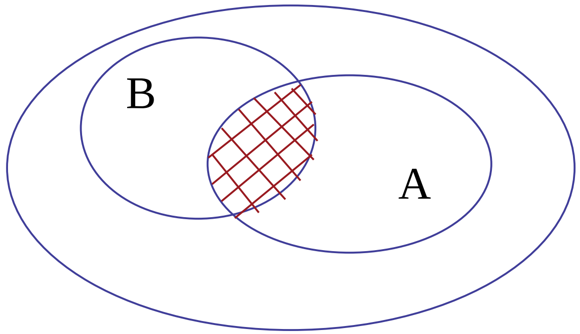
The reduced density matrix of the subsystem A

$$\rho_A = \mathrm{Tr}_B(\rho)$$

Then the entanglement entropy is defined as the von-Neumann entropy for A

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

Properties of Entanglement entropy



1. For pure state

$$S_A = A_B$$

2. For two subspace A and B, the strong subadditivity is

$$S_A + S_B \geq S_{A \cup B} + S_{A \cap B}$$

3. Leading divergence term is proportional to the area of the boundary ∂A

$$S_A = c_0 \frac{\text{Area}}{\epsilon^{d-1}} + O(\epsilon^{-(d-2)}),$$

where c_0 is a numerical constant; ϵ is the ultra-violet(UV) cut off in quantum field theories.

Rényi entropies

It is also useful to compute Rényi entropies

$$S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho^n$$

Then the entanglement entropy is given by

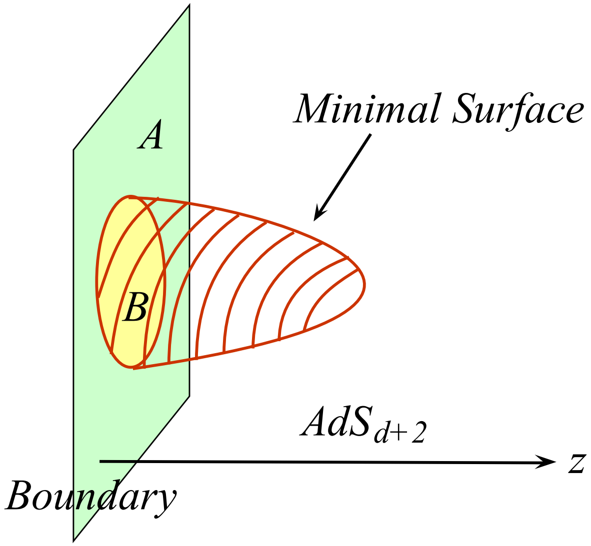
$$S_E = \lim_{n \to 1} S_n$$

Practically one may first compute $\mathrm{Tr}(\rho^n)$ by making use the replica trick and then

$$S_E = -\partial_n \mathrm{Tr} \rho^n |_{n=1}$$

Holographic Formula

For static background and fixed time divide the boundary into A and B. Extend this division $A \cup B$ to of the bulk spacetime. Extend ∂A to a surface γ_A in the entire spacetime such that $\partial \gamma_A = \partial A$.



$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}$$

S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96**, 181602 (2006) [hep-th/0603001].

For holographic entanglement entropy

1. The formula leads to the area law (for Einstein gravity).

2. The strong subadditivity can also be holographically proven (for static background)

3. For 2D CFT using AdS_3 one finds

$$S_A = \frac{c}{3} \ln \frac{\ell}{\epsilon}$$

where ℓ width of strip, $c = \frac{3R}{2G}$.

Higher Derivative Corrections

The holographic formula we have consudered is for Einstein gravity. Motivated by the Wald formula it is interesting to see how this formula is modified in the presence of higher derivative corrections to Einstein gravity.

Unlike the Wald formula there no a general expression when we have arbitrary higher derivative corrections. However the holographic entanglement entropy has been found only for the Lovelock gravities

For the Gauss-Bonnet gravity whose gravity action is

$$S_{GB} = -\frac{1}{16G_N}\int d^{d+2}x\sqrt{g}\left[R - 2\Lambda + \lambda(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\right],$$

The holographic entanglement entropy is argued to be

$$S_A = \mathrm{Min}_{\gamma_A}\left[\frac{1}{4G_N}\int_{\gamma_A} d^d x\sqrt{h}(1+2\lambda R_{int})\right],$$

where R_{int} is the intrinsic curvature of γ_A.

L. Y. Hung, R. C. Myers and M. Smolkin, On Holographic Entanglement Entropy and Higher Curvature Gravity, JHEP 1104 (2011) 025 [arXiv:1101.5813 [hep-th]].

J. de Boer, M. Kulaxizi and A. Parnachev, Holographic Entanglement Entropy in Lovelock Gravities, JHEP 1107 (2011) 109 [arXiv:1101.5781 [hep-th]].

See the talk of A. Sinha

Time-dependent backgrounds

So far we have considered static case where we have a time slice on which we can define minimal surfaces. In the time-dependent case there is no a natural choice of the time-slices.

In Lorentzian geometry there is no minimal area surface. In order to resolve this issue we use the covariant holographic entanglement entropy which is

$$S_A(t) = \frac{\text{Area}(\gamma_A(t))}{4G_N^{(d+2)}}$$

where $\gamma_A(t)$ is the extremal surface in the bulk Lorentzian spacetime with the boundary condition $\partial\gamma_A(t) = \partial A(t)$.

Strong subadditivity?

V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP **0707**, 062 (2007) [arXiv:0705.0016 [hep-th]].

Applications of entanglement entropy

The entanglement entropy can be used as an order parameter to study several aspects of quantum many-body physics.

It may characterize different phases and phase transitions and in particular quantum phase transitions.

Entanglement thermodynamics

Thermodynamics provides a useful tool to study a system when it is in the thermal equilibrium. In this limit the physics may be described in terms of few macroscopic quantities such as energy, temperaure, pressure, entropy.

There are also laws of thermodynamics which describe how these quantities behave under various conditions. In particular the first law of thermodynamics which is energy conservation, tells us how the entropy change as one changes the energy of the system.

There are several interesting phenomena which occur when the system is far from thermal equilibrium.

The entanglement entropy may provide a useful quantity to study excited quantum systems which are far from thermal equilibrium. For a generic quantum system it is difficult to compute the entanglement entropy. Nevertheless, at least, for those quantum systems which have holographic descriptions, one may use the holographic entanglement entropy to explore the behavior of the system.

Another quantity which can be always defined is the energy (or energy density) of the system. It is then natural to pose the question whether there is a relation between the entanglement entropy of an excited state and its energy.

For sufficiently small subsystem, the entanglement entropy is proportional to the energy of the subsystem. The proportionality constant is indeed given by the size of the entangling region.

J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, "Thermodynamical Property of Entanglement Entropy for Excited States," Phys. Rev. Lett. **110**, 091602 (2013) [arXiv:1212.1164 [hep-th]].

AdS/CFT

Gravity on an asymptotically locally AdS provides a holographic description for a strongly coupled quantum field with a UV fixed point.

The information of quantum state in the dual field theory is encoded in the bulk geometry. In particular the AdS geometry is dual to the ground state of the dual conformal field theory.

Exciting the dual conformal field theory from the ground state to an excited state holographically corresponds to modifying the bulk geometry from AdS solution to a general asymptotically locally AdS solution.

First law

The aim is to compute the entanglement entropy of an excited state for the case where the entangling region is sufficiently small.

Since the entanglement entropy for a small subsystem would probe the UV region of the theory, from holography point of view one only needs to know the asymptotic behavior of the bulk geometry.

On the other hand it is known that the most general form of the asymptotically locally AdS may be written in terms of the Fefferman-Graham coordinates as follows

$$ds^2_{d+1} = \frac{R^2}{r^2}\left(dr^2 + g_{\mu\nu}dx^\mu dx^\nu\right),$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x, r)$ with

$$h_{\mu\nu}(x, r) = h^{(0)}_{\mu\nu}(x) + h^{(2)}_{\mu\nu}(x)r^2 + \cdots + r^d\left(h^{(d)}_{\mu\nu}(x) + \hat{h}^{(d)}_{\mu\nu}(x)\log r\right) + \cdots$$

The log term is present for even d. The information about the excited state (or the bulk geometry) is encoded in the function $h_{\mu\nu}(x, r)$.

Let's compute the holographic entanglement entropy for a strip in an AdS geometry. A $d+1$ dimensional AdS solution in the Poincaré coordinates may be written as follows

$$ds^2 = \frac{R^2}{r^2}(dr^2 + \eta_{\mu\nu}dx^\mu dx^\nu), \qquad \mu, \nu = 0, 1, \cdots, d-1.$$

Let us consider an entangling region in the shape of a strip with the width of ℓ given by

$$-\frac{\ell}{2} \le x_1 \le \frac{\ell}{2}, \qquad 0 \le x_i \le L, \quad i = 2, \cdots, d-1.$$

The holographic entanglement entropy may be computed by minimizing a codimension two hypersurface in the bulk geometry whose intersection with the boundary coincides with the above strip.

Assuming that the bulk extension of the surface to be parameterized by $x_1 = x(r)$, the corresponding area is given by

$$A_0 = R^{d-1} L^{d-2} \int dr \frac{\sqrt{1 + x'^2}}{r^{d-1}}.$$

By making use of the standard procedure one may minimize the area to get

$$\ell = 2 \int_0^{\tilde{r}_t} dr \frac{(r/\tilde{r}_t)^{d-1}}{\sqrt{1 - (r/\tilde{r}_t)^{2(d-1)}}},$$

$$S_E^{(0)}(\tilde{r}_t) = \frac{R^{d-1} L^{d-2}}{4 G_N} \int_{\epsilon}^{\tilde{r}_t} \frac{dr}{r^{d-1}\sqrt{1 - (r/\tilde{r}_t)^{2(d-1)}}},$$

where $\tilde{r}_t$ is turning point and ϵ is a UV cut off. Thus one gets

$$S_E^{(0)} = \frac{L^{d-2} R^{d-1}}{4(d-2) G_N} \left[\frac{1}{\epsilon^{d-2}} - 2^{d-2} \pi^{(d-1)/2} \left(\frac{\Gamma\left(\frac{d}{2d-2}\right)}{\Gamma\left(\frac{1}{2d-2}\right)} \right)^{d-1} \frac{1}{\ell^{d-2}} \right],$$

Consider a deformation of the AdS geometry which in turn corresponds to dealing with an excited state in the dual field theory.

The aim is to compute the entanglement entropy of the strip for an excited state when the width of strip is sufficiently small in which the only UV regime of the system will be probed.

Using the notation of the Fefferman-Graham coordinates, we assume that $h_{\mu\nu}^{(n)}\ell^n \ll 1$. Note that in this limit, practically one needs to compute the minimal surface up to order of $\mathcal{O}(h)$.

For the above strip the induced metric in the Fefferman-Graham coordinates is

$$ds^2 = \frac{R^2}{r^2}\Big((1+g_{11}x'^2)dr^2 + 2g_{1i}x'drdx^i + g_{ij}dx^idx^j\Big).$$

Therefore to find the holographic entanglement entropy one needs to minimize the following area

$$A = R^{d-1}\int d^{d-2}xdr\ \frac{\sqrt{g(r)\ (1+G(r)\ x'^2)}}{r^{d-1}}$$

where $g(r) = \mathrm{det}(g_{ij})$ and $G(r) = g_{11} - g_{1i}g^{-1}_{ij}g_{j1}$.

1. Consider the case where the solution is static.

2. To find analytic expressions for our results we will assume that the components of the asymptotic metric are independent of x_1, the direction the width of strip is exended.

With these assumption the equation of motion of x leads to a constant of motion

$$\left(\frac{R}{r}\right)^{d-1} \frac{\sqrt{g(r)}\, G(r)\, x'}{\sqrt{1 + G(r)\, x'^2}} = \text{const} = c,$$

so that

$$x' = \frac{c}{\sqrt{G(r)\left[g(r)\, G(r)\, (\frac{R}{r})^{2(d-1)} - c^2\right]}}.$$

The constant c may be found in terms of the turning point where x' diverges.

Denoting by r_t the turning point, one finds

$$c^2 = g(r_t)\; G(r_t)\; \left(\frac{R}{r_t}\right)^{2(d-1)}$$

It is then straightforward to find the entanglement entropy and the width of the strip as follows

$$S_E = \frac{1}{4G_N}\int_0^{r_t} d^{d-2}x dr\, \left(\frac{R}{r}\right)^{2(d-1)} \sqrt{\frac{g(r)^2 G(r)}{g(r)\; G(r)\; (\frac{R}{r})^{2(d-1)} -\; c^2}}$$

$$\ell = 2\int_0^{r_t} dr\, \frac{c}{\sqrt{G(r)\Big[g(r)\; G(r)\; (\frac{R}{r})^{2(d-1)} -\; c^2\Big]}}$$

To evaluate the above expressions we note that at leading order one has

$$g(r) = 1 + \mathrm{Tr}(h_{ab}) - h_{11} + \mathcal{O}(h^2), \qquad G(r) = 1 + h_{11} + \mathcal{O}(h^2),$$

where $a, b = 1, 2, \cdots, d-1$. So that $g(r)G(r) = 1 + \mathrm{Tr}(h_{ab}) + \mathcal{O}(h^2)$.

In what follows in order to simplify the expressions, it is found useful to define the following parameters

$$\gamma(r) = \mathrm{Tr}(h_{ab}), \quad \beta(r) = h_{11}, \quad f(r, r_t) = \sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}.$$

In this notation at the first order in h one arrives at

$$\ell = \int_0^{r_t} \frac{(r/r_t)^{d-1}}{f(r, r_t)} \left[2 + \frac{\gamma(r_t) - \gamma(r)}{f^2(r, r_t)} - \beta(r)\right] dr$$

We are interested in the change of the entanglement entropy caused by the change of the state. We keep the entangling surface fixed.

Since ℓ is kept fixed while the geometry is deformed the turning point should also be changed. Indeed assuming $r_t = \tilde{r}_t + \delta r_t$ with $\tilde{r}_t$ being the turning point for the pure AdS case, one finds

$$\delta r_t = -\frac{1}{2a_d}\int_0^{\tilde{r}_t} \frac{(r/\tilde{r}_t)^{d-1}}{f(r,\tilde{r}_t)}\left[\frac{\gamma(\tilde{r}_t)-\gamma(r)}{f^2(r,\tilde{r}_t)} - \beta(r)\right]dr$$

where

$$a_d = \int_0^1 \frac{\xi^{d-1}}{\sqrt{1-\xi^{2(d-1)}}}d\xi,$$

Moreover the width of the strip ℓ is the same as that in pure AdS geometry which is $\ell = 2\tilde{r}_t a_d$.

It is straightforward to compute the entanglement entropy up to order of $\mathcal{O}(h)$. In fact expanding the expression of the entanglement entropy one finds

$$S_E = S_E^{(0)}(\tilde{r}_t) + \frac{R^{d-1}}{8G_N}\int_0^{\tilde{r}_t} dr\; d^{d-2}x\; \frac{\gamma(r) - f^2(r,\tilde{r}_t)\beta(r)}{r^{d-1}f(r,\tilde{r}_t)},$$

where $S_E^{(0)}(\tilde{r}_t)$ is the holographic entanglement entropy for the strip in a pure AdS$_{d+1}$ geometry.

By making use of the Fefferman-Graham expansion for the asymptotic form of the metric one arrives at

$$\Delta S_E = \frac{R^{d-1}}{8G_N}\int_0^{\tilde{r}_t} dr\left(\Gamma^{(0)} + \Gamma^{(2)} r^2 + \cdots + \Gamma^{(d)} r^d + \tilde{\Gamma}^{(d)} r^d \ln r\right),$$

where the change of the entanglement entropy is defined by

$$\Delta S_E = S_E - S_E^{(0)}(\tilde{r}_t)$$

also

$$\Gamma^{(n)} = \frac{\int d^{d-2}x\ \mathrm{Tr}(h_{ab}^{(n)})}{r^{d-1} f(r,\tilde{r}_t)} - \frac{f(r,\tilde{r}_t)}{r^{d-1}}\int d^{d-2}x\ h_{11}^{(n)}$$

$$\tilde{\Gamma}^{(d)} = \frac{\int d^{d-2}x\ \mathrm{Tr}(\tilde{h}_{ab}^{(d)})}{r^{d-1} f(r,\tilde{r}_t)} - \frac{f(r,\tilde{r}_t)}{r^{d-1}}\int d^{d-2}x\ \tilde{h}_{11}^{(d)}.$$

Using this expansion it is straightforward to perform the integration over r. Indeed for $d > 2$ one finds

$$\begin{aligned}\int_{\epsilon}^{\tilde{r}_t} dr\, \Gamma^{(n)} r^n \;&=\; \frac{1}{(d-2-n)\epsilon^{d-2-n}} \int d^{d-2}x \left(\mathrm{Tr}(h^{(n)}_{ab}) - h^{(n)}_{11} \right) \\ &\quad - \frac{F(d-1, d-1-n)}{\tilde{r}_t^{d-2-n}} \int d^{d-2}x \left(\mathrm{Tr}(h^{(n)}_{ab}) - \frac{d-1}{n+1} h^{(n)}_{11} \right) \\ &\equiv \frac{1}{(d-2-n)\epsilon^{d-2-n}}\, N^{(n)} + \frac{1}{\tilde{r}_t^{d-2-n}}\, M^{(n)},\end{aligned}$$

where ϵ is a UV cut off, and

$$F(m,n) = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1-n}{2m}, \frac{2m+1-n}{2m}, 1\right)}{n-1},$$

with ${}_2F_1$ being the hypergeometric function.

Note that for even d and $n = d-2$ one finds just a logarithmic divergence as $N^{(d-2)}$ $\ln \frac{\epsilon}{\tilde{r}_t}$ while for odd d and $n = d-1$ the result is finite and is given by $M^{(d-1)}\,\tilde{r}_t$.

For arbitrary d for $n = d$ it leads to a finite term given by $\tilde{r}_t^2 M^{(d)}$. More precisely, using the fact that in general at leading order $\mathrm{Tr}(h^{(d)}_{\mu\nu}) = \mathcal{A}$ with $\mathcal{A}$ being the trace anomaly one finds

$$\int_0^{\tilde{r}_t} dr\, \Gamma^{(d)} r^d = -F(d-1,-1)\, \tilde{r}_t^2 \int d^{d-2}x \left(h^{(d)}_{tt} + \mathcal{A} - \frac{d-1}{d+1} h^{(d)}_{11} \right).$$

Note that for odd d the anomaly term is zero. One should add that when d is an even number we have another term coming from $\tilde{\Gamma}^{(d)}$ which can similarly be calculated leading to an $\ln \tilde{r}_t$ contribution to the entanglement entropy.

Setting $\ell = 2\tilde{r}_t a_d$, one can find the variation of the entanglement entropy, ΔS_{EE}, as a function of ℓ. For odd d

$$\Delta S_E = \frac{R^{d-1}}{8G_N}\sum_{n<d-2}\left(\frac{1}{(d-2-n)\epsilon^{d-2-n}}\,N^{(n)} + \frac{(2a_d)^{(d-2-n)}}{\ell^{d-2-n}}\,M^{(n)}\right) + \frac{R^{d-1}M^{(d-1)}}{16G_N a_d}\,\ell - \frac{R^{d-1}F(d-1,-1)}{32a_d^2 G_N}\,\ell^2 \int d^{d-2}x\left(h^{(d)}_{tt} - \frac{d-1}{d+1}h^{(d)}_{11}\right)$$

while for even d one gets

$$\begin{aligned}\Delta S_E = {} & \frac{R^{d-1}}{8G_N}\sum_{n<d-2}\left(\frac{1}{(d-2-n)\epsilon^{d-2-n}}\,N^{(n)} + \frac{(2a_d)^{d-2-n)}}{\ell^{d-2-n}}\,M^{(n)}\right)\\ & + \frac{R^{d-1}N^{(d-2)}}{8G_N}\,\ln\frac{2\epsilon a_d}{\ell}\\ & - \frac{R^{d-1}F(d-1,-1)}{32a_d^2 G_N}\,\ell^2 \int d^{d-2}x\left(h^{(d)}_{tt} + \mathcal{A} - \frac{d-1}{d+1}h^{(d)}_{11}\right)\\ & - \frac{R^{d-1}F(d-1,-1)}{32a_d^2 G_N}\,\ell^2 \ln\frac{\ell}{2a_d}\int d^{d-2}x\left(\tilde{h}^{(d)}_{tt} - \frac{d-1}{d+1}\tilde{h}^{(d)}_{11}\right)\end{aligned}$$

Here we have used the fact that $\mathrm{Tr}(\tilde{h}^{(d)}_{\mu\nu}) = 0$.

When one excites the ground state to an excited state, the energy of the system is increased and generally one gets non-zero expectation value for the energy momentum tensor.

$$\langle T_{\mu\nu} \rangle = \frac{dR^{d-1}}{16\pi G_N}\, h_{\mu\nu}^{(d)}$$

The extra non-trivial contribution to the entanglement entropy is coming from expectation value of the energy-momentum tensor which does depend on the excited state we are considering.

S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," Commun. Math. Phys. **217**, 595 (2001) [hep-th/0002230].

More precisely one finds

$$\Delta S_E^{\text{finite}} = \sum_n (\cdots) \frac{M^{(n)}}{\ell^{d-2n}} - \frac{\pi F(d-1,-1)\ell}{2a_d^2 d} \left(\Delta E - \frac{d-1}{d+1} \int \Delta P_x dV_{d-1} + \frac{dR^{d-1}}{16\pi G_N} \int \mathcal{A} dV_{d-1} \right) + \cdots ,$$

where $(\cdots)$ stands for some numerical factors and $dV_{d-1} = \ell d^{d-2}x$. Moreover the energy and entanglement pressure are defined by

$$\Delta E = \int dV_{d-1} \langle T_{tt} \rangle, \qquad \Delta P_x = \langle T_{11} \rangle.$$

For the case of $h^{(0)}_{\mu\nu} = 0$ where one has

$$h_{\mu\nu}(x,r) = h^{(d)}_{\mu\nu}(x)\, r^d$$

the boundary is flat the anomaly term is zero and therefore one gets

$$\Delta S_E = \frac{\pi\ell}{4d}\frac{C_1}{C_0^2}\left(\Delta E - \frac{d-1}{d+1}\int dV_{d-1}\Delta p_x\right),$$

where

$$C_0 = \sqrt{\pi}\,\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}, \qquad C_1 = \sqrt{\pi}\,\frac{\Gamma\left(\frac{d}{d-1}\right)}{\Gamma\left(\frac{d+1}{2(d-1)}\right)},$$

One may define entanglement temperature in terms of the width of the strip. In the present case the corresponding temperature may be given by

$$T_E = \frac{4dC_0^2}{\pi C_1}\frac{1}{\ell}$$

Assuming that $h_{\mu\nu}^{(d)}$ to be constant one gets

$$\Delta E = T_E \Delta S_E + \frac{d-1}{d+1} V_{d-1} \Delta p_x$$

where V_{d-1} is the volume of the entangling region.

Due to its similarity with the first law of thermodynamics we would like to consider this expression as the first law of entanglement thermodynamics.

Universal features

In what extend the resultant first law is universal?

We will consider the holographic entanglement entropy for a system in the form of shpere to address this question.

To proceed let us first write down the boundary metric in the shperical coodinates (for fixed time)

$$
\begin{aligned}
ds^2 &= \frac{R^2}{r^2}(dr^2 + g_{ij}dx^i dx^j) \\
&= \frac{R^2}{r^2}(dr^2 + g_{\rho\rho}d\rho^2 + 2\rho g_{\rho\alpha}d\rho d\theta^\alpha + \rho^2 g_{\alpha\beta}d\theta^\alpha d\theta^\beta),
\end{aligned}
$$

where

$$
g_{\rho\rho} = \Omega^i\, g_{ij}\, \Omega^j, \qquad g_{\rho\alpha} = \Omega^i g_{ij}\frac{\partial\Omega^j}{\partial\theta^\alpha}, \qquad g_{\alpha\beta} = \frac{\partial\Omega^i}{\partial\theta^\alpha} g_{ij} \frac{\partial\Omega^j}{\partial\theta^\beta}
$$

Here Ω^is are the angular elements with the condition $\sum_i \Omega^i\Omega^i = 1$.

Now the aim is to study the entanglement entropy for a sphere with a radius ℓ in the boundary.

The extension of the region to the bulk will be parameterized by $\rho = \rho(r)$.

The induced metric on the codimension two hypersurface in the bulk is given by

$$ds^2 = \frac{R^2}{r^2}\Big[(1 + g_{\rho\rho}\rho'^2)\, dr^2 + 2\rho\rho' g_{\rho\alpha}\, dr\, d\theta^\alpha + \rho^2 g_{\alpha\beta}\, d\theta^\alpha d\theta^\beta\Big]$$

To compute the holographic entanglement entropy one needs to minimize the following area

$$A = R^{d-1} \int drd\Omega_{d-2}\, \rho^{d-2}\, \frac{\sqrt{g\, (1 + G\, \rho'^2)}}{r^{d-1}}$$

where $g = \det(g_{\alpha\beta})$ and $G = g_{\rho\rho} - g_{\rho\alpha}\, g^{-1}_{\alpha\beta}\, g_{\beta\rho}$.

The equation of motion

$$\left[\frac{1}{r^{d-1}}\frac{gG\rho^{d-2}\rho'}{\sqrt{g(1+G\rho'^2)}}\right]' = (d-2)\rho^{d-3}\frac{1}{r^{d-1}}\sqrt{g\,(1+G\rho'^2)}$$

It is easy to check that for the ground state where the dual gravity is given by an AdS$_{d+1}$ geometry a solution of the above equation is $\rho_0 = \sqrt{\tilde{r}_t^2 - r^2}$. Note that in this case $G = 1$ and

$$g^{(0)}_{\alpha\beta} = \frac{\partial\Omega^i}{\partial\theta^\alpha}\delta_{ij}\frac{\partial\Omega^j}{\partial\theta^\beta}.$$

Following our previous study the aim is to find the entanglement entropy for an excited state for a sufficiently small entangling region. To do so, one needs to expand the the expression for the area which at leading order it yields

$$A(\rho, r_t) = A(\rho_0, r_t) + \delta_g A(\rho_0, \tilde{r}_t)$$

Here

$$\begin{aligned} A(\rho_0, r_t) &= R^{d-1} \int \int_0^{r_t} dr d\Omega_{d-2}\, \rho_0^{d-2} \frac{\sqrt{g^{(0)}(1+\rho_0'^2)}}{r^{d-1}}, \\ \delta_g A(\rho_0, \tilde{r}_t) &= \frac{R^{d-1}}{2} \int \int_0^{\tilde{r}_t} dr d\Omega_{d-2}\, \rho_0^{d-2} \frac{\sqrt{g^{(0)}(1+\rho_0'^2)}}{r^{d-1}} \left[g_0^{\alpha\beta}\, \delta g_{\alpha\beta} + \frac{\rho_0'^2 \delta g_{\rho\rho}}{1+\rho_0'^2} \right], \end{aligned}$$

With these expressions it is easy to find the change of the entanglement entropy as follows

$$\Delta S_E = \frac{R^{d-1}}{8G_N}\int\int_0^{\tilde{r}_t}\frac{(\tilde{r}_t^2-r^2)^{\frac{d-3}{2}}\tilde{r}_t}{r^{d-1}}\left[\mathrm{Tr}(h_{ab})-\frac{\tilde{r}_t^2-r^2}{\tilde{r}_t^2}h_{\rho\rho}\right]dr\,d\Omega_{d-2}.$$

Note also that the radius of the entangling sphere is found to be $\ell=\tilde{r}_t$.

Now we need to use the Fefferman-Graham expansion for the metric to find an expansion for the change of the entanglement entropy.

The result has the same structure as that in the strip case. Namely there are divergent terms which must be regulated by introducing a UV cut off and they all vanish when the bounday is flat.

Consider the case where $h_{\mu\nu} = h^{(d)}_{\mu\nu} r^d$ then one arrives at

$$T_E \, \Delta S_E = \Delta E - \frac{d-1}{d+1} \int \Delta P_\rho \, dV_{d-1}$$

where $T_E = \frac{d}{2\pi\ell}$ and $dV_{d-1} = \rho^{d-2} d\rho \, d\Omega_{d-2}$.

$$T_E\,\Delta S_E = \Delta E - \frac{d-1}{d+1}\int \Delta P_\perp \, dV_{d-1}$$

1. The numerical factor in the definition of the entanglement temperature is different from that in the strip case.

2. The final form of the first law is the same.

3. The numerical factor in front of the pressure term is universal.

3. Only the entanglement pressure normal to the entangling surface appears in the first law.

General form

So far we have considered the static case where the corresponding background geometry was time independent.

It is, however, possible to show that the final results also hold for time dependent cases.

As long as we are interested in a sufficiently small subsystem we could still use the static solution leading to the same result for the first law.

Consider a time dependent excitation state above a vaccum solution. From the bulk point of view it corresponds a time dependent deviation from AdS solution.

There are several sources which contribute to the change of the holographic entanglement entropy. The change may be caused by the change of the turning point, the change of the solution and the change of the metric.

The interesting point is that at leading order which is what we are interested in the change of entanglement entropy is completey given by the change of metric

$$\Delta S_E = \frac{1}{4G_N}\int d^{d-1}x\sqrt{\det(g^{(0)}_{\text{in}})}(g^{(0)}_{\text{in}\,ab})^{-1}g^{(1)}_{\text{in}\,ab},$$

where $g^{(0)}_{\text{in}}$ and $g^{(1)}_{\text{in}}$ are the induced metrics on the codimension two hypersurface in the bulk for the cases of AdS geometry and the perturbation above it, respectively.

The result is the same as that we considered in the previous section. Therefore the first law we have inroduced may also be applyed for the time dependent case.

M. Nozaki, T. Numasawa and T. Takayanagi, "Holographic Local Quenches and Entanglement Density," arXiv:1302.5703 [hep-th].

Other laws of entanglement thermodynaics

Based on the holographic description of the entanglement entropy and for explicit examples we have found a relation between entanglement entropy, energy and entanglement pressure which using the similarity with the thermodynamics could be thought of as the first law of entanglement thermodynamics.

It is then natural to pose the question whether there are other laws similar to what we have in the thermodynamics.

Second law

There is a natural statement for the second law of entanglement thermodynamics: the strong subadditivity.

According to the strong subadditivity for any given two subsystems A and B one has

$$S_{E(A)} + S_{E(B)} \geq S_{E(A\cup B)} + S_{E(A\cap B)}.$$

It is worth noting that although the entanglement entropy is divergent due to UV effects, the divergent parts of the entanglement entropy drop from both sides. In fact this inequality is also satisfied by the finite part of the entanglement entropy.

So far our suggestions and statements about the laws of entanglement thermodynamics were based on rigorous computations.

To proceed for other possible laws we note that although we will use an explicit example to explore them.

The most important part of our study is the definition of the entanglement temperature.

From dimensional analysis and also from our experiences in thermodynamics and hydrodynamics it is natural to consider the inverse of the typical size of the entangling region as the temperature.

But there is a non-universal numerical factor in its definition!

As long as we are considering entangling regions with a fixed shape the numerical factor is universal.

Zeroth law

Apart from this ambiguity, in what follows for a fixed shape we suggest a statement which could be considered as the zeroth law of entanglement thermodynamics.

Consider two entangling regions given by two strips with the width of ℓ_1 and ℓ_2, respectively.

When they joined together we get another strip whose width at most could be $\ell_3 = \ell_1 + \ell_2$, Or

$$\ell_1 + \ell_2 \geq \ell_3$$

Using the definition of the entanglement temperatures before and after joining one gets

$$\frac{1}{T_{1E}} + \frac{1}{T_{2E}} \geq \frac{1}{T_{3E}}$$

It is easy to argue that such a relation could also be satisfied when the entangling regions are spheres.

For a special case where the system is isotropic one can find this relation from strong subadditivity.

$$\Delta S_{1E} + \Delta S_{2E} \geq \Delta S_{3E}$$

which results, taking into account $T_E \Delta S_E \sim \Delta E$

$$\frac{V_{d-1}^{(1)}\mathcal{E}}{T_{1E}} + \frac{V_{d-1}^{(2)}\mathcal{E}}{T_{2E}} \geq \frac{V_{d-1}^{(3)}\mathcal{E}}{T_{3E}}$$

Then (for strip)

$$\frac{1}{T_{1E}^2} + \frac{1}{T_{2E}^2} \geq \frac{1}{T_{3E}^2}$$

Therefore one finds

$$\left(\frac{1}{T_{1E}} + \frac{1}{T_{2E}}\right)^2 \geq \frac{1}{T_{1E}^2} + \frac{1}{T_{2E}^2} \geq \left(\frac{1}{T_{3E}}\right)^2$$

Third law

Let us now proceed to introduce the third law of entanglement thermodynamics.

Consider the finite part of the entanglement entropy of a strip for an excited state up to order of $\mathcal{O}(T_E^{-2})$

$$
\begin{aligned}
S_E^{\text{finite}} \;=\; & \frac{R^{d-1}}{8G_N}\left[(\tilde{B}_0 L^{d-2} + B_0 M^{(0)})T_E^{d-2} + \sum_{n=1} B_n M^{(n)} T_E^{d-2-n}\right] \\
& + \frac{1}{T_E}\left(\Delta E + \frac{d-1}{d+1}\Delta P_x \Delta V_{d-1}\right)
\end{aligned}
$$

where $\tilde{B}_0, B_n$ are numerical factors.

$$S_E^{\text{finite}} \sim T_E^{d-2} \quad \text{for large } T_E$$

The finite part of entanglement entropy goes to infinity for sufficiently higher entanglement temperature.

Due to a natural UV cut off in the theory there is a natural cut off for temperature preventing to get infinite entanglement entropy.

Note that as we increase the temperature, the dominant divergent parts comes from the ground state which corresponds to the AdS geometry. it is then possible to argue that the above statement is also valid for other shape of the entangling region.

Laws of entanglement thermodynamics

- Zeroth law: The entanglement temperature is proportional to the inverse of the typical size of the entangling region and for two subsystem A and B one has

$$\frac{1}{T_{(A)E}} + \frac{1}{T_{(B)E}} \geq \frac{1}{T_{(A\cup B)E}}.$$

- First law: There is a relation between the energy of the system and the entanglement entropy as follows

$$\Delta E = T_E \Delta S_E + \frac{d-1}{d+1} V_{d-1} \Delta P_\perp,$$

where $\Delta P_\perp$ is the entanglement pressure normal to the entangling surface.

- Second law: Entanglement entropy enjoys strong subadditivity

$$S_{E(A)} + S_{E(B)} \geq S_{E(A\cup B)} + S_{E(A\cap B)}$$

- Third law: There is an upper bound on the entanglement temperature preventing to have an infinite entanglement entropy.

An explicit example

The AdS Schwarzschild background

$$ds^2 = \frac{R^2}{\rho^2}\left(-f(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + \sum_{i=1}^{d-1} dx_i^2\right), \qquad f(\rho) = 1 - \left(\frac{\rho}{\rho_H}\right)^d$$

where ρ_H is the radius of horizon. By making use of the coordinate transformation $\frac{dz}{z} = \frac{d\rho}{\rho f^{1/2}}$, one may recase the metric to the Fefferman-Graham coordites as follows

$$ds^2 = \frac{R^2}{r^2}(dr^2 + g_{\mu\nu}dx^\mu dx^\nu),$$

whose asymptotic behvior of the metric components are

$$g_{tt} = -1 + h_{tt}^{(d)} r^d = -1 + \frac{4(d-1)}{d}\rho_H^d r^d, \qquad g_{aa} = 1 + h_{aa}^{(d)} r^d = 1 + \frac{4}{d}\rho_H^d r^d$$

So $\Delta E = \frac{4(d-1)}{d}\rho_H^d V_{d-1}$ and $\Delta P_x = \frac{4}{d}\rho_H^d$. From first law one finds

$$T_E \Delta S_E = \frac{4(d-1)}{d+1}\rho_H^d V_{d-1}.$$

What about charged system?

An example could be Reissner-Nordstrom black hole (or system with charged matter field) whose solution for $d \geq 2$ is

$$ds^2 = \frac{R^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{d} dx_i^2\right), \quad F_{rt} = -QR\sqrt{2d(d-1)}r^{d-2},$$
$$f(r) = 1 - (1 + Q^2 r_H^{2d})\left(\frac{r}{r_H}\right)^{d+1} + Q^2 r^{2d},$$

One may compute holographic entanglement entropy for this background and the charge effects are just through the metric's components.

Is there any way to see the charges?

There are two ways to introduce the charges into the equation.

1. One may consider the fluxes through the entangleing surface

$$S = \frac{A}{4G} + \sigma\Phi$$

It is an order parameter which can probe different charges in the system.

S. A. Hartnoll and D. Radicevic, "Holographic order parameter for charge fractionalization," Phys. Rev. D **86**, 066001 (2012) [arXiv:1205.5291 [hep-th]].

2. There could be another way to probe the background charges. To proceed let's make a double Wick rotation and then fixed the time.

Consider a codimension two hypersurface in the bulk parametrized by coodrinates ξ_a for $a = 1, \cdots, d$. Then one may define two natural quantities: the induced metric and the pull back of the gauge field on the world volume of the hypersurface which are given by

$$\tilde{g}_{ab} = \frac{\partial x^\mu}{\partial \xi_a}\frac{\partial x^\nu}{\partial \xi_b} g_{\mu\nu}, \qquad\qquad F_{ab} = \frac{\partial x^\mu}{\partial \xi_a}\frac{\partial x^\nu}{\partial \xi_b} F_{\mu\nu},$$

Then the geometric entropy can be defined in terms of the induced metric as

$$S_G = \int d^d\xi \sqrt{\mathrm{det}(\tilde{g})}$$

It can be obtained from entanglement entropy by a double Wick rotation.

On the other hand motivated by DBI action in the string theory it is natural to define the following quantity

$$\Gamma = \frac{1}{G_{d+2}} \int d^d\xi \, \sqrt{\det\left(\tilde{g} + RF_{ab}\right)}$$

where R is a typical scale of the theory (*e.g.* the radius of curvature).

An advantage of this definition is that, it is directly sensitive to the background charge.

For sufficietnly small charges one may expand the square root which at leading order one arrives at

$$\Gamma = \frac{1}{G_{d+2}} \int d^d\xi \, \sqrt{\det(\tilde{g})} \left(1 - \frac{1}{4} R^2 F_{ab}^2\right),$$

which, in turns, shows that in this limit it essentially contains the same information as the geometric entropy.

Geometric entropy

Consider a finite temperature four dimensional quantum field theory on $S^1 \times S^3$.

$$d\Omega_d^2 = d\theta + \sin^2\theta(d\psi^2 + \sin^2\psi \, d\phi^2),$$

Let us change the periodicity of ϕ into $0 \leq \phi \leq 2\pi k$ which results to conical singularities at $\psi = 0, \pi$ for $k \neq 1$ with the deficit angle $2\pi(1-k)$. Let us denote by $Z[k]$ the partition function of the theory on this singular space. Then one may define a density matrix as follows

$$\mathrm{Tr}\rho^k = \frac{Z[k]}{(Z[1])^k},$$

Using the definition of von-Neumann entropy, the geometric entropy is defined by

$$S_G = -\mathrm{Tr}(\rho \log \rho) = -\partial_k \log\left(\frac{Z[k]}{(Z[1])^k}\right)\Bigg|_{k=1}.$$

M. Fujita, T. Nishioka and T. Takayanagi, "Geometric Entropy and Hagedorn/Deconfinement Transition," JHEP **0809**, 016 (2008) [arXiv:0806.3118 [hep-th]].

Actually setting $k = \frac{1}{n}$, the proceduer reduces to the computation of the partition function of the model on the orbifold S^3/Z_n.

It can be done for the finite temperature $\mathcal{N} = 4$ SYM theory in four dimensions at certain limits.

O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, "The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories," Adv. Theor. Math. Phys. **8**, 603 (2004) [hep-th/0310285].

P. Basu and S. R. Wadia, "R-charged AdS(5) black holes and large N unitary matrix models," Phys. Rev. D **73**, 045022 (2006) [hep-th/0506203].

D. Yamada and L. G. Yaffe, "Phase diagram of N=4 super-Yang-Mills theory with R-symmetry chemical potentials," JHEP **0609**, 027 (2006) [hep-th/0602074].

T. Harmark and M. Orselli, "Quantum mechanical sectors in thermal N=4 super Yang-Mills on R x S**3," Nucl. Phys. B **757**, 117 (2006) [hep-th/0605234].

Summary

1. Entanglement entropy is a good order parameter

2. There is very nice simple holographic description of entanglement entropy

3. One may define a framework for entanglement entropy such as thermo-dynamics

4. One may also define new objects which are sensitive to background charges