The Geometry of the Quantum Hall Effect

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7th Crete Regional Meeting in String Theory

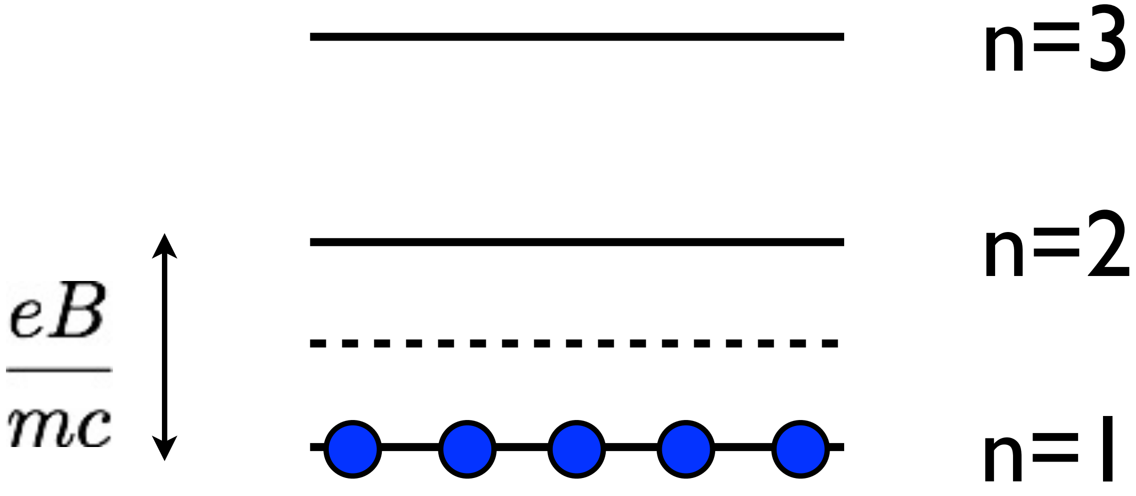
Refs:

DTS arXiv:1306.0638
ongoing work with Siavash Golkar and Dung Nguyen
Carlos Hoyos, DTS arXiv:1109.2651
DTS, M. Wingate cond-mat/0509786

Plan

- Review of quantum Hall physics
- Summary of results
- Nonrelativistic diffeomorphism
- Construction of the action

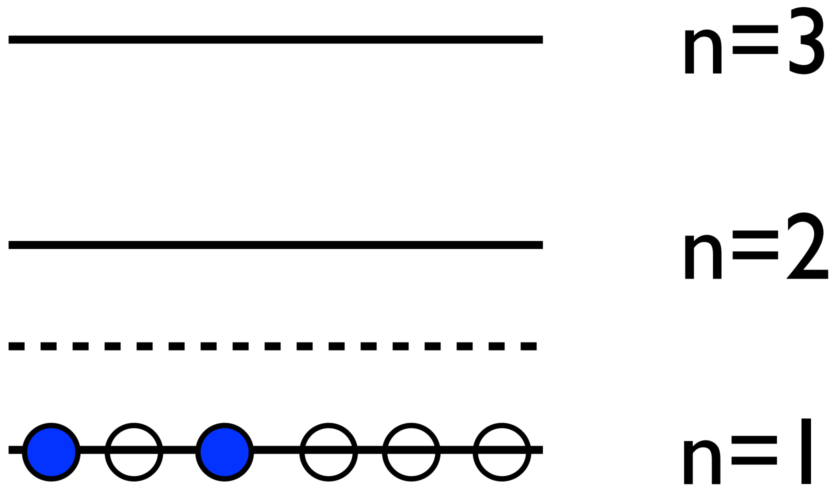
Integer quantum Hall state



degeneracy $\frac{B}{2\pi} \times \mathrm{Area}$

Energy gap

Fractional quantum Hall state



Huge ground state degeneracy without interactions

FQHE: interactions lift degeneracy

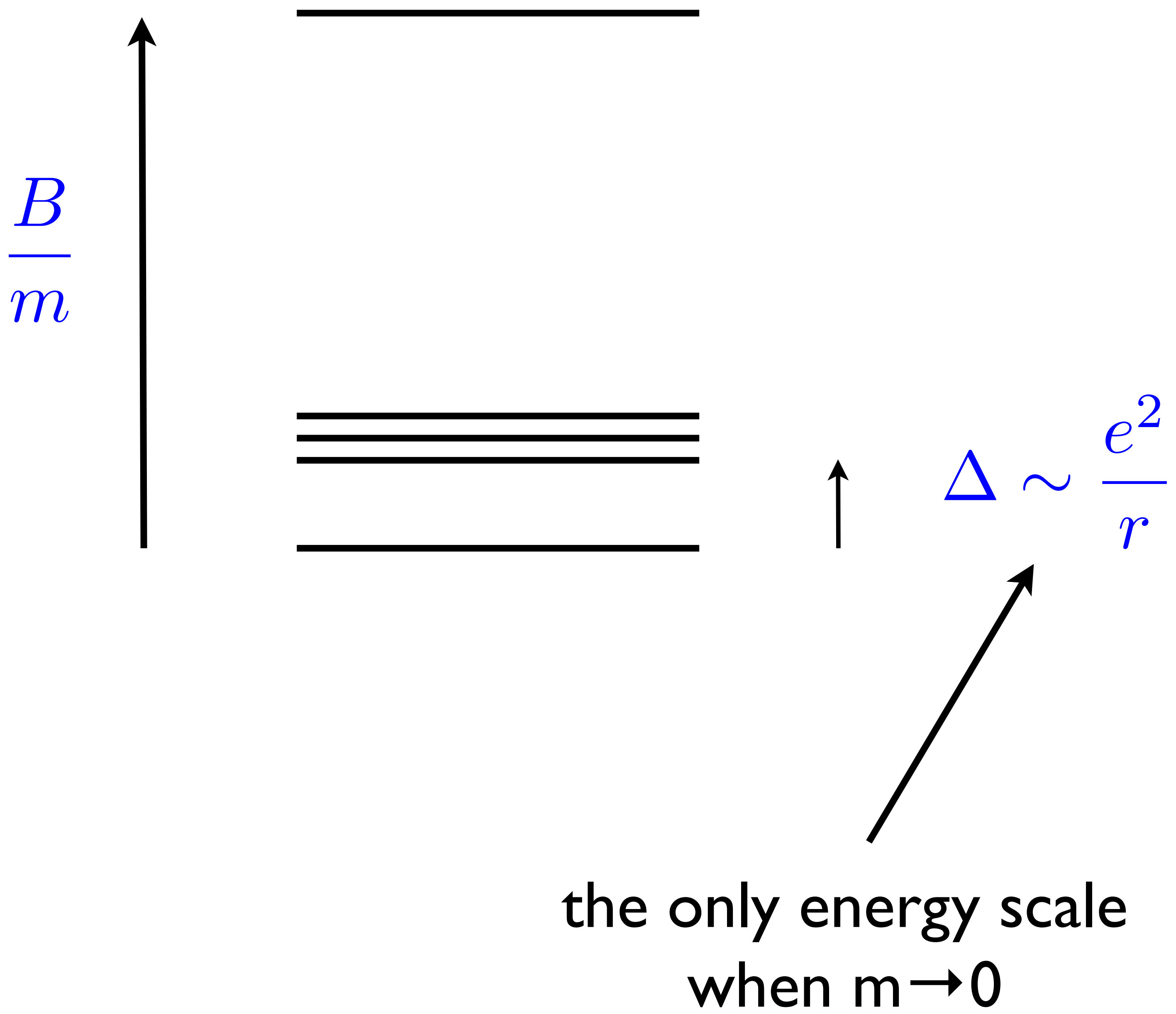
exp: the system is gapped at some values of filling fraction

$$\nu = \frac{n}{B/2\pi}$$

$$\nu = \frac{1}{3}, \frac{1}{5}, \cdots$$

Laughlin's filling factors

Massless limit



Laughlin's wave function

- In the symmetric gauge LLL states are

$$\psi(z) = f(z)e^{-|z^2|/4\ell^2}$$

Laughlin's guess for the ground state wave fn $\nu = 1/3$

$$\psi(z) = \prod_{\langle ij \rangle}(z_i - z_j)^3 \prod_i e^{-|z_i|^2/4\ell^2}$$

Not exact, although seems to be very good approximation
implies equal-time correlators, not at unequal times

Effective field theory

- Effective field theory: captures low-energy dynamics
- What are the low-energy degrees of freedom of a quantum Hall state?
  - there are none (in the bulk): energy gap
  - Thus the effective Lagrangian is polynomial over external fields and derivatives (generating functional)

Chern-Simons action

- To lowest order in derivatives:

$$S = \frac{\nu}{4\pi}\int d^3x\,\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$$

encodes Hall conductivity

$$J^\mu = \frac{\delta S}{\delta A_\mu} \qquad \rho = \frac{\nu}{2\pi}B$$

$$J_y = \sigma_{xy}E_x \qquad \sigma_{xy} = \frac{\nu}{2\pi}\frac{e^2}{\hbar}$$

Another formulation of CS theory

$$\mathcal{L} = \frac{\nu}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - j^\mu(\partial_\mu\varphi - A_\mu + a_\mu)$$

$$j^\mu = \frac{\delta S}{\delta A_\mu}$$ is the current

at the same time is the Lagrange multiplier enforcing

$$a_\mu = A_\mu + \partial_\mu\varphi$$

Note: φ is the phase of the condensate of composite bosons (ZHK)

Universality beyond CS

Higher-derivatives corrections: of dynamical, not topological nature, hence non universal?

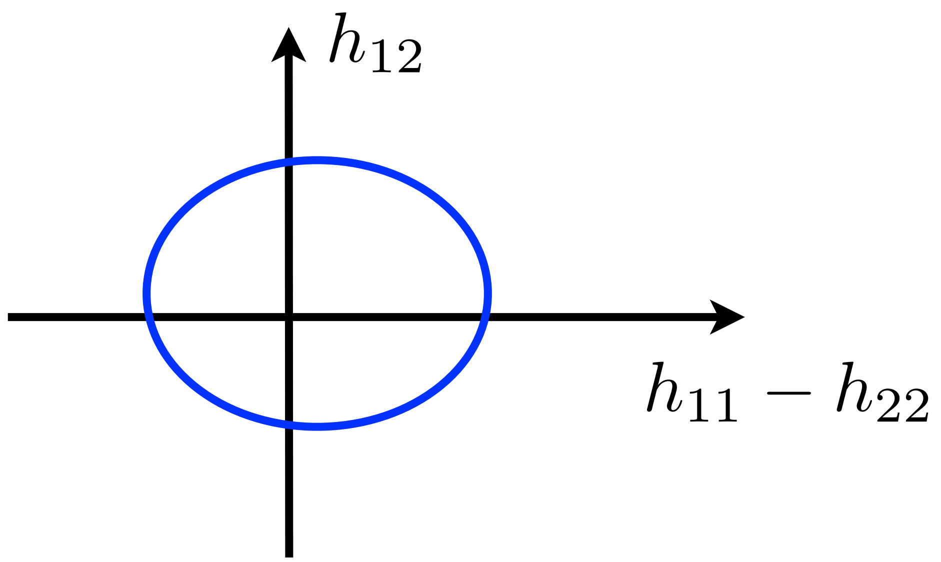
But there is universality beyond the CS action

Hall viscosity

Following the evolution of the QH state with changing metric

$$h_{ij} = h_{ij}(t), \, \det h = 1$$

2-dim space



nonzero Berry curvature

$$\langle T_{11} - T_{22} \rangle \sim \eta_A \dot{h}_{12}$$

universal (not renormalized by interactions)

However in CS theory $T_{\mu\nu} = 0$

Symmetries of NR theory

DTS, M.Wingate 2006

Microscopic theory

$$S = \int d^3x \sqrt{h} \left[\frac{i}{2} \psi^\dagger \overleftrightarrow{D}_t \psi - \frac{h^{ij}}{2m} D_i \psi^\dagger D_j \psi + \frac{g}{4m} \frac{F_{12}}{\sqrt{h}} \psi^\dagger \psi \right]$$

$$D_\mu \psi \equiv (\partial_\mu - i A_\mu) \psi$$

Invariance under time-independent diff $\xi = \xi(\mathbf{x})$:

$$\delta\psi = -\xi^k \partial_k \psi$$

$$\delta A_0 = -\xi^k \partial_k A_0$$

$$\delta A_i = -\xi^k \partial_k A_i - A_k \partial_i \xi^k$$

$$\delta h_{ij} = -\nabla_i \xi_j - \nabla_j \xi_i$$

NR diffeomorphism

- These transformations can be generalized to be time-dependent: $\xi=\xi(t,\mathbf{x})$

$$\delta\psi = -\xi^k\partial_k\psi$$

$$\delta A_0 = -\xi^k\partial_k A_0 - A_k\dot{\xi}^k + \frac{g}{4}\varepsilon^{ij}\partial_i(h_{jk}\dot{\xi}^k)$$

$$\delta A_i = -\xi^k\partial_k A_i - A_k\partial_i\xi^k - mh_{ik}\dot{\xi}^k$$

$$\delta h_{ij} = -\nabla_i\xi_j - \nabla_j\xi_i$$

Galilean transformations: special case $\xi^i=v^i t$

$g=0$ version can be understood as NR reduction of relativistic diffeomorphism invariance

NR reduction

Start with complex scalar field

$$S = -\int dx\,\sqrt{-g}(g^{\mu\nu} D_\mu \phi^* D_\nu \phi + m^2 \phi^* \phi)$$

$$D_\mu \phi = (\partial_\mu - i\mathcal{A}_\mu)\phi$$

Take nonrelativistic limit:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \dfrac{2\alpha_0}{mc^2} & \dfrac{\alpha_j}{mc} \\ \dfrac{\alpha_i}{mc} & h_{ij} \end{pmatrix}$$

$$\phi = e^{-imcx^0} \frac{\psi}{\sqrt{2mc}}$$

$$S = \int d^3x\,\sqrt{h}\left[\frac{i}{2}\psi^\dagger \overleftrightarrow{D}_t \psi - \frac{h^{ij}}{2m} D_i \psi^\dagger D_j \psi\right]$$

$$D_\mu \psi = \partial_\mu \psi - i(\mathcal{A}_\mu + \alpha_\mu)\psi$$

Relativistic diffeomorphism

$$g_{\mu\nu} = \begin{pmatrix} -1 + \dfrac{2\alpha_0}{mc^2} & \dfrac{\alpha_j}{mc} \\ \dfrac{\alpha_i}{mc} & h_{ij} \end{pmatrix}$$

under diff $\quad A_\mu = \mathcal{A}_\mu + \alpha_\mu$

ξ^0 : gauge transform $\qquad \phi = e^{-imcx^0} \dfrac{\psi}{\sqrt{2mc}}$

ξ^i : general coordinate transformations

$$\delta A_0 = -\xi^k \partial_k A_0 \ -A_k \dot{\xi}^k$$

$$\delta A_i = -\xi^k \partial_k A_i - A_k \partial_i \xi^k \ -m h_{ik} \dot{\xi}^k$$

Interactions

- Interactions can be introduced that preserve nonrelativistic diffeomorphism
  - interactions mediated by fields
- For example, Yukawa interactions

$$S = S_0 + \int d^3x \, \sqrt{h} \, \phi \psi^\dagger \psi + \int d^3x \, \sqrt{h} (h^{ij} \partial_i \phi \partial_j \phi + M^2 \phi)$$

$$\delta\phi = -\xi^k \partial_k \phi$$

Is CS action invariant?

- CS action is gauge invariant, Galilei invariant
- but *not* diffeomorphism invariant

$$\delta S_{\text{CS}} = \frac{\nu m}{2\pi} \int d^3x \, \epsilon^{ij} E_i h_{jk} \dot{\xi}^k \qquad g = 0$$

A_μ does not transform like a one-form

$$\delta A_\mu = -\xi^k \partial_k A_\mu - A_k \partial_\mu \xi^k$$

$$\delta A_0 = -\xi^k \partial_k A_0 \quad -A_k \dot{\xi}^k + \frac{g}{4} \varepsilon^{ij} \partial_i (h_{jk} \dot{\xi}^k)$$

$$\delta A_i = -\xi^k \partial_k A_i - A_k \partial_i \xi^k - m h_{ik} \dot{\xi}^k$$

$$A_\mu = \mathcal{A}_\mu + \alpha_\mu$$

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$$A_\mu = \mathcal{A}_\mu + \alpha_\mu$$

Requirements for EFT

$$S_g[A_0,\, A_i,\, h_{ij}]$$

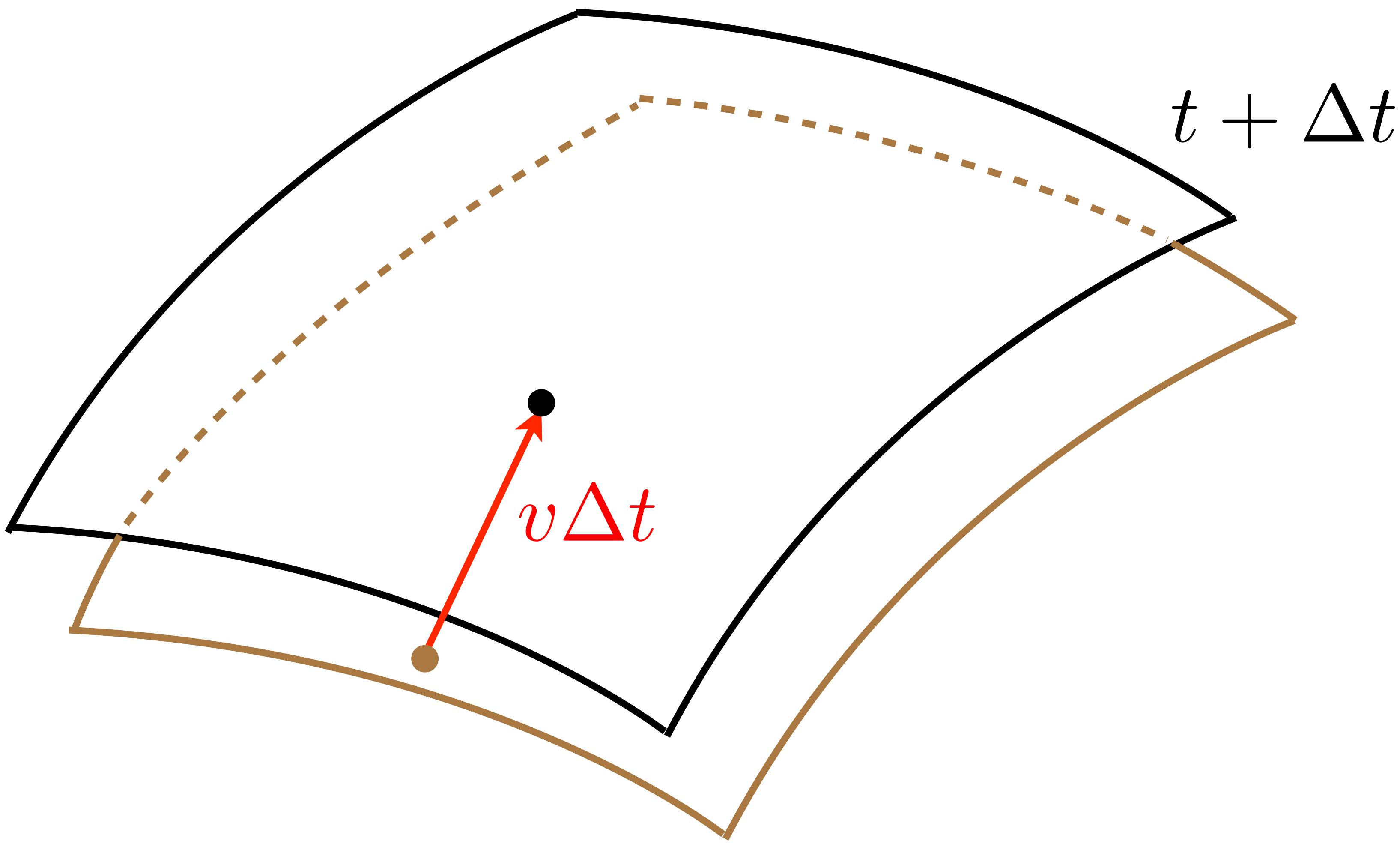
- Respect general coordinate invariance
- Reproduce all topological properties of the quantum Hall state
- Have regular limit of $m \to 0,\; g = 2$

LLL degenerate with zero energy
for any metric and B
(Aharonov-Casher)

What kind of geometry

- System does not live in a 3D Riemann space
- 2D Riemann manifold at any time slice
  - can parallel transport along equal-time slices, but not between different times

Velocity vector v



Use v to transform objects from one time slice to another

Cartan 1923-1924

Reformulation of Newton's theory of gravity

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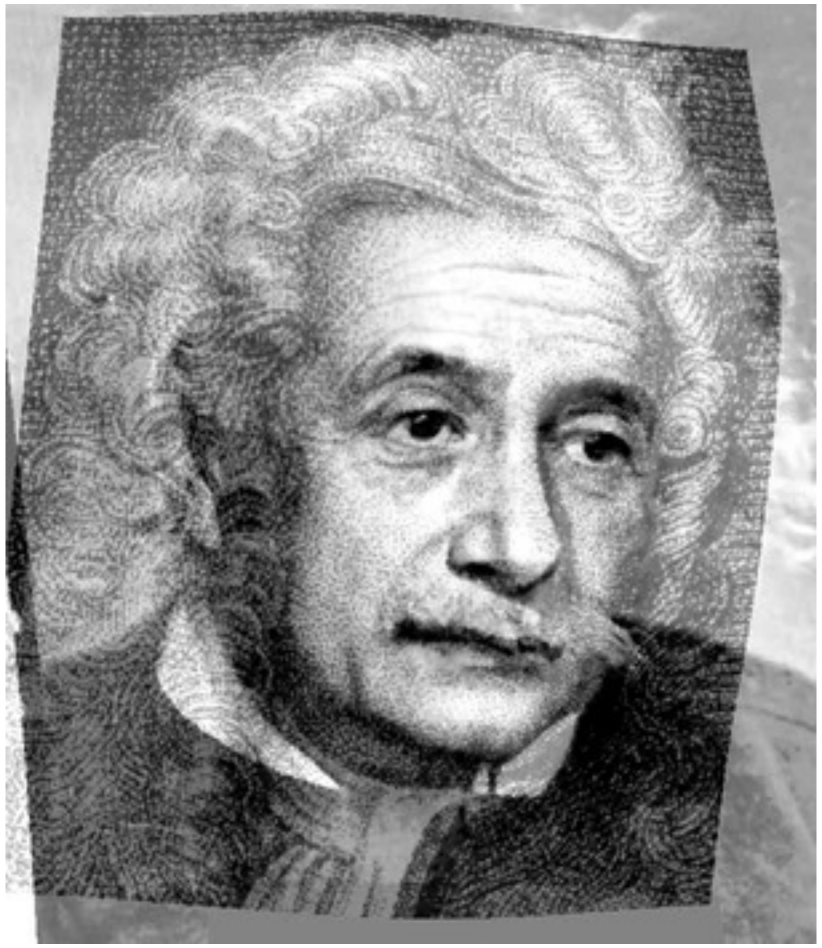
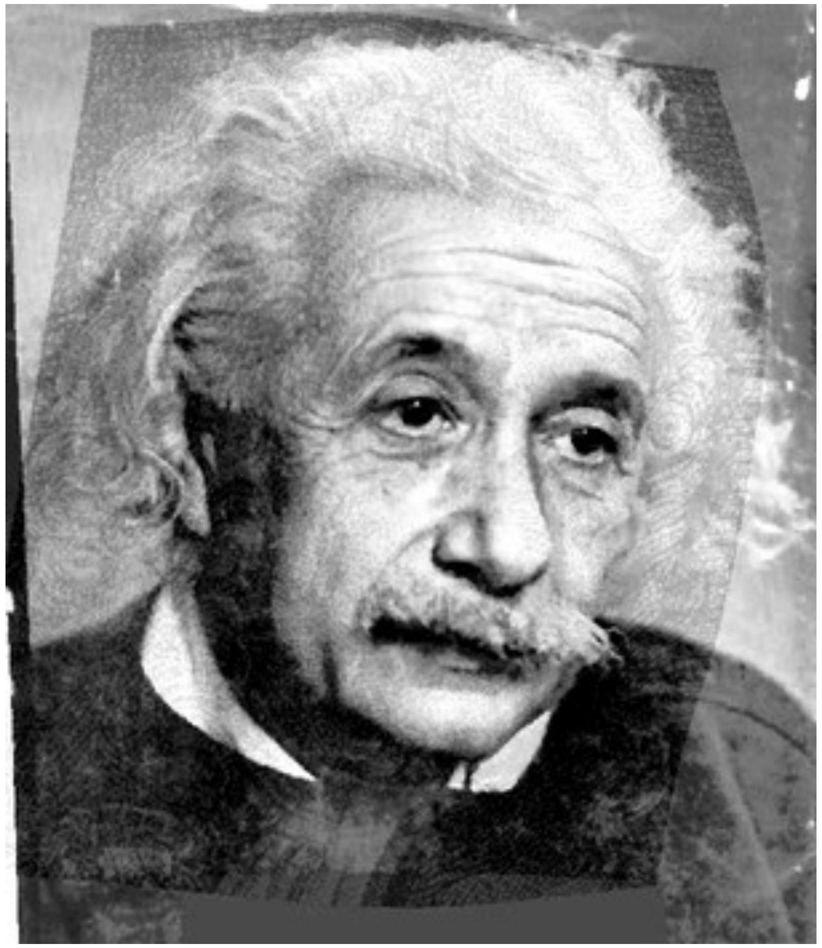


Cartan 1923-1924

Reformulation of Newton's theory of gravity



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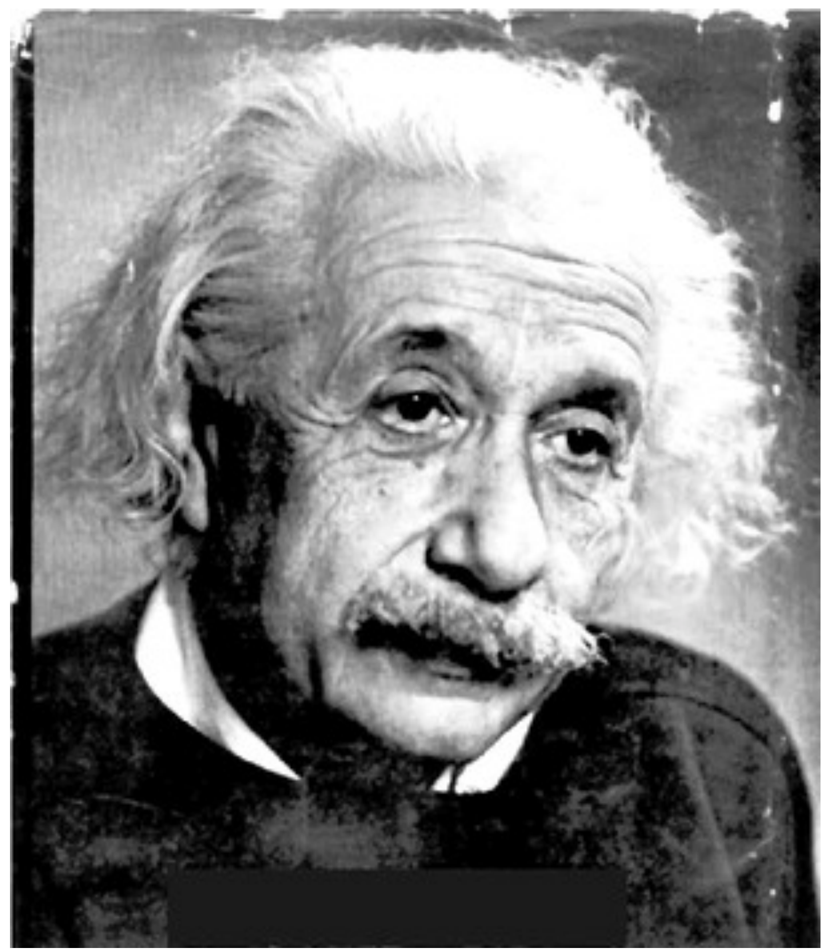


Newton-Cartan geometry

$(h^{\mu\nu},\, n_\mu,\, v^\mu)$ $\qquad dn = 0 \qquad h^{\mu\nu} n_\nu = 0$

$n_\mu v^\mu = 1$

$h^{\mu\nu} h_{\nu\lambda} = \delta^\mu_\lambda - v^\mu n_\lambda \qquad h_{\mu\nu} v^\nu = 0$

$dn = 0 \Rightarrow n = dt$ choose t to be time coordinate

$$h^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & h^{ij} \end{pmatrix} \qquad h_{\mu\nu} = \begin{pmatrix} v^2 & -v_j \\ -v_i & h_{ij} \end{pmatrix}$$

Connection in Newton-Cartan geometry

$$\Gamma^{\lambda}_{\mu\nu} = v^{\lambda}\partial_{(\mu}n_{\nu)} + \frac{1}{2}h^{\lambda\rho}(\partial_{\mu}h_{\rho\nu} + \partial_{\nu}h_{\rho\mu} - \partial_{\rho}h_{\mu\nu})$$

$$\nabla_{\lambda}h^{\mu\nu} = 0 \qquad \nabla_{\mu}n_{\nu} = 0 \qquad h_{\alpha[\mu}\nabla_{\nu]}v^{\alpha} = 0$$

Higher-dimensional interpretation

$$x^M = (x^-, x^\mu)$$

$$g_{MN} = \begin{pmatrix} 0 & n_\nu \\ n_\mu & h_{\mu\nu} \end{pmatrix} \qquad g^{MN} = \begin{pmatrix} 0 & v^\nu \\ v^\mu & h^{\mu\nu} \end{pmatrix}$$

$$\Gamma^L_{MN} = \frac{1}{2} g^{LR} (\partial_M g_{RN} + \partial_N g_{RM} - \partial_R g_{MN})$$

$$\Gamma^\lambda_{\mu\nu} = v^\lambda \partial_{(\mu} n_{\nu)} + \frac{1}{2} h^{\lambda\rho} (\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu})$$

Improved gauge potentials

- With v one can construct a gauge potential that transforms as a one-form

$$\tilde{A}_i = A_i + mv_i$$

$$\tilde{A}_0 = A_0 - \frac{mv^2}{2} - \frac{g}{4}\varepsilon^{ij}\partial_i v_j$$

$$\delta\tilde{A}_\mu = -\xi^k\partial_k\tilde{A}_\mu - \tilde{A}_k\partial_\mu\xi^k$$

What is v? Should be dynamically determined

Effective field theory

$$S = \frac{\nu}{4\pi}\int d^3x\, \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \int d^3x\, \sqrt{h}\, \rho v^\mu D_\mu \varphi$$

$$+ \int d^3x\, \frac{g-2}{8m} \epsilon^{\mu\nu\lambda} n_\mu \tilde{F}_{\nu\lambda} \; + S_0[\rho, v^i, h_{ij}]$$

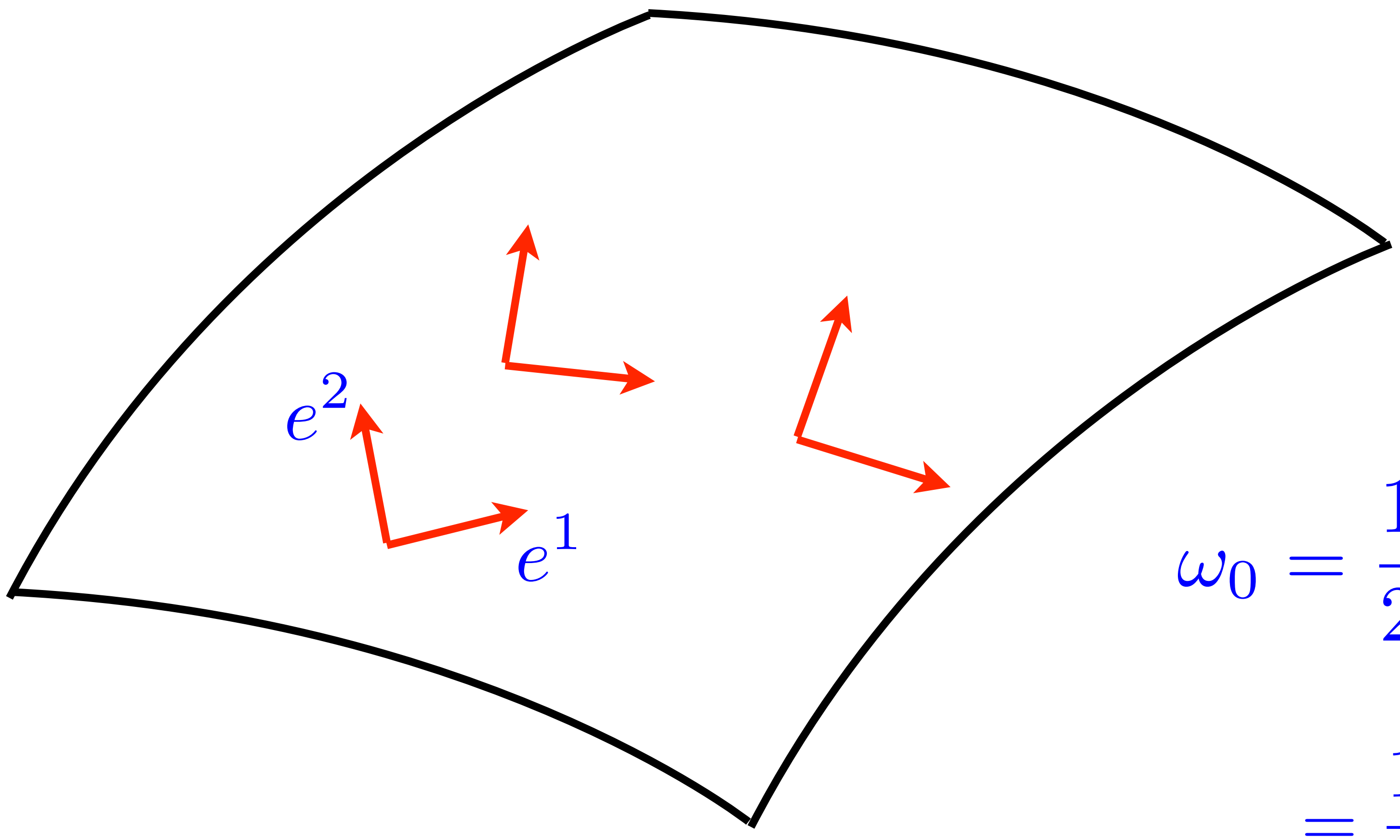
$$D_\mu \varphi = \partial_\mu \varphi - \tilde{A}_\mu + a_\mu - s\omega_\mu$$

related to "shift"

Integrating out $\rho,\ v^i,\ a_\mu$

$\Rightarrow$ effective action for $A_\mu,\ h^{ij}$

Spin connection



$$\omega_\mu = \frac{1}{2}\epsilon^{ab} e^{a\nu} \nabla_\mu e^b_\nu$$

$$\omega_0 = \frac{1}{2}\epsilon^{ab} e^{ai} \nabla_0 e^b_i$$

$$= \frac{1}{2}\epsilon^{ab} e^{ai} \partial_t e^b_i + \frac{1}{2}\epsilon^{ij}\partial_i v_j$$

$$\partial_1\omega_2 - \partial_2\omega_1 = \frac{1}{2}\sqrt{g}\, R$$

nonzero
in flat space

Wen-Zee shift

mixed CS $$\frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu\omega_\lambda = \frac{\kappa}{4\pi}\sqrt{g}\,A_0 R + \cdots$$

Total particle number on a manifold:

$$Q = \int \mathrm{d}^2x\,\sqrt{g}\,j^0 = \int \mathrm{d}^2x\,\sqrt{g}\left(\frac{\nu}{2\pi}B + \frac{\kappa}{4\pi}R\right) = \nu N_\phi + \kappa\chi$$

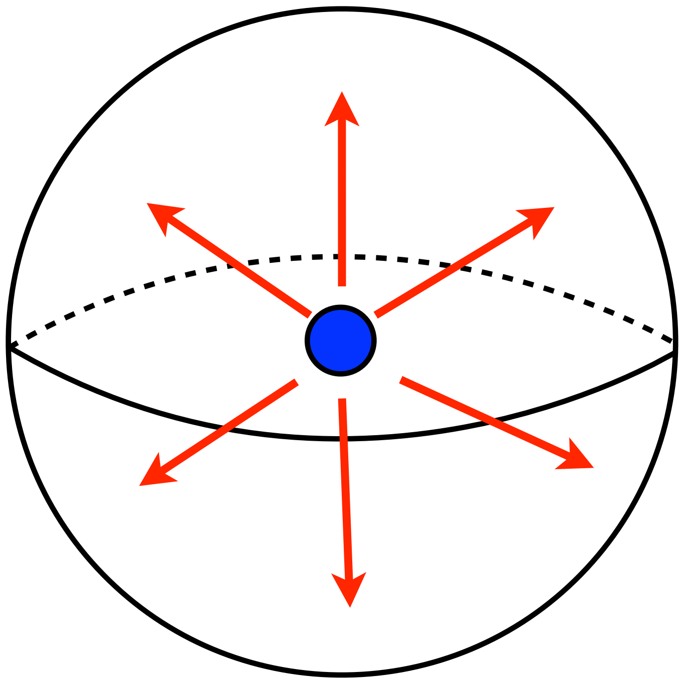
On a sphere: $$Q = \nu(N_\phi + \mathcal{S}), \quad \mathcal{S} = \frac{2\kappa}{\nu}$$

'shift'

IQH states: $\mathcal{S} = \nu$

Laughlin's states: $\mathcal{S} = 1/\nu$

Shift for IQH states



$N_\phi + 5$

$N_\phi + 3$

$N_\phi + 1$

$$Q = nN_\phi + n^2 = n(N_\phi + n)$$

Physical consequences

- Kohn's theorem
- Hall viscosity
- Hall conductivity: universal to order q^2
- Structure factor

Kohn's theorem

Constant B,
Response to homogeneous, time dependent E(t) independent of interactions: motion of center of mass

$$m\dot{v}_i = E_i + \epsilon_{ij} v_j B$$

Hall viscosity from WZ term

Avron, Seiler, Zograf

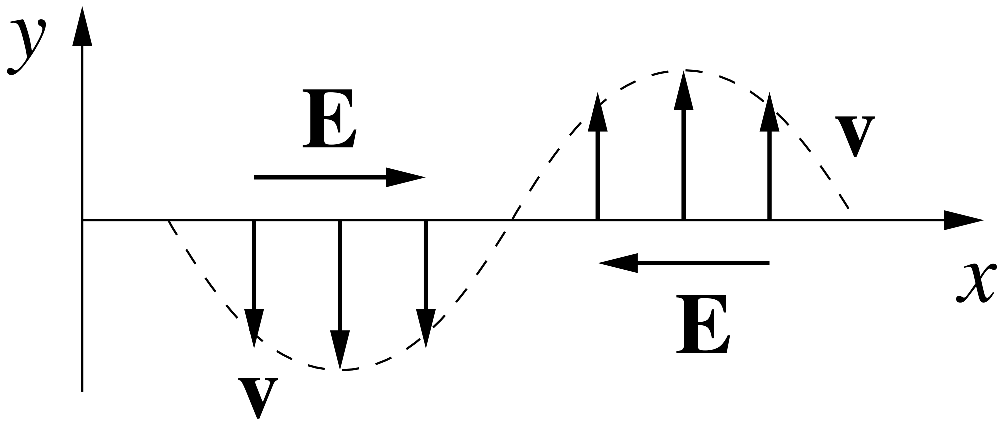
$$S_{\mathrm{WZ}} = -\frac{\kappa B}{16\pi}\epsilon^{ij} h_{ik}\partial_t h_{jk} + \cdots$$

$$\langle T_{xy}(T_{xx} - T_{yy})\rangle \sim i\eta^a\omega$$

$$\eta^a = \frac{1}{4}\mathcal{S}\rho$$

derived by N.Read
previously

$\sigma_{xy}(q)$



$$E_x = E\,\mathrm{e}^{iqx}$$

$$j_y = \sigma_{xy}(q) E_x$$

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 + C_2 (q\ell)^2 + \mathcal{O}(q^4 \ell^4)$$

$$C_2 = \frac{\mathcal{S}}{4} - 1$$

correct for ν=1 state

Structure factor and shift

- Non-universal part of the action: leading contributions are

$$\sigma_{\bar{z}\bar{z}} F(i\partial_t) \sigma_{zz} \qquad\qquad \sigma_{\mu\nu} = \pounds_v h_{\mu\nu}$$

positivity of spectral densities of stress-stress correlators:

equal time density-density corr

$$\lim_{k \to 0} \frac{\bar{s}(k)}{(k\ell)^4} \geq \frac{|\mathcal{S} - 1|}{8}$$ (Haldane 2009)

Inequality saturated by Laughlin's wave function

Conclusion and outlook

- Quantum Hall states naturally live in Newton-Cartan geometry
- Symmetry determines the q^2 correction to Hall conductivity, other physical quantities
- Further questions:
  - edge states?
  - Relationship with conformal field theories?
  - Meaning of Laughlin's wave function?
  - Implications for holographic realizations?

What is Hall viscosity?

Standard fluid dynamics: $\partial_t \rho + \partial_i j^i = 0$ continuity eq.

$\partial_t j^i + \partial_j T^{ij} = 0$ Navier-Stokes eq.

$$j^i = \rho v^i$$

$$T^{ij} = \rho v^i v^j + P\delta^{ij} - \eta V_{ij} \qquad V_{ij} = \frac{1}{2}(\partial_i v^j + \partial_j v^i)$$

In 2 spatial dimensions, it is possible to write

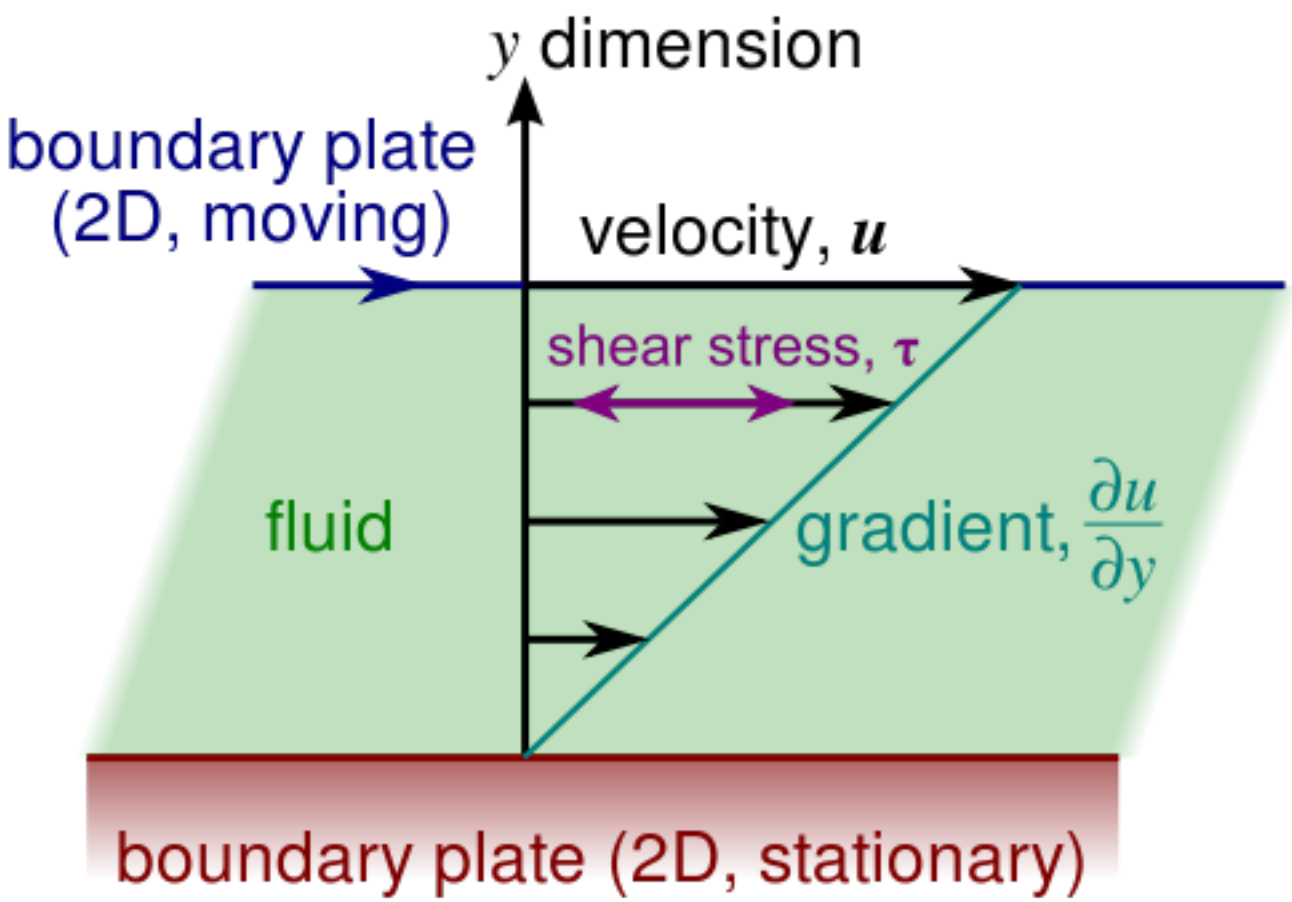
$$T^{ij} = \cdots - \eta_H(\epsilon^{ik}V^{kj} + \epsilon^{jk}V^{ki})$$

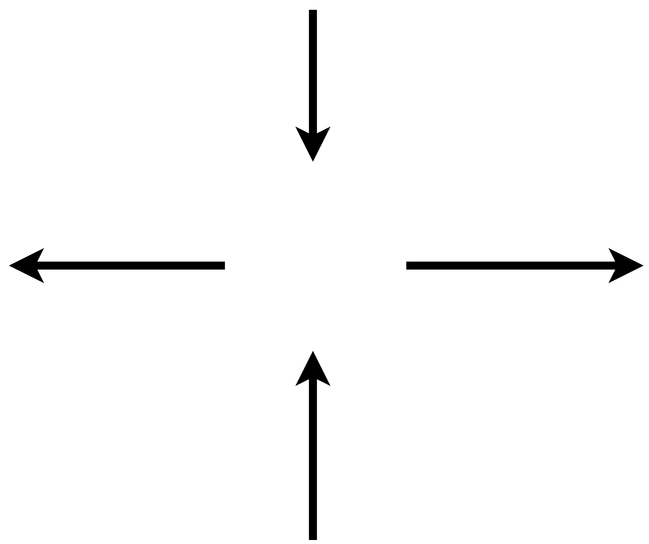
breaks parity

dissipationless

Hall viscosity (Avron Seiler Zograf)

Hall viscosity in picture

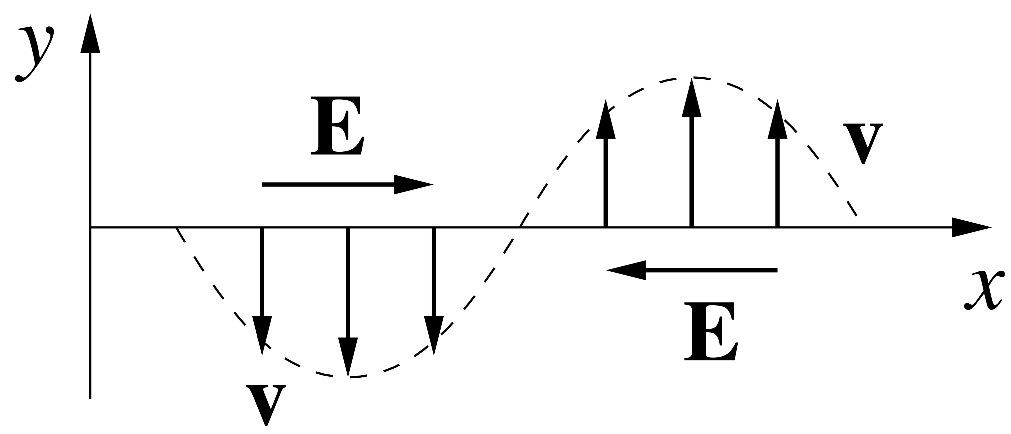




Hall shear stress

Physical interpretation

- First term: Hall viscosity



$$\partial_x v_y + \cancel{\partial_y v_x} \neq 0$$

$$T_{xx} = T_{xx}(x) \neq 0$$

additional force $F_x \sim \partial_x T_{xx}$
Hall effect: additional contribution to v_y

Physical interpretation (II)

- 2nd term: more complicated interpretation

Fluid has nonzero angular velocity

$$\Omega(x) = \frac{1}{2}\partial_x v_y = -\frac{cE_x'(x)}{2B}$$

$$\delta B = 2mc\Omega/e$$

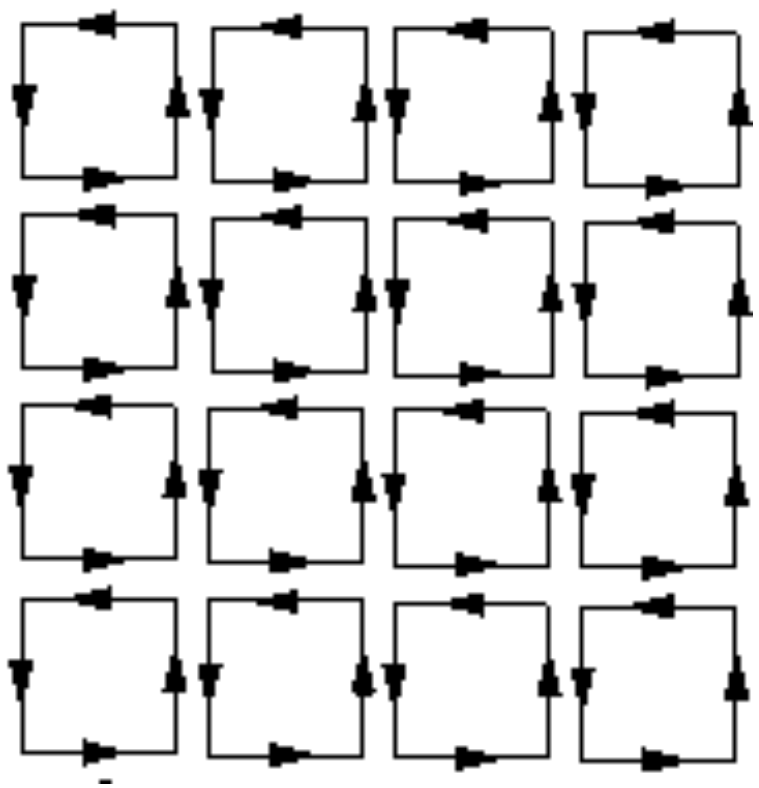
Coriolis=Lorentz

Hall fluid is diamagnetic: $d\epsilon = -M dB$

M is spatially dependent M=M(x)

Extra contribution to current $\mathbf{j} = c\,\hat{\mathbf{z}} \times \nabla M$

Current ~ gradient of magnetization



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