Geometry and Thermodynamics of Black Holes in Magnetic Fields

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We investigate the global structure of rotating black holes immersed in an external magnetic field; i.e. a "Magnetised Kerr-Newman Spacetime." In general, there is an ergoregion extending to infinity, but this is avoided in a special case. We discuss the thermodynamics, and also extensions to black holes in supergravity.

Based on work with Gary Gibbons, Abid Mujtaba, and more recent work with Yi Pang, Mirjam Cvetič and Zain Saleem.

Astrophysical Black Hole in an External Magnetic Field

- Black holes are known to occur at the centre of most galaxies. These will in general have large angular momentum, owing to the accretion processes involving infalling matter.

- Blandford and Znajek proposed a mechanism in which the magnetic field generated by the circulation of charged particles in an equatorial accretion disc leads to electrostatic fields that accelerate charged particles near the horizon, generating radiation that itself gives rise to electron-positron pair production. This in turn can lead to a net extraction of energy from the black hole via a Penrose-type process, thus possibly providing a powerful energy source at the galactic nucleus.

- Wald considered a neutral rotating black hole (Kerr solution) in the background of an external magnetic field. As with Blandford and Znajek, the magnetic field was treated as a "test field"; i.e. the back-reaction of the magnetic field on the geometry was neglected.

- Wald argued that if the black hole has mass m and angular momentum $j = am$, then in the presence of an external magnetic field B it becomes energetically favourable for the black hole to acquire an electric charge Q, given by

  $$Q = 2jB\,.$$

  This could occur via a vacuum breakdown through pair production.

Exact Solution with Back Reaction?

- The notion of a black hole immersed in an "originally uniform magnetic field" is a little problematic in general relativity, since the total electromagnetic energy will be infinite, and spacetime will necessarily become highly non-Minkowskian at infinity.

- An exact solution that is perhaps the closest to this is provided, in the absence of rotation, by the Schwarzschild-Melvin metric, with

$$
\begin{aligned}
ds^2 &= H\,[-f dt^2 + f^{-1} dr^2 + r^2 d\theta^2] + \frac{r^2 \sin^2\theta}{H}\, d\phi^2\,,\\
f &= 1 - \frac{2m}{r}\,, \qquad H = (1 + \tfrac{1}{4} B^2 r^2 \sin^2\theta)^2\,,\\
A &= \frac{B r^2 \sin^2\theta}{\sqrt{H}}\, d\phi\,.
\end{aligned}
$$

- This is asymptotic to the Melvin universe at large r. (i.e. when $r >> m$). It approaches the Minkowski metric near the axis ($\theta = 0$ or $\theta = \pi$), but is highly non-Minkowskian away from the axis. This is most clearly seen in cylindrical coordinates

$$\rho = r \sin\theta\,, \qquad z = r \cos\theta\,.$$

The Melvin Universe

- In cylindrical coordinates, Melvin solution ($m = 0$) is

$$\begin{aligned} ds^2 &= H\,(-dt^2 + dz^2 + d\rho^2) + H^{-1}\,\rho^2\,d\phi^2\,, \\ F &= dA = B\,H^{-1}\,\rho d\rho \wedge d\phi\,, \qquad H = (1 + \tfrac{1}{4}B^2\,\rho^2)^2\,. \end{aligned}$$

- If $B^2\rho^2 << 1$ (i.e. much closer to the z axis than the Melvin radius $\rho_{\text{Melvin}} = 2/B$), the metric and field strength become

$$\begin{aligned} ds^2 &= -dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2 = -dt^2 + dz^2 + dx^2 + dy^2\,, \\ F &= B\,\rho d\rho \wedge d\phi = B\,dx \wedge dy\,, \end{aligned}$$

  where $x = \rho\cos\phi$, $y = \rho\sin\phi$. This is just Minkowski spacetime with a uniform magnetic field B along the z axis.

- Melvin solution has $|\text{Riem}|^2 = \frac{1}{4}B^4(80 - 24B^2\rho^2 + 3B^4\rho^4)H^{-4}$. It is everywhere non-singular, and it describes a parallel bundle of magnetic flux held together by its own gravitational attraction.

- Schwarzschild-Melvin is a non-rotating black hole immersed in this asymptotic background.

Rotating Generalisations?

- Rotating generalisations have been obtained before. The solutions are too complicated to work with reliably by hand, so it is useful first to reconstruct from scratch, using algebraic computing. We used a solution-generating technique to "magnetise" Kerr-Newman, via a Kaluza-Klein reduction to three dimensions. The starting point is Einstein-Maxwell theory in four dimensions:

$$\mathcal{L}_4 = R - F^2\,.$$

We then assume a metric with a spacelike Killing vector $\partial/\partial\phi$ and reduce to three dimensions:

$$\begin{aligned} ds_4^2 &= e^{2\varphi}\, d\bar{s}_3^2 + e^{-2\varphi}\,(d\phi + 2\bar{\mathcal{A}})^2\,,\\ A &= \bar{A} + \chi\,(d\phi + 2\bar{\mathcal{A}})\,. \end{aligned}$$

This gives the three-dimensional Lagrangian

$$\bar{\mathcal{L}}_3 = \bar{R} - 2(\partial\varphi)^2 - 2e^{\varphi}\,(\partial\chi)^2 - e^{-4\varphi}\bar{\mathcal{F}}^2 - e^{-2\varphi}\bar{F}^2\,,$$

where $\bar{\mathcal{F}} = d\bar{\mathcal{A}}$ and $\bar{F} = d\bar{A} + 2\chi\, d\bar{\mathcal{A}}$.

- Dualising $\bar{A}$ and $\bar{\mathcal{A}}$; $e^{-2\varphi}\,\bar{*}\bar{F} = d\psi$ and $e^{-4\varphi}\,\bar{*}\bar{\mathcal{F}} = d\sigma - 2\chi\, d\psi$ gives 3D gravity coupled to a nonlinear sigma model:

$$\mathcal{L}_3 = \bar{R} - 2(\partial\varphi)^2 - 2e^{\varphi}\,[(\partial\chi)^2 + (\partial\psi)^2] - 2e^{2\varphi}\,(\partial\sigma - 2\chi\partial\psi)^2\,.$$

$SU(2,1)$ Transformations, and Magnetisation

- The scalar sigma model metric

$$d\Sigma^2 = d\varphi^2 + e^{2\varphi}\,[d\chi^2 + d\psi^2] + e^{4\varphi}\,(d\sigma - 2\chi d\psi)^2$$

is $\widetilde{CP}^2 = SU(2,1)/U(2)$, the non-compact (negative curvature) version of CP^2. Defining matrices ${E_a}^b$ with just a 1 at row a, column b, and $\mathcal{H} = {E_0}^0 - {E_2}^2$, we can parameterise a coset representative, and $SU(2,1)$ element, as

$$\mathcal{V} = e^{\varphi\mathcal{H}} e^{-2\mathrm{i}\,\sigma {E_0}^2} e^{\sqrt{2}\chi({E_0}^1 + {E_1}^2)} e^{-\mathrm{i}\,\sqrt{2}\psi({E_0}^1 - {E_1}^2)}\,.$$

We have

$$\mathcal{V}^\dagger \eta \mathcal{V} = \eta\,, \qquad \eta = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\,,$$

with η being the invariant metric of $SU(2,1)$. The Lagrangian can be written as $\mathcal{L}_3 = R - \mathrm{tr}(\mathcal{M}^{-1}\,\partial\mathcal{M})^2$ where $\mathcal{M} = \mathcal{V}^\dagger\mathcal{V}$. $\mathcal{L}_3$ is manifestly invariant under $\mathcal{M} \longrightarrow \mathcal{M}' = U^\dagger \mathcal{M} U$ where U is any constant $SU(2,1)$ matrix, obeying $U^\dagger \eta U = \eta$.

Magnetisation of Kerr-Newman Black Hole

- We start from the rotating, charged Kerr-Newman black hole:

$$\begin{aligned} ds_4^2 &= -f\,dt^2 + R^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \frac{\Sigma \sin^2\theta}{R^2}\,(d\phi - \omega dt)^2\,, \\ A &= \Phi_0\,dt + \Phi_3\,(d\phi - \omega dt)\,, \end{aligned}$$

where

$$\begin{aligned} R^2 &= r^2 + a^2\cos^2\theta\,, \qquad \Delta = (r^2+a^2) - 2mr + q^2\,, \\ \omega &= \frac{a(2mr - q^2)}{\Sigma}\,, \qquad \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta\,, \\ f &= \frac{R^2\Delta}{\Sigma}\,, \qquad \Phi_0 = \frac{qr(r^2+a^2)}{\Sigma}\,, \qquad \Phi_3 = -\frac{aqr\sin^2\theta}{R^2}\,. \end{aligned}$$

- We now reduce to three dimensions on $\partial/\partial\phi$, apply the magnetising transformation, which turns out to be

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \frac{B}{\sqrt{2}} & 1 & 0 \\ \frac{B^2}{4} & \frac{B}{\sqrt{2}} & 1 \end{pmatrix}\,,$$

and then retrace the steps back to four dimensions.

Further $SU(2,1)$ Transformations

- Amongst the other possible transformations within the $SU(2,1)$ symmetry group are electrifications (adding an external electric field):

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \frac{(B+\mathrm{i}\,E)}{\sqrt{2}} & 1 & 0 \\ \frac{(B^2+E^2)}{4} & \frac{(B-\mathrm{i}\,E)}{\sqrt{2}} & 1 \end{pmatrix} .$$

- Electric/magnetic duality transformations, which in four dimensions take the form

$$F \longrightarrow F' = F\,\cos\alpha + *F\,\sin\alpha\,,$$

This is implemented in 3 dimensions by the $SU(2,1)$ matrix

$$U = \begin{pmatrix} e^{-\frac{\mathrm{i}}{3}\alpha} & 0 & 0 \\ 0 & e^{\frac{2\mathrm{i}}{3}\alpha} & 0 \\ 0 & 0 & e^{-\frac{\mathrm{i}}{3}\alpha} \end{pmatrix} .$$

Magnetisation of Kerr-Newman Black Hole

- Applied to Schwarzschild, it indeed gives Schwarzschild-Melvin.
- Applied to the Kerr-Newman metric (with electric and magnetic charges q and p, it gives a rather indigestible result:

$$
\begin{aligned}
ds_4^2 &= H\Big[-f dt^2 + R^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big)\Big] + \frac{\Sigma \sin^2\theta}{H R^2}(d\phi - \omega dt)^2\,,\\
A &= \Phi_0\, dt + \Phi_3\,(d\phi - \omega dt)\,,
\end{aligned}
$$

where

$$
H = 1 + \frac{H_{(1)}B + H_{(2)}B^2 + H_{(3)}B^3 + H_{(4)}B^4}{R^2}
$$

with

$$
\begin{aligned}
H_{(1)} &= 2aqr\sin^2\theta - 2p(r^2+a^2)\cos\theta\,,\\
H_{(2)} &= \tfrac{1}{2}[(r^2+a^2)^2 - a^2\Delta\,\sin^2\theta]\,\sin^2\theta + \tfrac{3}{2}\tilde{q}^2(a^2 + r^2\cos^2\theta)\,,\\
H_{(3)} &= -pa^2\Delta\sin^2\theta\cos\theta - \frac{qa\Delta}{r}[r^2(3-\cos^2\theta)\cos^2\theta + a^2(1+\cos^2\theta)] + \frac{aq(r^2+a^2)^2(1+\cos^2\theta)}{2r}\\
&\quad -\tfrac{1}{2}p(r^4-a^4)\sin^2\theta\cos\theta + \frac{q\tilde{q}^2a[(2r^2+a^2)\cos^2\theta + a^2]}{2r} - \tfrac{1}{2}p\tilde{q}^2(r^2+a^2)\cos^3\theta\,,\\
H_{(4)} &= \tfrac{1}{16}(r^2+a^2)^2R^2\sin^4\theta + \tfrac{1}{4}ma^2r(r^2+a^2)\sin^6\theta + \tfrac{1}{4}ma^2\tilde{q}^2r(\cos^2\theta - 5)\sin^2\theta\cos^2\theta\\
&\quad +\tfrac{1}{4}m^2a^2[r^2(\cos^2\theta-3)^2\cos^2\theta + a^2(1+\cos^2\theta)^2]\\
&\quad +\tfrac{1}{8}\tilde{q}^2(r^2+a^2)(r^2+a^2+a^2\cos^2\theta)\sin^2\theta\cos^2\theta + \tfrac{1}{16}\tilde{q}^4[r^2\cos^2\theta + a^2(1+\sin^2\theta)^2]\cos^2\theta
\end{aligned}
$$

and we have defined $\tilde{q}^2 \equiv q^2 + p^2$.

The function ω is given by

$$\omega = \frac{(2mr - \tilde{q}^2)a + \omega_{(1)} B + \omega_{(2)} B^2 + \omega_{(3)} B^3 + \omega_{(4)} B^4}{\Sigma},$$

where

$$\begin{aligned}
\omega_{(1)} &= -2qr(r^2+a^2) + 2ap\Delta\cos\theta\,,\\
\omega_{(2)} &= -\tfrac{3}{2}a\tilde{q}^2(r^2+a^2+\Delta\cos^2\theta)\,,\\
\omega_{(3)} &= 4qm^2a^2r + \tfrac{1}{2}ap\tilde{q}^4\cos^3\theta - \tfrac{1}{2}qr(r^2+a^2)[r^2-a^2+(r^2+3a^2)\cos^2\theta]\\
&\quad +\tfrac{1}{2}ap(r^2+a^2)[3r^2+a^2-(r^2-a^2)\cos^2\theta]\cos\theta + \tfrac{1}{2}q\tilde{q}^2r[(r^2+3a^2)\cos^2\theta - 2a^2]\\
&\quad +\tfrac{1}{2}ap\tilde{q}^2[3r^2+a^2+2a^2\cos^2\theta]\cos\theta - am\tilde{q}^2(2aq+pr\cos^3\theta)\\
&\quad +qm[r^4-a^4+r^2(r^2+3a^2)\sin^2\theta] - apmr[2R^2+(r^2+a^2)\sin^2\theta]\,,\\
\omega_{(4)} &= \tfrac{1}{2}a^3m^3r(3+\cos^4\theta) - \tfrac{1}{16}a\tilde{q}^6\cos^4\theta - \tfrac{1}{8}a\tilde{q}^4[r^2(2+\sin^2\theta)\cos^2\theta + a^2(1+\cos^2\theta)]\\
&\quad +\tfrac{1}{16}a\tilde{q}^2(r^2+a^2)[r^2(1-6\cos^2\theta+3\cos^4\theta) - a^2(1+\cos^4\theta)] - \tfrac{1}{4}a^3m^2\tilde{q}^2(3+\cos^4\theta)\\
&\quad +\tfrac{1}{4}am^2[r^4(3-6\cos^2\theta+\cos^4\theta) + 2a^2r^2(3\sin^2\theta - 2\cos^4\theta) - a^4(1+\cos^4\theta)]\\
&\quad +\tfrac{1}{8}am\tilde{q}^4r\cos^4\theta + \tfrac{1}{4}am\tilde{q}^2r[2r^2(3-\cos^2\theta)\cos^2\theta - a^2(1-3\cos^2\theta-2\cos^4\theta)]\\
&\quad +\tfrac{1}{8}amr(r^2+a^2)[r^2(3+6\cos^2\theta-\cos^4\theta) - a^2(1-6\cos^2\theta-3\cos^4\theta)]\,.
\end{aligned}$$

- This solution is often assumed to be asymptotic to the Melvin universe. In general, it isn't. Look at g_{tt} :
- At large r, for generic θ, the metric component g_{tt} *does* behave just like in Melvin, with

$$g_{tt} \longrightarrow -\frac{1}{16}B^4r^4\sin^4\theta\,.$$

  But, near to the z axis at large distance, g_{tt} goes (arbitrarily) positive, signaling an ergoregion extending out to infinity.

- This is evident in the cylindrical coordinates $\rho = r\sin\theta$ and $z = r\cos\theta$ where, for fixed ρ, a large z expansion gives

$$g_{tt} \longrightarrow \quad \frac{16B^6(q+amB)^2\rho^2}{W(\rho)^2}\, z^2 - \frac{4B^6(q+amB)[8qm + aB(q^2+4m^2)]\rho^2}{W(\rho)^2}\, z + \mathcal{O}(z^0)\,,$$

So not asymptotically Melvin near the axis, and an infinite ergoregion.

- The energy of a particle with 4-momentum p_μ, measured with respect to a future-directed timelike Killing vector K, is

$$E = -K^\mu\, p_\mu$$

Here, we are thinking of $K = \partial/\partial t$. If K becomes spacelike, i.e. if there is an ergoregion, then the energy can become negative. Can then extract energy by the **Penrose Process**, in which a particle decays into two, one with negative energy that falls into the black hole, and the other, with more energy than the original particle, that escapes.

- In the original Kerr-Newman black hole there is a unique choice of asymptotically-timelike Killing vector (i.e. $K = \partial/\partial t$). In the magnetised solution we find that provided we take $q = -amB$, there is a family of possible choices $K = \partial/\partial t + \Omega\, \partial/\partial\phi$, if the angular velocity Ω lies in an appropriate range.

$q = -amB$ Magnetised Kerr-Newman

- With $q = -amB$, the problem of an ergoregion at infinity can be avoided. Using the coordinate $\tilde{\phi} = \phi - \Omega\, t$, first look at the off-diagonal metric component $g_{t\tilde{\phi}}$ at large z:

$$g_{t\tilde{\phi}} = \frac{2(8\Omega + 12am^2B^4 + a^3m^2B^6)\rho^2}{(4 + a^2m^2B^4 + B^2\rho^2)^2} + \mathcal{O}(\frac{1}{z})\,.$$

Choosing $\Omega = \Omega_s$, where

$$\Omega_s = -\tfrac{1}{8}am^2B^4(12 + a^2B^2)\,,$$

we find at large z that

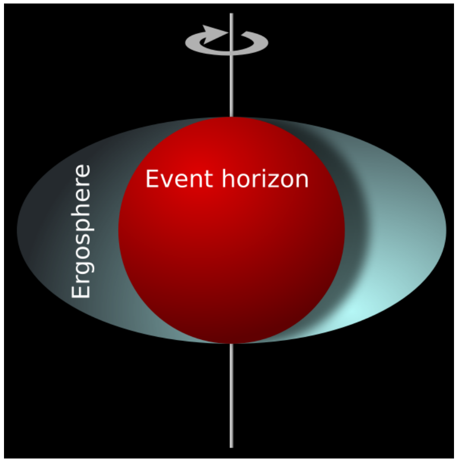
$$\begin{aligned} g_{t\tilde{\phi}} &= -\frac{8amB^2(4 + a^2m^2B^4)\rho^2}{(4 + a^2m^2B^4 + B^2\rho^2)^2}\,\frac{1}{z} + \mathcal{O}(\frac{1}{z^2})\,,\\ g_{tt} &= -\tfrac{1}{16}(4 + a^2m^2B^4 + B^2\rho^2)^2 + \mathcal{O}(\frac{1}{z})\,, \end{aligned}$$

and so the metric near the axis is then genuinely asymptotic to the static Melvin universe.

Asymptotically-Melvin Magnetised Kerr-Newman

- In the asymptotically static Melvin frame with timelike Killing vector $K = \partial/\partial t + \Omega_s\,\partial/\partial\phi$, the $q = -amB$ magnetised Kerr-Newman-Melvin solution will still have an ergoregion in the form of an oblate spheroid outside the outer horizon (very like in the Kerr solution):

- In the Kerr solution, $\partial/\partial t$ is the **unique** Killing vector that is timelike at infinity, and hence it is the unique choice as generator of time translations:



- In the magnetised Kerr-Newman solution there is in fact a range of angular velocities Ω around $\Omega = \Omega_s$ for which the ergoregion is of only finite extent, and confined to the vicinity of the black hole.

Conserved Charge and Angular Momentum

- These can be calculated most simply in the KK reduced 3D language. The physical charge is given by the Gaussian integral

$$
\begin{aligned}
Q &= \frac{1}{4\pi}\int_{S^2} {*F} = \frac{\Delta\phi}{4\pi}\int_{S^2} e^{-4\varphi}\,\bar{*}\bar{F}\,,\\
&= \frac{\Delta\phi}{4\pi}\int d\psi = \frac{\Delta\phi}{4\pi}\Big[\psi\Big]_{\theta=0}^{\theta=\pi} = q(1-\tfrac{1}{4}q^2B^2) + 2amB\,.
\end{aligned}
$$

$\Delta\phi$ is the period of the azimuthal coordinate ϕ, determined by requiring no conical singularities at N and S poles of S^2:

$$
\Delta\phi = 2\pi\Big[1+\tfrac{3}{2}q^2B^2 + 2aqmB^3 + (a^2m^2+\tfrac{1}{16}q^4)B^4\Big]\,.
$$

- For angular momentum, following Wald, for every Killing vector ξ the Noether method gives a conserved charge

$$
\mathcal{Q}[\xi] = \frac{1}{16\pi}\int_{S^2} {*\mathcal{P}}\,, \qquad \mathcal{P} = d\xi + 4(\xi^\mu A_\mu)\,F\,.
$$

Taking the azimuthal Killing vector with $\xi = \partial/\partial\tilde{\phi}$, where $\tilde{\phi} = (2\pi/\Delta\phi)\,\phi$ has canonical 2π period, then as a 1-form, and in the 3D language, we have

$$\xi = \frac{\Delta\phi}{2\pi} e^{-2\varphi} (d\phi + 2\bar{\mathcal{A}}) \qquad \text{and hence}$$

$$\mathcal{P} = \frac{\Delta\phi}{\pi}\Big[e^{-2\varphi} \bar{\mathcal{F}} + 2\chi \bar{F} - (e^{-2\varphi} d\varphi - 2\chi d\chi) \wedge (d\phi + 2\bar{\mathcal{A}})\Big] .$$

This gives

$$*\mathcal{P} = \frac{\Delta\phi}{\pi} \Big[d\sigma \wedge (d\phi + 2\bar{\mathcal{A}}) - \bar{*}(d\varphi - 2e^{2\varphi} \chi d\chi)\Big] ,$$

and so the angular momentum is

$$J = \frac{1}{16\pi} \int_{S^2} *\mathcal{P} = \left(\frac{\Delta\phi}{4\pi}\right)^2 \int d\sigma = \left(\frac{\Delta\phi}{4\pi}\right)^2 \Big[\sigma\Big]_{\theta=0}^{\theta=\pi}$$

$$= am + \tfrac{1}{2}q^3 B + \tfrac{9}{2}amq^2B^2 + \tfrac{1}{2}qB^3(12a^2m^2 - q^4) + \tfrac{1}{16}amB^4(48a^2m^2 - 11q^4) .$$

(Note: This includes the electromagnetic field contribution as well as the purely gravitational "Komar" contribution from $*d\xi$. Each by itself is singular in the Melvin background, but the sum is non-singular.)

Thermodynamics

- We can expect that the first law should take the form

$$dE = TdS + \Omega dJ + \Phi dQ - \mu dB\,,$$

We have calculated the conserved charges Q and J, and the calculation of the entropy S and temperature T is straightforward. Unusual asymptotics of the Melvin background mean that calculating the mass E, the angular velocity Ω the electrostatic potential Φ, and the magnetic moment μ is trickier.

- The mass is given by (??) $E = (\Delta\phi/2\pi)\, m$. We can seek solution of the first law for Ω, Φ and μ. This works, and is non-trivial since we have 4 equations for 3 unknowns.

- Solution is complicated. At leading order, we have

$$\begin{aligned} \mu &= aq(1 + a^2 m^2 B^4) + \mathcal{O}(q^2)\,, \\ \Omega &= \frac{a}{r_+^2 + a^2} - \frac{2qBr_+}{r_+^2 + a^2} + \mathcal{O}(B^2)\,, \\ \Phi &= \frac{qr_+}{r_+^2 + a^2} - \frac{3aq^2B}{2(r_+^2 + a^2)} + \mathcal{O}(B^2)\,. \end{aligned}$$

Magnetic moment $\mu \sim aq \sim JQ/M$, so at leading order we recover the well-known black hole gyromagnetic ratio $g \sim 2$.

- Satisfies Smarr-type relation $E = 2TS + 2\Omega J + \Phi Q + \mu B$.

Magnetised Black Holes in STU Supergravity

- We can magnetise black holes in supergravities too. Currently we (Cvetič, Gibbons, Saleem, CNP) are looking in the four-dimensional STU model ($\mathcal{N} = 2$ supergravity coupled to 3 vector multiplets). Now have four independent $U(1)$ gauge fields, so four charges and four magnetic fields. We no use the $O(4,4)$ symmetry of the associated KK reduced 3-dimensional sigma model to generate the magnetised solutions.

- As well as magnetising electric black holes, it is also now of interest to magnetise magnetically-charged black holes. Consider static black holes for simplicity. The metric is

$$\begin{aligned}
ds^2 &= H\, ds_3^2 + H^{-1} \sin^2\theta\, (d\phi - \omega dt)^2\,, \\
ds_3^2 &= -r(2-2m)dt^2 + R_1 r_2 r_3 r_4 \Big(d\theta^2 + \frac{R_1 r_2 r_3 r_4}{r(r-2m)}\, dr^2 \Big)\,, \\
\omega &= \sum_{i=1}^4 \Big[-\frac{p_i B_i}{2r_i} + \frac{p_i B_1 B_2 B_3 B_4 [r_i + (r-2m)\cos^2\theta] r}{8 B_i\, r_i} \Big]\,, \\
H^2 &= \frac{1}{r_1 r_2 r_3 r_4} \prod_{i=1}^4 \Big((1 + \tfrac{1}{2} B_i p_i \cos\theta)^2 + \frac{B_i^2 r_1 r_2 r_3 r_4}{r_i^2} \sin^2\theta \Big]\,, \\
r_i &= r + 2m \sinh^2\delta_i\,, \qquad p_i = 2m \sinh\delta_i \cosh\delta_i\,.
\end{aligned}$$

Conical Singularities and Force Balancing

- To avoid conical singularity at the North (South) pole, ϕ should be identified with different periods at each:

$$\text{N pole}: \quad \Delta\phi = 2\pi \prod_i (1 + \tfrac{1}{2} B_i p_i)\,,$$

$$\text{S pole}: \quad \Delta\phi = 2\pi \prod_i (1 - \tfrac{1}{2} B_i p_i)\,.$$

  In general, there is no single choice of period $\Delta\phi$ that eliminates both conical singularities. This reflects the fact that there is a net force on the black hole, and it must therefore be supported by a "strut," which is described by an energy-momentum tensor with a delta-function singularity on the axis. This happens, for example, for a magnetic Reissner-Nordström black hole in an external magnetic field. (Or the S-dual, an electric black hole in an external electric field.)

- With four separate gauge fields in the STU model we can tune the fields and the charges to achieve a force balance:

$$\prod_i (1 + \tfrac{1}{2} B_i p_i) = \prod_i (1 - \tfrac{1}{2} B_i p_i)\,.$$

  This provides families of non-singular magnetic black holes in external magnetic fields. These provide interesting models for investigating thermodynamics of systems in external backgrounds.

Conclusions and Open Problems

- A rotating generalisation of the asymptotically-Melvin magnetised Schwarzschild solution does exist, but it must carry a very specific electric charge, related to its mass and angular momentum and the external B field. Otherwise, it has an ergoregion extending to infinity close to the axis.

- The thermodynamics is quite subtle, owing to the unusual asymptotic behaviour of the Melvin type background. Charge and angular momentum can be calculated from conserved currents, but ab initio computation of angular velocity, electric potential and magnetic moment is more difficult. Frame-dragging of the magnetic field is rather complex.

- More extensive analysis of the global structure is needed. Also, a detailed investigation of "asymptotically Melvin" boundary conditions for fields.

- Melvin/CFT type dualities?

- Magnetisation of black holes in supergravities opens up new possibilities; non-singular magnetised magnetic black holes, etc.

- Recent work (Cvetič, Guica, Saleem) has shown how related solution-generating transformations can be used to interpolate between black hole backgrounds and "subtracted geometries" that can be used for studying the microscopic entropy. We are investigating these further.