SECOND ORDER TRANSPORT FROM ANOMALIES

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INTRODUCTION AND MOTIVATION

• Consider a system in $3+1$ dimensions described by relativistic hydrodynamics.

• The variables of fluid dynamics:
local velocities: $u^\mu(x)$,
temperature: $T(x)$,
conserved charges / chemical potentials: $\mu^a(x)$

• The stress tensor $T^{\mu\nu}(x)$ and conserved currents $J^\mu(x)_a$ are related to the fluid variables by the constitutive relations.

These relations can be organized in terms of a derivative expansion.

• The equations of motion of fluid dynamics are the conservation laws for the stress tensor and the currents.

For a charged relativistic fluid to first order in the derivative expansion:

$$
\begin{aligned}
T^{\mu\nu} &= (E+P)u^\mu u^\nu + P g^{\mu\nu} + \pi^{\mu\nu} \\
\pi^{\mu\nu} &= -2\eta\sigma^{\mu\nu} - \zeta\Theta P^{\mu\nu} \\
\sigma^{\mu\nu} &= P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2} - \frac{\nabla\cdot u}{3} g_{\alpha\beta}\right) \\
\Theta &= \nabla\cdot u \\
P^{\mu\nu} &= u^\mu u^\nu + g^{\mu\nu}
\end{aligned}
$$

E, P energy density and pressure
η shear viscosity.
Θ bulk viscosity.
$P^{\mu\nu}$ is the projector onto the spatial directions.
$u \cdot u = -1$

- The charge current

$$
\begin{aligned}
J^\mu &= qu^\mu + \Delta V^\mu + \xi_I I^\mu + \xi_B B^\mu \\
V^\mu &= (E^\mu - TP^{\mu\rho}\partial_\rho \nu) \\
E_\mu &= \mathcal{F}_{\mu\nu} u^\nu, \\
B^\mu &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{F}_{\alpha\beta}, \\
I^\mu &= \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta, \\
\nu &= \frac{\mu}{T}
\end{aligned}
$$

The background electric and magnetic field are given by E_μ, B_μ.
Δ is the charge diffusivity.

- We work in the Landau frame

$$
u^\mu T_{\mu\nu} = -E u_\nu, \qquad u^\mu J_\mu = -q
$$

• Focus on the parity odd terms.
These terms are all related to the anomaly in the conservation of the current Son and Surowka (2009), Neiman and Oz (2010)

$$\begin{aligned}\nabla_\mu J^\mu &= -\frac{C}{8}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}+\frac{C_2}{32\pi^2}\epsilon^{\alpha\beta\gamma\delta}R^\mu_{\nu\alpha\beta}R^\nu_{\mu\gamma\delta}\\ &= CE_\mu B^\mu+\frac{C_2}{32\pi^2}\epsilon^{\alpha\beta\gamma\delta}R^\mu_{\nu\alpha\beta}R^\nu_{\mu\gamma\delta}.\end{aligned}$$

Then

$$\begin{aligned}\xi_I &= C\nu^2T^2\left(1-\frac{2q}{3(E+P)}\nu T\right)+T^2C_2\left(-2+\frac{4\nu qT}{E+P}\right),\\ \xi_B &= C\nu T\left(1-\frac{q}{2(E+P)}\nu T\right)+C_2\frac{qT^2}{E+P}\end{aligned}$$

- Question:

Can parity odd transport coefficients at the next order (2 derivatives) in the derivative expansion of the constitutive relations be related to the anomaly.

- What motivated this question?

Observation of chiral shear modes.

• Examine the hydrodynamic modes in the shear sector.
Keep stress tensor to the first order in the derivative expansion.
Examine the fluctuations about the equilibrium.

• Consider the equilibrium fluid configuration.

$$u^\mu = (1, 0, 0, 0), \qquad g_{\mu\nu} = (-1, 1, 1, 1)$$

$E^{(0)}, P^{(0)}$ are constant in space time.
Turn on perturbations in the spatial components of the velocities.

$$u^\mu = (1, v^x(t, y), 0, v^z(t, y))$$

Velocities depend on only in the t and the y directions.

- To the linear order in perturbations,

$$T^{tx} = (E^{(0)} + P^{(0)})v^x, \qquad T^{tz} = (E^{(0)} + P^{(0)})v^z,$$
$$T^{xy} = -\eta\partial_y v^x, \qquad T^{zy} = -\eta\partial_y v^z$$

- The equations of motion for the x, z components of the stress tensor

$$\partial_t T^{tx} + \partial_y T^{yx} = 0, \qquad \partial_t T^{tz} + \partial_y T^{yz} = 0$$

result in closed equations for the velocities

$$\partial_t v^x - \frac{\eta}{E^{(0)} + P^{(0)}}\partial_y^2 v^z = 0,$$
$$\partial_t v^z - \frac{\eta}{E^{(0} + P^{(0)}}\partial_y^2 v^z = 0$$

- These are the 2 degenerate shear hydrodynamic modes with dispersion relation

$$\omega = -i\frac{\eta}{E^{(0)} + P^{(0)}}k^2$$

• The dispersion relation of these modes is determined by the shear viscosity.

• AdS/CFT implies that the shear modes can be obtained by examining the lowest shear channel quasi-normal modes of the graviton fluctuations in the background of the Schwarzschild black hole in AdS_5 .

Thus the low lying shear modes in the bulk correspond to the shear hydrodynamic modes.

• Let us now examine the holographic dual of $\mathcal{N} = 4$ Yang-Mills at finite chemical potential and temperature:
the charged AdS_5 Reissner-Nördstrom black hole.

• Sahoo and Yee (2010) observed that the degeneracy of the quasi-normal modes corresponding to the shear modes for the case of a charged AdS_5 Reissner-Nördstrom black holes splits.

• The dispersion relations splits into two chiral shear modes given by

$$\omega = -i\frac{1}{4\pi T_H}k^2 \pm \frac{i\kappa\mu^3}{24\pi T_H^2 s}k^3$$

where κ is the coefficient of the Chern-Simons term in the gravitational action

$$\kappa \int d^5 x A \wedge F \wedge F$$

μ is the chemical potential, s is the entropy density.

- Recall the charged black hole corresponds to the dual theory at finite temperature and finite chemical potential.
- How can the degeneracy of the 2 chiral modes be split ?

It certainly cannot be explained by single derivative terms in the stress tensor or charge current.

• Erdmenger et. al (2008) and Banerjee et. al. (2008) evaluated contributions to the stress tensor of the dual field theory in this system directly using holography.

• Found the existence of the following 2 derivative term

$$
\begin{aligned}
\pi^{(2)}_{\mu\nu} &= \Phi_1 \nabla_{\langle\mu} l_{\nu\rangle} \\
l^{\mu} &= \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \\
A^{\langle\mu\nu\rangle} &= P^{\mu\alpha} P^{\nu\beta} \left(\frac{A_{\alpha\beta} + A_{\beta\alpha}}{2} - \frac{g_{\alpha\beta}}{3} A^{\rho}_{\rho} \right)
\end{aligned}
$$

• Φ_1 is proportional to the coefficient of the Chern-Simons term and the (chemical potentials $)^3$.

• The presence of the 2 derivative term explains the splitting of the chiral shear modes.

• With this additional term in the stress tensor let us again examine the fluctuations about the equilibrium fluid configuration.
Consider again the equilibrium fluid configuration.

$$u^\mu = (1, 0, 0, 0), \qquad g_{\mu\nu} = (-1, 1, 1, 1)$$

$E^{(0)}, P^{(0)}$ are constant in space time.
Turn on perturbations in the spatial components of the velocities.

$$u^\mu = (1, v^x(t, y), 0, v^z(t, y))$$

The dependence of the velocities is only in the t and the y directions.

• To the linear order in perturbations,

$$T^{tx} = (E+P)v^x, \qquad T^{tz} = (E+P)v^z,$$
$$T^{xy} = -\eta\partial_y v^x + \frac{\Phi_1}{2}\partial_y^2 v^z, \qquad T^{zy} = -\eta\partial_y v^z - \frac{\Phi_1}{2}\partial_y^2 v^x$$

• The equations of motion for the stress tensor now yield the following closed equations for the velocities.

$$-i\omega(E^{(0)}+P^{(0)})v^x + k^2\eta v^x - i\frac{\Phi_1}{2}k^3 v^z = 0$$
$$-i\omega(E^{(0)}+P^{(0)})v^z + k^2\eta v^z + i\frac{\Phi_1}{2}k^3 v^x = 0$$

• The dispersion relation from these equations are

$$\omega = -i\frac{\eta}{E^{(0)} + P^{(0)}}k^2 \pm i\frac{\Phi_1}{2(E^{(0)} + P^{(0)})}k^3$$

• Substituting the value of Φ_1 found by Erdmenger et. al and Banerjee et. al we can recover the quasi-normal modes of Sahoo and Yee.

• Thus in fluid dynamics with this parity odd 2 derivative term the 2 shear modes split and become chiral.

• Another observation from holography is that the current dual to the gauge field in the bulk obeys the following conservation law

$$\partial_\mu J^\mu = \kappa \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

The current obeys the anomalous conservation law.

• All this had to do with the strong coupling hydrodynamics of $\mathcal{N} = 4$.

• Lets ask the question:
Is it true that in any fluid dynamics if the charge current obeys the anomalous conservation law the second order transport Φ_1 and other parity odd transport coefficients can be related to the anomaly.

KUBO FORMULAE

• The transport coefficient Φ_1 affects dispersion relation obtained by linearized fluctuations.

There must be a Kubo formula involving a 2 point function for it.

• Let us assume the only parity odd coefficient is Φ_1. (Can be generalized with all the parity odd coefficients).

• The following equilibrium configuration

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1), \qquad u^{\mu} = (1, 0, 0, 0)$$

The temperature T, chemical potential ν are constant in space time.
Energy $E^{(0)}$ and pressure $P^{(0)}$ are constant is space time.

• Now consider the following perturbations from this background.

$$\delta g_{tx} = h_{tx}, \quad \delta g_{tz} = h_{tz}, \quad \delta g_{yx} = h_{yx}, \quad \delta g_{yz} = h_{yz}.$$

The velocity perturbation is given by

$$\delta u^\mu = (0, v^x, 0, v^z)$$

All perturbations are assumed to have dependence only in the time t and y direction.

• To second order in derivatives and to linear order in the perturbations we have

$$
\begin{aligned}
T^{tx} &= (E^{(0)} + P^{(0)})v^x + P^{(0)} h_{tx},\\
T^{tz} &= (E^{(0)} + P^{(0)})v^z + P^{(0)} h_{tx},\\
T^{ty} &= 0,\\
T^{yx} &= -P^{(0)} h_{yx} - \eta(\partial_y v^x + \partial_t h_{yx}) + \frac{1}{2}\Phi_2 \partial_y^2 (v^z + h_{zt})
\end{aligned}
$$

• The equations of motion in the x direction

$$\partial_t T^{tx} + \partial_t h_{tx} E^{(0)} + \partial_y T^{yx} = 0.$$

Fourier transforming and in the zero frequency limit obtain the Ward identity

$$\lim_{\omega \to 0} T^{yx}(k) = 0.$$

- Eliminate v^x, v^z

$$T^{yx} = -P^{(0)}h_{yx} - \eta\left(\frac{\partial_y T^{tx} - P^{(0)}\partial_y h_{tx}}{E^{(0)} + P^{(0}}\right) + \frac{\Phi_1}{2}\left(\frac{\partial_y^2 T^{tz} + E^{(0)}\partial_y^2 h_{tz}}{E^{(0)} + P^{(0)}}\right)$$

- Fourier transform, take zero frequency limit, differentiate the Ward identity with respect to h_{zt} :

$$\frac{k^2}{E^{(0)} + P^{(0)}}\left[\Phi_1(\langle T^{tz}(k)T^{tz}(-k)\rangle + E^{(0)})\right] = -ik\frac{2\eta}{E^{(0)} + P^{(0)}}\langle T^{tx}(k)T^{tz}(-k)\rangle,$$

- We have

$$\lim_{k\to 0,\omega\to 0}\langle T^{tz}(k)T^{tz}(-k)\rangle = P^{(0)},$$

$$\lim_{k\to 0,\omega\to 0}\langle T^{tx}(k)T^{tz}(-k)\rangle = ik\left(\frac{C}{3}(\nu^{(0)}T^{(0)})^3 - 2C_2(T^{(0)})^3\nu^{(0)}\right),$$

C is the coefficient of the chiral anomaly for the charge current, C_2 is the coefficient of the mixed anomaly. Landsteiner et. al

• This result is obtained by: Considering a theory of free chiral fermions with a chiral chemical potential

The retarded correlator receives contributions at one-loop which is proportional to the anomalies.

Note that the chiral chemical potential is necessary for the retarded correlator to be non-vanishing.

• These correlators can also be evaluated by AdS/CFT in the background of the Reissner-Nördstrom black hole.

• The one-loop answer, when applied to the case of the fermions in $\mathcal{N}=4$. agrees with that obtained at strong coupling.

• Substituting these expressions one obtains

$$\Phi_1 = \frac{2\eta}{E^{(0)}+P^{(0)}}\left(\frac{C}{3}(\nu^{(0)}T^{(0)})^3 - 2C_2(T^{(0)})^3\nu^{(0)}\right)$$

• A similar Kubo formula can be derived for the transport coefficient

$$\pi^{(2)}_{\mu\nu} = \Phi_2 \nabla_{\langle\mu} B_{\nu\rangle}$$

B_μ is the magnetic field.

• Both these transport coefficients are obtained by examining time independent equilibrium, $\omega \to 0$ limit.

These are called 'non-dissipative' coefficients as opposed to coefficients which are obtained as

$$T = \lim_{\omega \to 0} \frac{\mathrm{Green's\,function}}{\omega}$$

like the shear viscosity, the 'dissipative coefficients'. These transport coefficients vanish at time independent equilibrium.

• An ideal method of obtaining 'non-dissipative' type of coefficients is the equilibrium partition function method developed by Jensen et. al , Banerjee et. al .

• We will apply this method to obtain some of the parity odd coefficients at the second order.

PARITY ODD COEFFICIENTS AT 2ND ORDER

• We first write down the number of independent parity odd transport coefficients for a charged non-conformal fluid at the 2nd order in derivative expansion.

Type	number	non-vanishing
pseudo scalars	6	4
pseudo vectors	9	2
pseudo tensors	12	6

• The explicit forms of each terms are listed in the paper.

• The pseudo scalars occur in the 2nd derivative constitutive terms of the trace of the stress tensor.

The pseudo vectors occur in the current.

The pseudo tensors occur in the traceless part of the stress tensor.

• Basis of vectors to construct the second order terms:
Vorticity

$$l^{\mu} = \epsilon^{\mu\nu\alpha\beta} u_{\nu} \partial_{\alpha} u_{\beta}$$

Magnetic field

$$B^{\mu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_{\nu} \mathcal{F}_{\alpha\beta}$$

Parity odd vectors first order in derivatives.

Parity even vectors.

$$u^{\mu}, \quad \partial_{\mu} T, \quad \partial_{\mu} \nu$$

The electric field in the combination

$$V^{\mu} = E^{\mu} - T P^{\mu\rho} \partial_{\rho} \nu$$

vanishes on time independent equilibrium.

- Tensors: The shear tensor

$$\sigma_{\mu\nu} = \nabla_{\langle\mu} u_{\nu\rangle}$$

which vanishes on time independent equilibrium and

$$A_{\langle\mu\nu\rangle} = P^{\alpha}_{\mu} P^{\beta}_{\nu} \left(\frac{A_{\alpha\beta} + A_{\beta\alpha}}{2} - \frac{P^{\gamma\theta} A_{\gamma\theta}}{3} G_{\alpha\beta} \right),$$

- Scalar

$$\Theta = \nabla_{\mu} u^{\mu}$$

vanishes on time independent equilibrium.

- Using these basic quantities:

one can put together the various independent terms that will occur at 2nd order in the constitutive relations.

• We will list out the terms which do not vanish in time independent equilibrium.

Pseudo-scalars:

$$\mathcal{S}_1 = l^\mu \partial_\mu \nu \qquad : \chi_1$$
$$\mathcal{S}_2 = B^\mu \partial_\mu \nu \qquad : \chi_2$$
$$\mathcal{S}_3 = l^\mu \partial_\mu T \qquad : \chi_3$$
$$\mathcal{S}_4 = B^\mu \partial_\mu T \qquad : \chi 4$$

where $l^\mu = \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$, $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{F}_{\alpha\beta}$

Pseudo-vectors:

$$\mathcal{V}^\mu_{(1)} = \epsilon^{\mu\nu\alpha\beta} u_\nu B_\alpha l_\beta \qquad : \Delta_1$$
$$\mathcal{V}^\mu_{(2)} = \epsilon^{\mu\nu\alpha\beta} u_\nu (\partial_\alpha \nu)(\partial_\alpha T) \qquad : \Delta_2$$

Pseudo-tensors:

$$\tau^{(1)}_{\mu\nu} = \nabla_{\langle\mu} I_{\nu\rangle} \qquad : \Phi_1$$

$$\tau^{(2)}_{\mu\nu} = \nabla_{\langle\mu} B_{\nu\rangle} \qquad : \Phi_2$$

$$\tau^{(3)}_{\mu\nu} = I_{\langle\mu}\partial_{\nu\rangle}\nu \qquad : \Phi_3$$

$$\tau^{(4)}_{\mu\nu} = B_{\langle\mu}\partial_{\nu\rangle}\nu \qquad : \Phi_4$$

$$\tau^{(5)}_{\mu\nu} = I_{\langle\mu}\partial_{\nu\rangle} T \qquad : \Phi_5$$

$$\tau^{(6)}_{\mu\nu} = B_{\langle\mu}\partial_{\nu\rangle} T \qquad : \Phi_6$$

• The rest of the transport coefficients involve structures which vanish in time independent equilibrium.

• We will determine the coefficients $\Phi_1, \cdots \Phi_6, \Delta_2$ in terms of the anomalies. We will also obtain 3 constraints among the 5 coefficients $\chi_1, \cdots \chi_4$ and Δ_1 which also involves the anomalies.

• Note that using the Kubo formula we have already found Φ_1, Φ_2.
Thus it can be used as a check.

EQUILIBRIUM PARTITION METHOD

• The equilibrium partition method relies on the existence of an time independent fluid configuration for the background

$$\begin{aligned} ds^2 &= -e^{2\sigma}(dt + a_i dx^i)^2 + g_{ij} dx^i dx^j \\ \mathcal{A} &= \mathcal{A}_\mu dx^\mu \end{aligned}$$

$\sigma, a_i, g_{ij}, \mathcal{A}_\mu$ are functions of only the spatial co-ordinates.

• We will see: the relation of the local temperature and the chemical potential to the background is

$$T = T_0 e^{-\sigma}, \qquad \nu = \frac{\mathcal{A}_0}{T_0}$$

The strategy to obtain transport coefficients:

• Write down the most general partition function to a given order in derivative expansion.

It will be a function of the background: $\sigma, a_i, g_{ij}, \mathcal{A}_\mu$ and their derivatives.

• Vary the partition function Z to obtain the general form of $T^{\mu\nu}|_{equilibrium}$ and $J^{\mu}|_{equilibrium}$.

• Write down the most general form for the constitutive relations for the stress tensor and the charge current up to a given order in the derivative expansion.

This will define the transport coefficients in a given basis.
We choose a basis which satisfies the Landau frame condition.

• Evaluate this stress tensor and the charge current on the equilibrium configuration.

This results in $T^{\mu\nu}|_{equilibrium}$ and $J^{\mu}|_{equilibrium}$.

- Equate

$$T^{\mu\nu}|_{equilibrium} = T^{\mu\nu}|_{equilibrium}$$
$$J^{\mu}|_{equilibrium} = J^{\mu}|_{equilibrium}.$$

- The last step constrains the transport coefficients or determines them.

- This procedure constrains only those transport coefficients which do not vanish on time independent equilibrium configurations.

• Let us see how the the procedure works at the zeroth derivative order.

• The partition function is given by

$$\ln Z = \int d^3x \sqrt{g_3} \frac{e^{\sigma}}{T_0} P(T_0 e^{-\sigma}, \mathcal{A}_0 e^{-\sigma})$$

The pre-factor appears due to dimensional reduction: the radius of the thermal circle.

The dependence of the function as $e^{-\sigma} T_0$ and $\mathcal{A}_0 e^{-\sigma}$ is taken for convenience.

Note that it is local temperature T and the local chemical potential μ.

- Varying the partition function with respect to the backgrounds

$$
\begin{aligned}
T^{ij} &= P g^{ij}, \\
T_{00} &= e^{2\sigma}(P - T\partial_T P - \mu\partial_\mu P) \\
J^0 &= e^{-\sigma}\partial_\mu P
\end{aligned}
$$

The rest of the components vanish.

• The zeroth derivative, perfect fluid form for the stress tensor and the charge current

$$\begin{aligned} T^{\mu\nu} &= (E+P)u^\mu u^\nu + Pg^{\mu\nu}, \\ J^\mu &= qu^\mu \end{aligned}$$

• Equating them yields

$$\begin{aligned} e^{-\sigma}(1,0,0,0) &= u^\mu \\ P &= P \\ -P + T\partial_T P + \mu\partial_\mu P &= E \\ \partial_\mu P &= q \end{aligned}$$

- Note that we found the equilibrium fluid configuration also.

Comparision with thermodynamics also yields the expression for the local temperature in terms of background.

$$T(x) = e^{-\sigma} T_0, \qquad \mu(x) = \frac{A_0}{T_0}$$

• This procedure can be carried to the first order in derivatives.

• The CPT invariant partition function at first order is determined entirely by the anomaly coefficients C, C_2

- The first order correction to the equilibrium fluid configuration

$$[\delta u_{(1)}]_0 = 0, \quad \delta T_{(1)} = 0, \quad \delta\nu_{(1)} = 0,$$
$$[\delta u_{(1)}]^0 = -a_i[\delta u_{(1)}]^i,$$
$$[\delta u_{(1)}]^i = \left(\frac{b_1}{2}\right)\bar{l}^i + b_2\bar{B}^i,$$

where

$$F_{jk} \equiv \partial_j A_k - \partial_k A_j,$$
$$\bar{l}^i = -\frac{e^\sigma}{2}\epsilon^{ijk} f_{jk},$$
$$b_1 = \frac{T^3}{E+P}\left(\frac{2C\nu^3}{3} - 4C_2\nu\right),$$
$$b_2 = \frac{T^2}{E+P}\left(\frac{C\nu^2}{2} - C_2\right),$$

$$A_i = \mathcal{A}_i - \mathcal{A}_0 a_i$$

• Let us now proceed to the 2nd order.

The most general second order parity odd partition function

$$Z_{(2)} = \int \sqrt{g_3} \left[M_1(T,\nu)\, \epsilon^{ijk} \partial_i \nu F_{jk} + T_0 M_2(T,\nu)\, \epsilon^{ijk} \partial_i \nu f_{jk} \right].$$

$f_{ij} = \partial_i a_j - \partial_j a_k$.

• From this one sees the second order parity odd correction

$$[T^{(2)}]^{ij}|_{equilibrium} = 0$$

• The other components of the stress tensor and current at second order can also be obtained.

• The second order correction to the stress tensor from fluid dynamic considerations

$$
\begin{aligned}
[T_{(2)}]_{\mu\nu} &= \sum_{i=1}^{6} \Phi_i \, \tau^{(i)}_{\mu\nu} + P_{\mu\nu} \left[\sum_{i=1}^{4} \chi_i \, \mathcal{S}_i \right] , \\
J^{\mu}_{(2)} &= \sum_{i=1}^{2} \Delta_i \, \mathcal{V}^{\mu}_{(i)} .
\end{aligned}
$$

We have kept only the parity odd terms which do not vanish on time independent equilibrium configurations.

• Examine the corrections at the 2nd order to the traceless part of the stress tensor at equilibrium.

$$T^{ij}|_{equilibrium} = [T_{(0)}]^{ij} - 2\eta\sigma^{ij}_{(1)} + [T_{(2)}]^{ij}$$

• Corrections arise from two sources.

(1) Substituting the 1st order correction of the fluid velocity in the shear tensor.
(2) Substituting the 0th order fluid velocity configuration in the second order terms $[T_{(2)}]^{ij}$.

There are no corrections to the traceless part from the 2nd order corrections to the fluid velocity, thermodynamic functions substituted in the zeroth order stress tensor.

- From the equilibrium partition function we know that

$$[T_{(2)}]^{ij}|_{equlibrium} = 0$$

Thus the two terms (1) +(2) should vanish.

- Corrections from (2)

We examine the 2nd order term obtained by substituting the zeroth order equilibrium velocity configuration into $[T_{(2)}]^{ij}$.

They can be organized as

$$[T_{(2)}]^{ij}_{odd}|_{equilibrium} = \sum_{a=1}^{6} \Phi_a [\tau^{(a)}]^{ij}$$

where

$$[\tau^{(1)}]^{ij} = g^{il} g^{jm} \left[\frac{\nabla_l \bar{l}_m + \nabla_m \bar{l}_l}{2} - \frac{g_{lm}}{3} (\nabla_k \bar{l}^k) - \frac{g_{lm}}{3} (\partial_k \sigma) \bar{l}^k \right],$$

$$[\tau^{(2)}]^{ij} = g^{il} g^{jm} \left[\frac{\nabla_l \bar{B}_m + \nabla_m \bar{B}_l}{2} - \frac{g_{lm}}{3} (\nabla_k \bar{B}^k) - \frac{g_{lm}}{3} (\partial_k \sigma) \bar{B}^k \right],$$

$$[\tau^{(3)}]^{ij} = g^{il} g^{jm} \left[\frac{(\nabla_l \bar{\nu}) \bar{l}_m + (\nabla_m \bar{\nu}) \bar{l}_l}{2} - \frac{g_{lm}}{3} (\nabla_k \bar{\nu} \bar{l}^k) \right],$$

$$[\tau^{(4)}]^{ij} = g^{il} g^{jm} \left[\frac{(\nabla_l \bar{\nu}) \bar{B}_m + (\nabla_m \bar{\nu}) \bar{B}_l}{2} - \frac{g_{lm}}{3} (B^k \nabla_k \bar{\nu}) \right],$$

$$[\tau^{(5)}]^{ij} = g^{il} g^{jm} \left[\frac{(\nabla_l \bar{T}) \bar{l}_m + (\nabla_m \bar{T}) \bar{l}_l}{2} - \frac{g_{lm}}{3} (\bar{l}^k \nabla_k \bar{T}) \right]$$

$$[\tau^{(6)}]^{ij} = g^{il} g^{jm} \left[\frac{(\nabla_l \bar{T}) \bar{B}_m + (\nabla_m \bar{T}) \bar{B}_l}{2} - \frac{g_{lm}}{3} (\bar{B}^k \nabla_k \bar{T}) \right],$$

where $\bar{l}^i \equiv -\frac{e^\sigma}{2} \epsilon^{ijk} f_{jk}$ and $\bar{B}^i \equiv \frac{1}{2} \epsilon^{ijk} (F_{jk} + A_0 f_{jk})$.

- Corrections from (1)

Substituting the 1st order velocity correction into the shear tensor, it organizes as

$$
\begin{aligned}
-2\eta\delta\sigma^{ij} = -2\eta\Bigg[& \frac{b_1}{2}[\tau^{(1)}]^{ij} + b_2[\tau^{(2)}]^{ij} + \frac{1}{2}\left(\frac{\partial b_1}{\partial \nu}\right)[\tau^{(3)}]^{ij} \\
& + \left(\frac{\partial b_2}{\partial \nu}\right)[\tau^{(4)}]^{ij} \\
& + \frac{1}{2}\left(-\frac{b_1}{T} + \frac{\partial b_1}{\partial T}\right)[\tau^{(5)}]^{ij} + \left(-\frac{b_2}{T} + \frac{\partial b_2}{\partial T}\right)[\tau^{(6)}]^{ij}\Bigg].
\end{aligned}
$$

where

$$
b_1 = \frac{T^3}{E+P}\left(\frac{2C\nu^3}{3} - 4C_2\nu\right), \quad b_2 = \frac{T^2}{E+P}\left(\frac{C\nu^2}{2} - C_2\right),
$$

• Demanding that the total contribution to the spatial part of the stress tensor vanishes gives the following unique solution to the transport coefficients.

$$\Phi_1 = \eta\, b_1, \quad \Phi_2 = 2\eta\, b_2, \quad \Phi_3 = \eta \left(\frac{\partial b_1}{\partial \nu} \right), \quad \Phi_4 = 2\eta \left(\frac{\partial b_2}{\partial \nu} \right),$$

$$\Phi_5 = \eta \left[-\frac{b_1}{T} + \frac{\partial b_1}{\partial T} \right], \quad \Phi_6 = 2\eta \left[-\frac{b_2}{T} + \frac{\partial b_2}{\partial T} \right],$$

• The values for Φ_1, Φ_2 agrees with that obtained from the Kubo formula.

• On examining the trace part of the stress tensor and the other components of the stress tensor and the charge current from the partition function.

One more transport coefficient Δ_2 can be determined.

3 relations among the 5 transport coefficients $\Delta_2, \chi_1, \chi_2, \chi_3, \chi_4$ can be obtained.

RESULTS

- The final result of the analysis.

$$\Phi_1 = \eta\, b_1, \quad \Phi_2 = 2\eta\, b_2, \quad \Phi_3 = \eta \left(\frac{\partial b_1}{\partial \nu} \right), \quad \Phi_4 = 2\eta \left(\frac{\partial b_2}{\partial \nu} \right),$$

$$\Phi_5 = \eta \left[-\frac{b_1}{T} + \frac{\partial b_1}{\partial T} \right], \quad \Phi_6 = 2\eta \left[-\frac{b_2}{T} + \frac{\partial b_2}{\partial T} \right],$$

$$\Delta_2 = -\frac{\Delta b_1}{2},$$

$$T^2 R_1 \left[\chi_3 - \frac{\zeta}{2}\left(\frac{\partial b_1}{\partial T} - \frac{2b_1}{T}\right)\right] - R_2 \left[\chi_1 - \frac{\zeta}{2}\left(\frac{\partial b_1}{\partial \nu} - 2b_2 T\right)\right] = 0,$$

$$T^2 R_1 \left[\chi_4 - \zeta\left(\frac{\partial b_2}{\partial T} - \frac{b_2}{T}\right)\right] + R_2 \left[\chi_2 - \zeta\left(\frac{\partial b_1}{\partial \nu}\right)\right] = 0,$$

$$R_1 T \Delta_1 + \left[\chi_2 - \zeta\left(\frac{\partial b_2}{\partial \nu}\right)\right] - \frac{q}{(E+P)}\left[\chi_1 - \frac{\zeta}{2}\left(\frac{\partial b_1}{\partial \nu} - 2b_2 T\right)\right] = 0,$$

Δ is the charge diffusivity : occurs at the 1st order in the charge current.

ζ the bulk viscosity.

$$b_1 = \frac{T^3}{E+P}\left(\frac{2C\nu^3}{3} - 4C_2\nu\right), \quad b_2 = \frac{T^2}{E+P}\left(\frac{C\nu^2}{2} - C_2\right),$$

$$R_1 = \left(\frac{\partial P}{\partial E}\right)_q, \quad R_2 = \left(\frac{\partial P}{\partial q}\right)_E.$$

Note that C is the gauge anomaly coefficient and C_2 is the coefficient of the mixed gauge-gravitational anomaly.

CONCLUSIONS

- We have related 7 parity odd transport coefficients directly to the anomaly.
- 5 others are constrained by 3 relations. The constrains involve the anomaly coefficient.
- These results were obtained by the equilibrium partition function method.
- 2 transport coefficients were also obtained by Kubo formulae.

• Our results are in agreement with the recent holographic evaluation of the transport coefficients of conformal fluids by Amando et. al (2013)

• They are also in agreement with the calculations of Kharzeev and Yee (2011) who used constrains from the positivity of entropy production in conformal fluids.

• It will be interesting to obtain Kubo formulae for the remaining 2nd order transport coefficients.

The rest will involve 3 point functions.

• One consequence of these relations is that if in any fluid dynamics there is an anomalous conservation law, eg. Conditions at RHIC due to a chiral chemical potential. Such transport coefficients will be non-zero and their effects like chiral shear modes though sub-leading will be present.

• This insight into the behaviour of general fluids was possible by first examining the fluid dynamics for system which admits a holographic dual and studying its consequences.