Conformal Defects *and* *(some of their)* Applications

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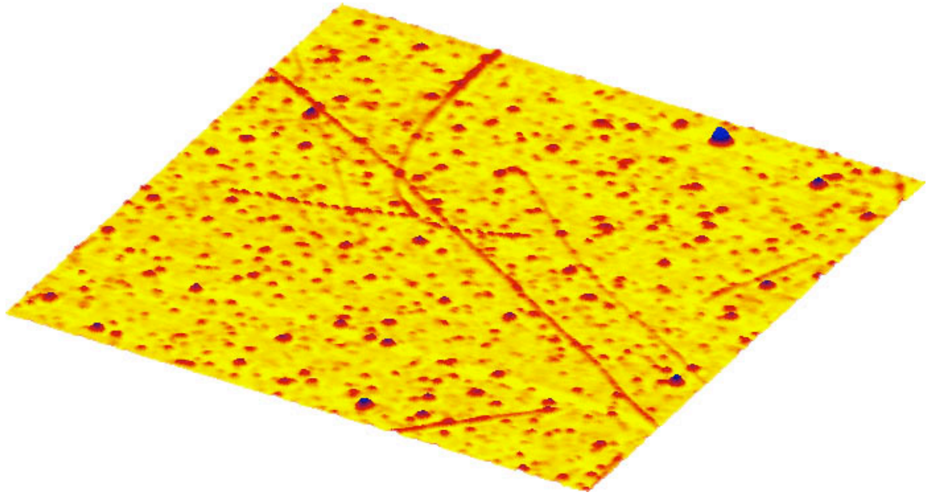
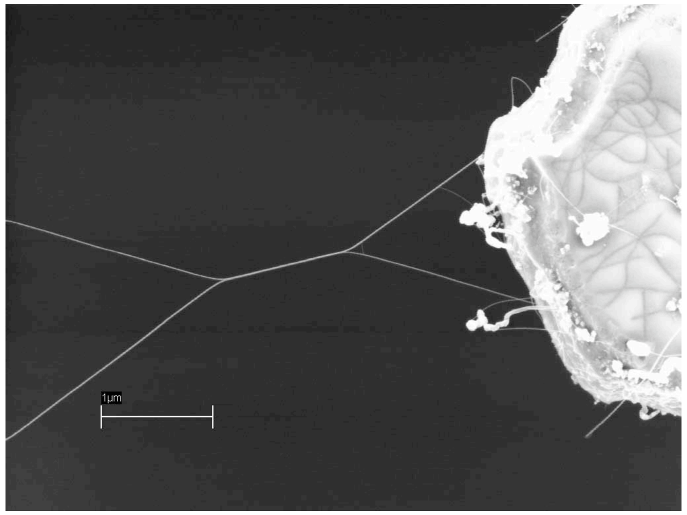
Conformal Defects have been studied extensively by Condensed-Matter physicists for about 20 years

They describe critical behavior of impurities in low-*d* quantum systems (quantum dots in quantum wires)

Impressive advances in nano-electronics makes it possible nowadays to engineer these in the lab



nanotubes grown by an ethylene-hydrogen process on Si/Si-oxide



Nanotubes over Slits on Si-nitride membranes

Photos from U. of Basel " Nano-electronics" group

A (small) group coming from string theory, or more formal QFT, has been interested in them for just over 10 years

Why ?

- Natural extension of CFT_2 to non-local observables (like Wilson loop in gauge theories, but much richer);

  Interesting new mathematical structure(s)

- Hope that these may play a role in string theory, similar to D-branes (?)

  Some (modest) observations, more to come ??

Outline of this talk

- Conformal defects & interfaces: a short review

- CFT maps ("functors") and fusion

- Rational extension of $O(d,d, Z)$

  with I. Brunner, D. Roggenkamp   (arXiv: 1205.4647;   1303.3616)

- Calabi's diastasis as interface entropy

  with I. Brunner, M. Douglas, L. Rastelli   (arXiv: 1307.xxxx)

1. **Conformal defects & interfaces:** a short review

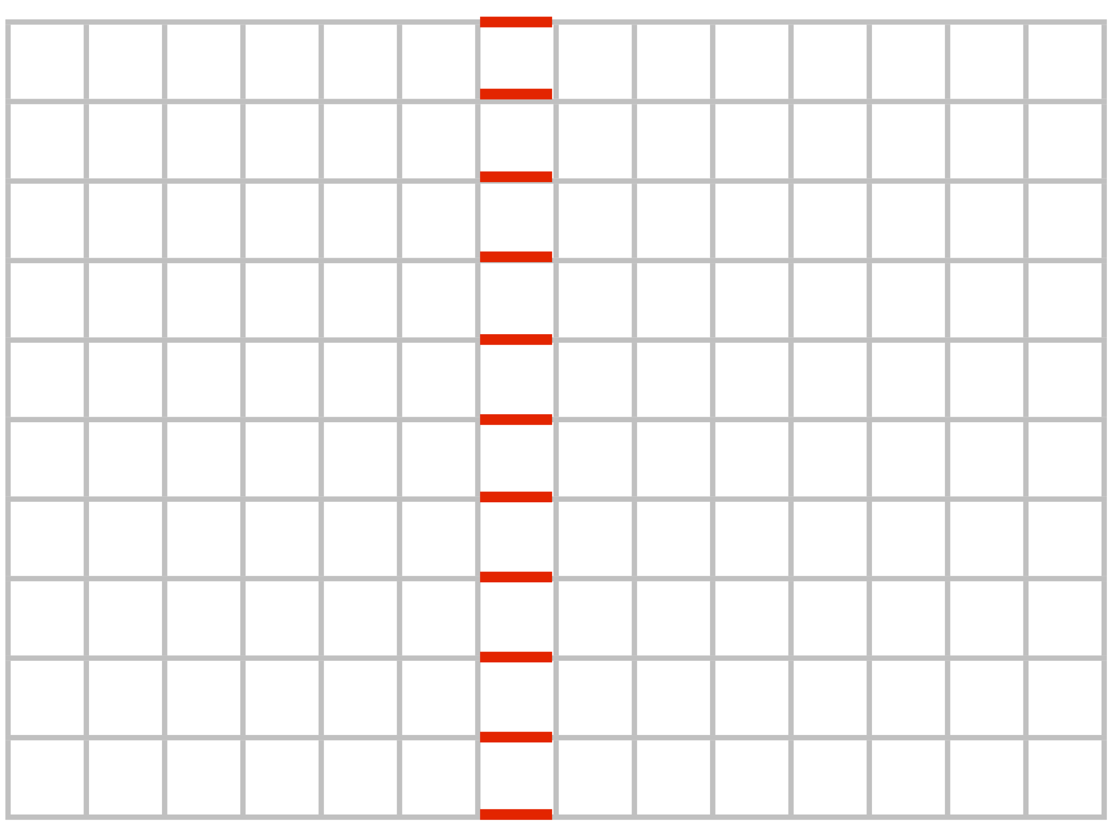
They are close relatives of boundary critical phenomena:

- Quantum impurities in 1D systems

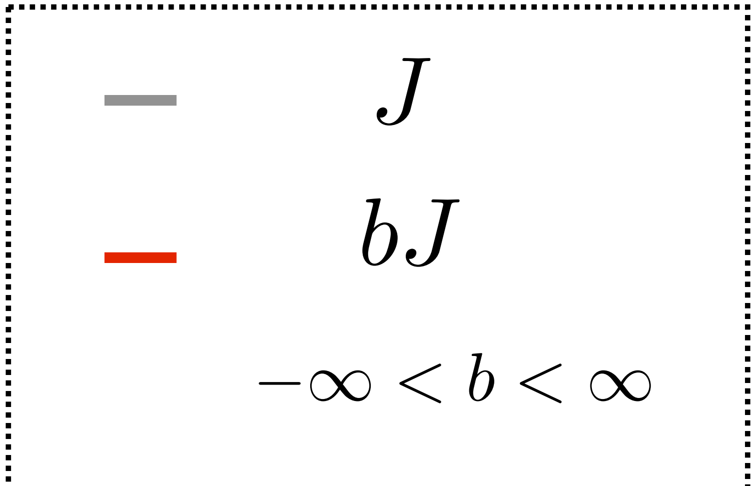
  Kane, Fisher '92 ; ....

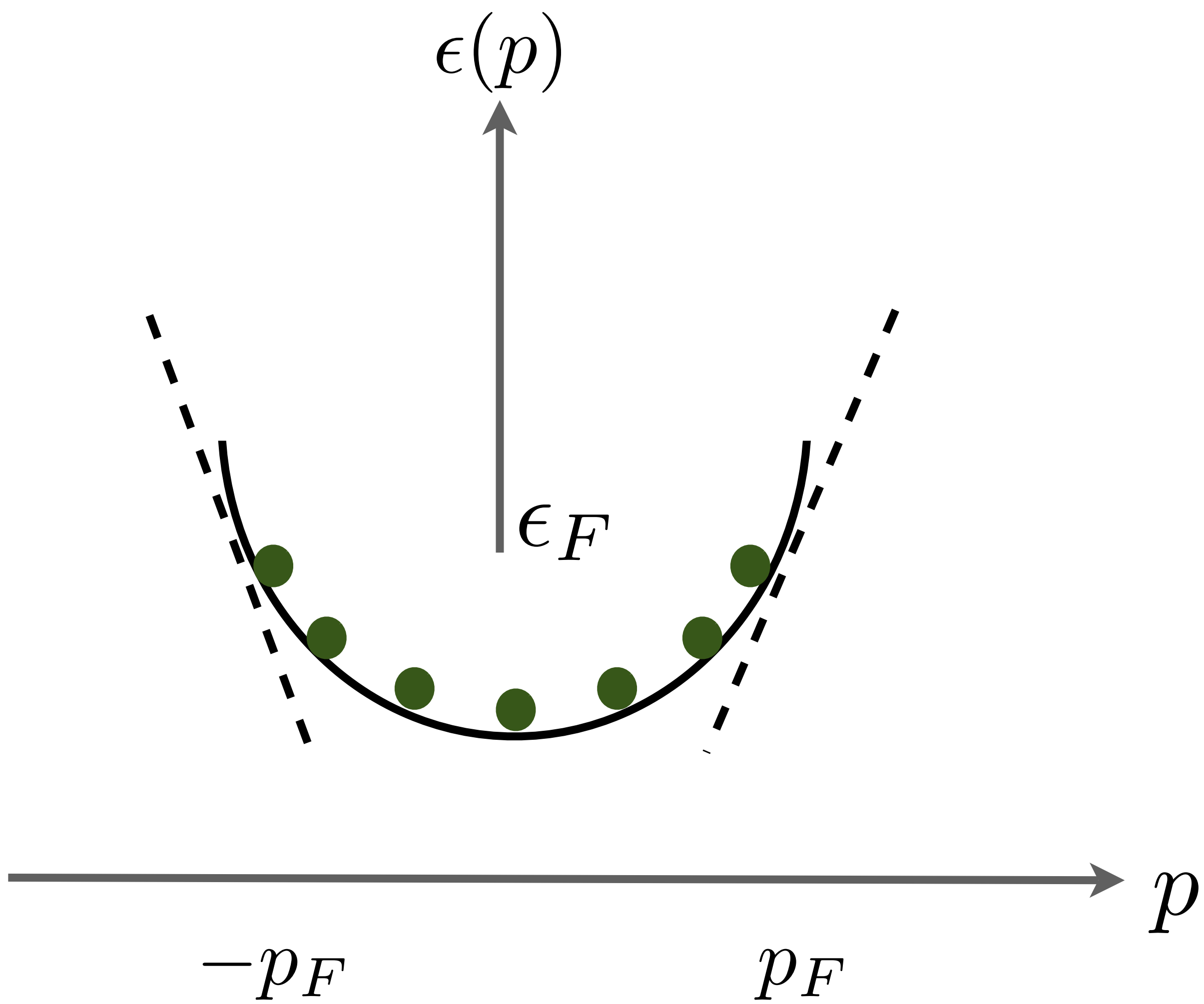
- Line defects in classical 2D systems

  Affleck, Oshikawa '95 ; ....



Ising model with couplings:





example : Transport in Tomonaga-Luttinger liquid

linear dispersion near two Fermi points

$\Longrightarrow$ two relativistic fermions ψ , $\tilde{\psi}$

Bosonization:

$$\rho = \frac{i}{\pi}\partial_x\phi \ ; \quad j = \frac{i}{\pi}\partial_t\phi$$ ← charge ; current densities

$$\psi =: \exp(-i\phi + \frac{i}{v_F}\int^x \partial_t\phi) : \qquad \tilde{\psi} =: \exp(i\phi + \frac{i}{v_F}\int^x \partial_t\phi) :$$

$$\phi \equiv \phi + \pi$$ ← periodic identification

the T-L Hamiltonian is:

$$H = \int \frac{dx}{2\pi} \left[\frac{1}{v_F} (\partial_t \phi)^2 + v_F (\partial_x \phi)^2 + G (\partial_x \phi)^2 \right]$$

free fermions

charge-charge interaction

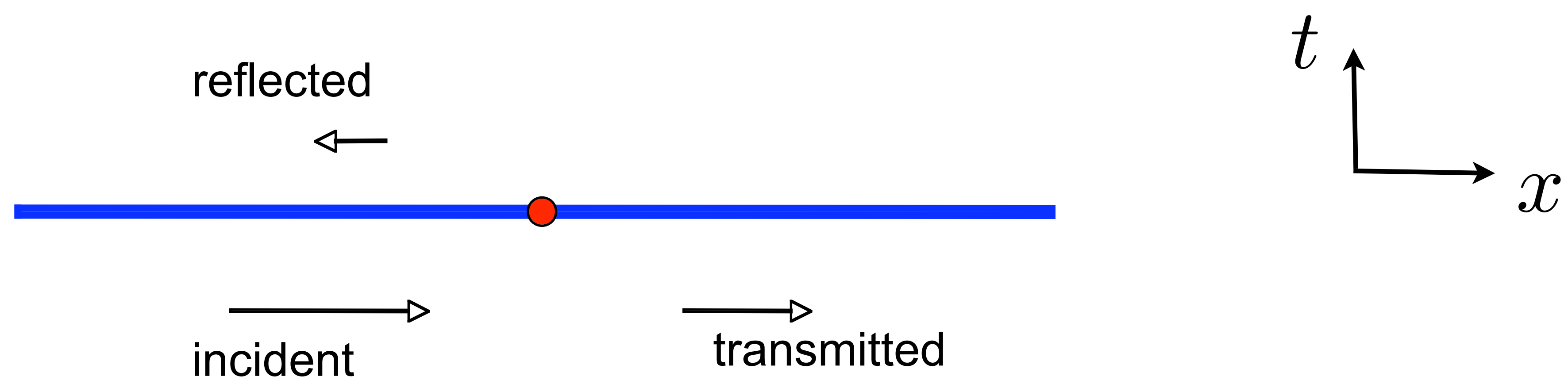
Rescaling to bring to standard form changes periodicity of field:

$$\phi \equiv \phi + 2\pi R \qquad \text{where} \qquad 2R = (1 + G/v_F)^{1/4}$$

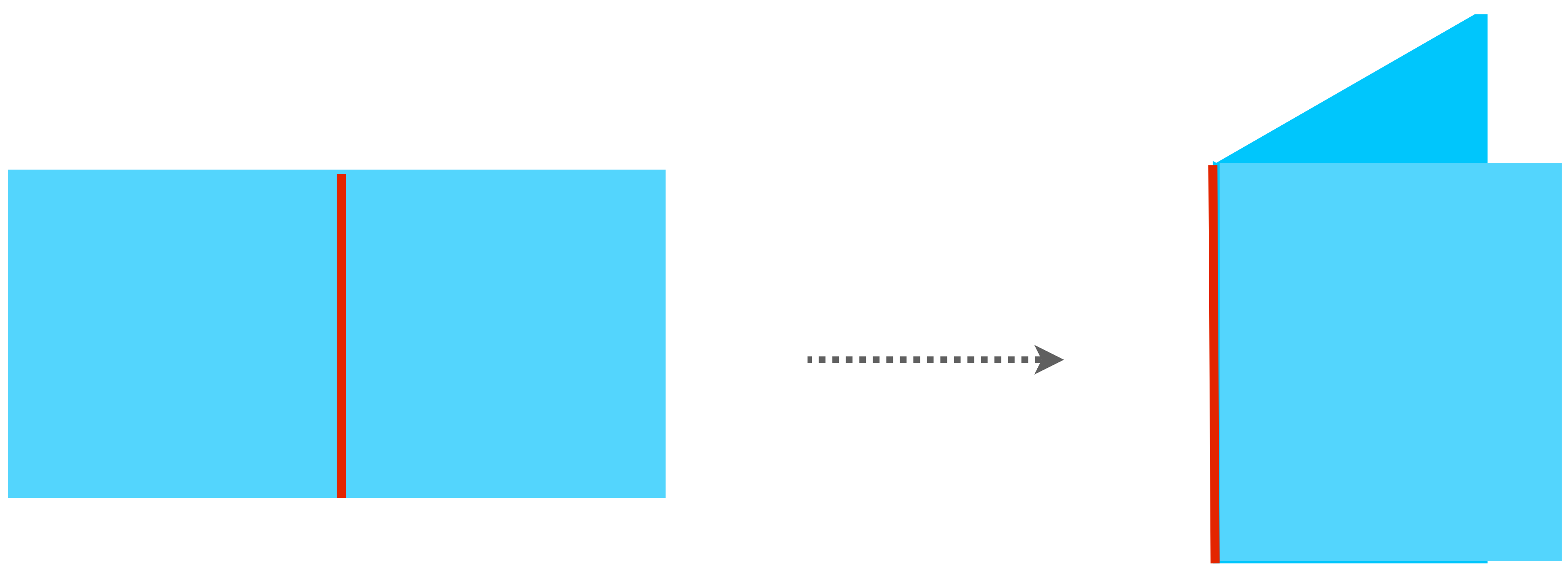
depends on material of q-wire

NB: Simplest example of non-Fermi liquid for $G \neq 0$

Consider simple impurity: what are the possible (low-E) fixed points?



Folding trick $\Longrightarrow$ conformal boundary of c=2 theory



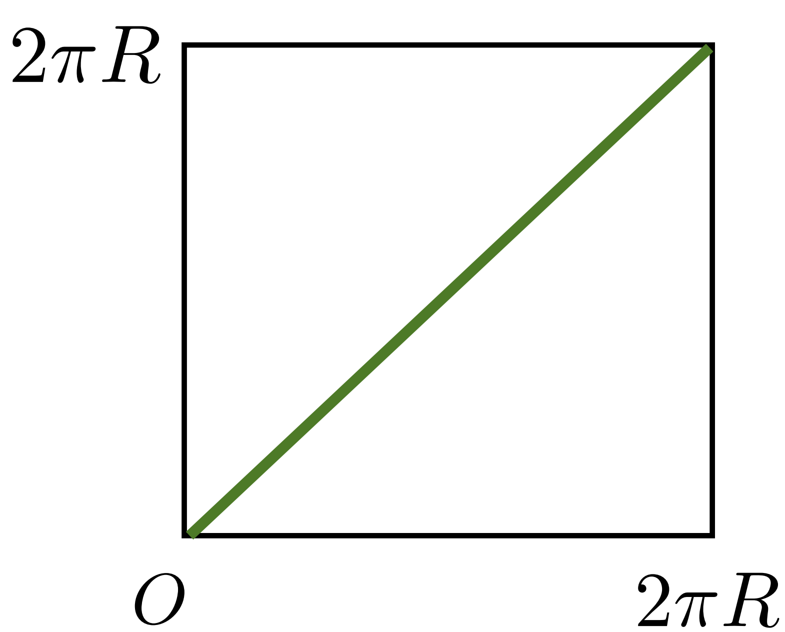
Charge conservation $\Longrightarrow$ $\partial_t \phi^1 = \partial_t \phi^2$

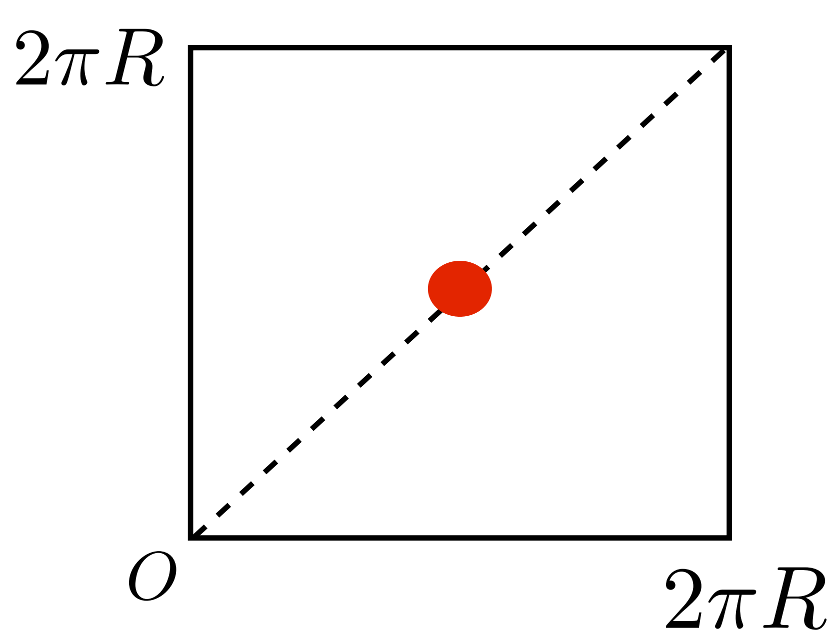
$\Longrightarrow$ two possibilities:

$\partial_t(\phi^1 + \phi^2) = 0$ (full reflection) D0-brane

$\partial_x(\phi^1 + \phi^2) = 0$ (full transmission) D1-brane

Pictorially:





Stability depends on R :

most relevant operators:

back-scattering: *open-string KK mode* $\Delta = \frac{1}{2R}$

tunneling: *open-string winding mode* $\Delta = 2R$

KK mode becomes relevant for *R*$>1/2$, i.e. **repulsive** interactions
Ballistic transport of charge in this case impossible.

Conversely, tunneling relevant *for* **attractive** interactions
Impurity renormalizes away in this case.

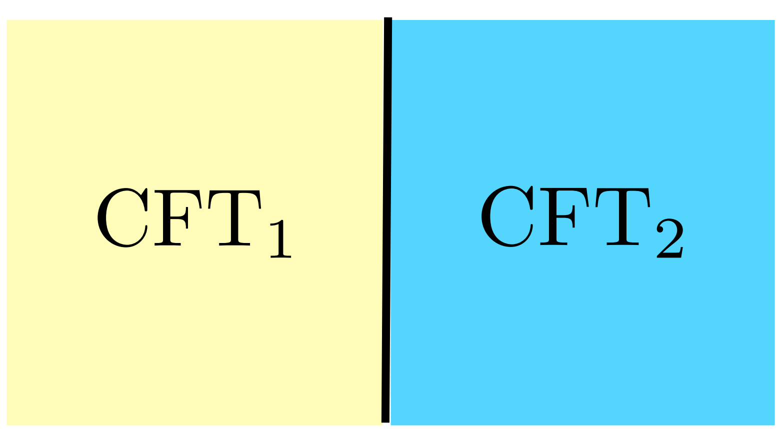
Simple generalizations :

- Many bands ("channels") $\Longrightarrow$ many fields, but Fermi velocities need not be the same; CFT methods must then be modified.

- Spin current : $SU(2)_k$ rather than $U(1)$ current algebra

- Junction of N quantum wires $\Longrightarrow$ boundary in c=N theory

- **Interfaces between different CFTs**



Boundary of

$$\mathcal{P}(CFT_1) \otimes CFT_2$$

CB, de Boer, Dijkgraaf, Ooguri '02

Generic defect :

- A n-dimensional space of quantum states

  (n = 2j+1 for magnetic impurity; or the # of states of an electron in a quantum dot)

- An interaction Hamiltonian $H_{\rm imp}$ which is is an *n* x *n* matrix, with entries depending on the local bulk fields.

$$P e^{-i \oint H_{imp}(\phi, \partial\phi)}$$

No higher derivatives in geometric (sigma-model) limit

RG flow: $H_{imp}^{\rm UV} \longrightarrow H_{imp}^{\rm IR}$

An interesting example in the G_k WZW model:

$$H_{imp} = \frac{1}{k} \sum_{a=1}^{\dim G} M^a J^a$$

Coupling constants: $n \times n$ matrices

$$J^a = \sum_{r \in Z} J^a_r e^{-irt - |r|\epsilon/2}$$

$$\beta^a(M) = -\frac{dM^a}{d\log\epsilon} = -\frac{1}{2k}\left[M^b,\, i f^{abc} M^c - [M^a, M^b]\right] + O\left(\frac{1}{k^2}\right)$$

CB, Gaberdiel '04

CB, Monnier '10

(A subset of !) fixed points :

M^a = n - dim generators of G

Choosing $G = SU(2)$ and $M^a = \lambda\,\sigma^a$

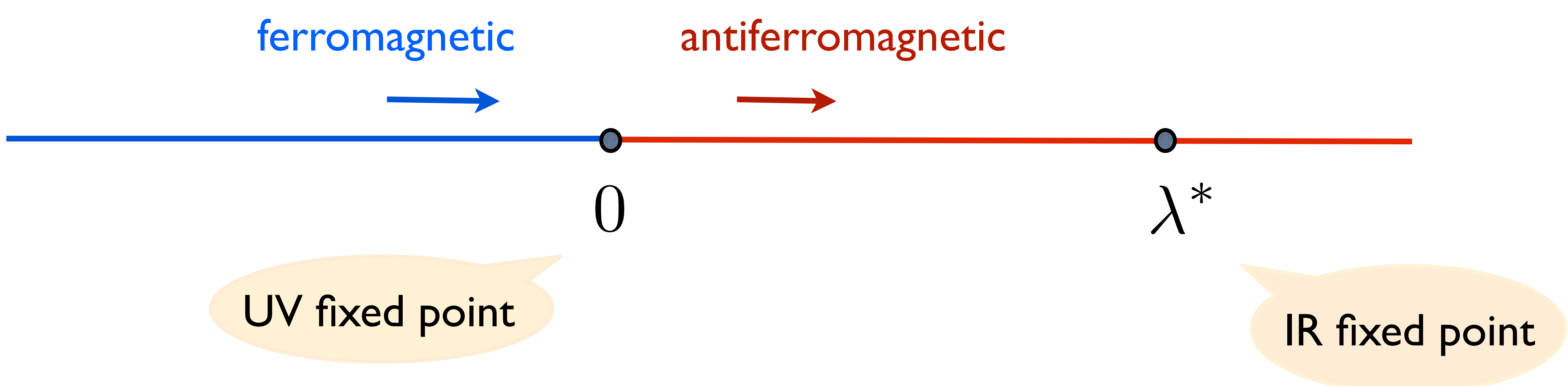
impurity spin

gives the famous **Kondo** flow (*screening of a magnetic impurity by the conduction electrons in a metal*):

Wilson '75

Andrei '80; Wiegmann '80

Affleck, Ludwig '91



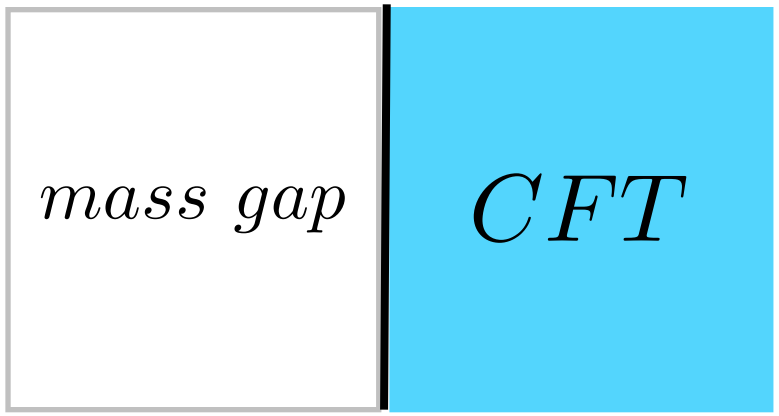
Several other examples have been worked out :

- Quella, Schomerus '02 — WZW interfaces with *k* jump
- Mikhailov, Schafer-Nameki '07, Benichou '11 — $AdS_5 \times S^5$ sigma model
- Bazhanov, Lukyanov, Zamolodchikov '94-98, Runkel '07 — Integrable flows of minimal-model defects
- Sarkissian '09, Drukker, Gaiotto, Gomis '10 — Liouville, Toda defects (&AGT)

....

2. Fusion and general properties

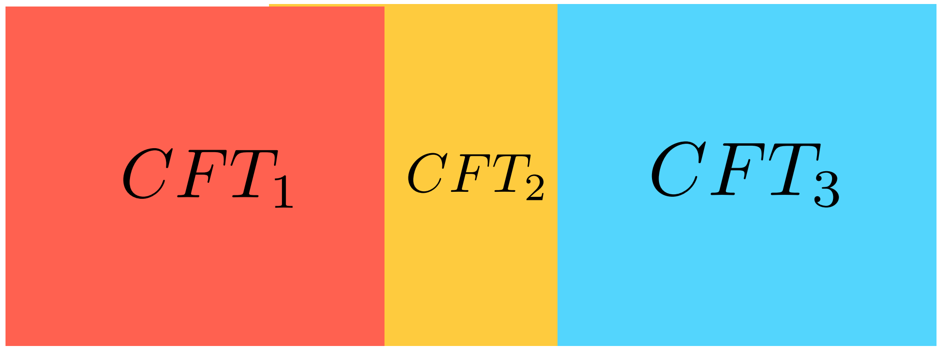
- Boundaries are special cases of interfaces



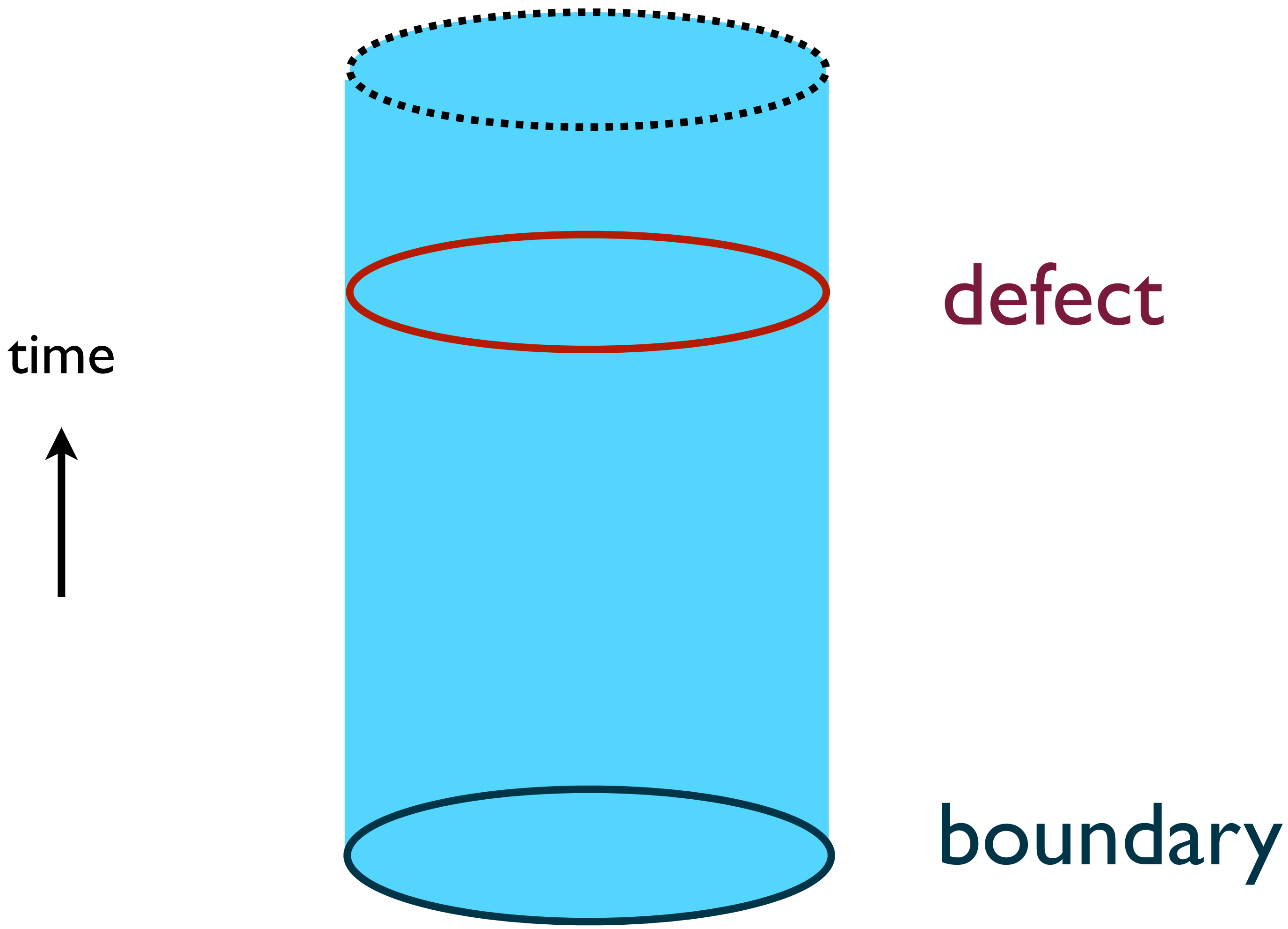
- Interfaces are boundaries of folded (*tensor-product*) theories

But non-trivial interfaces can be both added and **fused** (composed):

they form an "**algebra**"



To make this precise, exchange the roles of space and time & go to the cylinder:



Associate an operator acting on the states of the CFT

$$\mathcal{O} = tr(Pe^{-i\oint H_{\rm imp}})$$

$$[T_{++} - T_{--}, \mathcal{O}] = 0$$

Associate a state of the CFT on the circle, such that

$$(T_{++} - T_{--})|\mathcal{B}\rangle = 0 \ .$$

Special cases of interfaces :

- Fully **reflecting** (factorized boundaries)

$$(T_{++} - T_{--})\,\mathcal{O}_{\rm refl} \;=\; \mathcal{O}_{\rm refl}\,(T_{++} - T_{--}) \;=\; 0$$

- Fully **transmitting** ("topological")

$$[\mathcal{O}_{\rm top}, T_{++}] = [\mathcal{O}_{\rm top}, T_{--}] = 0$$

Petkova, Zuber '00

NB: Can be deformed freely without crossing other defects or operator insertions.

- **Chiral** (like previously-met WZW defects)

$$[T_{--}\,, \mathcal{O}_{\rm chir}] \;=\; 0$$

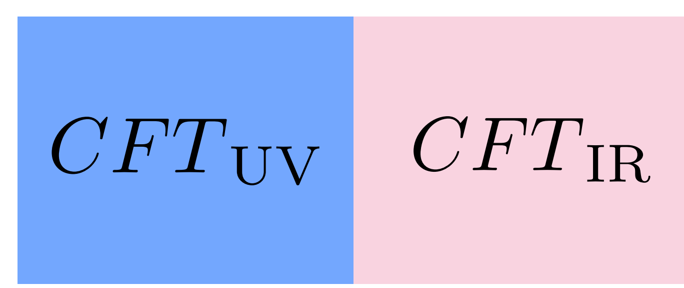
<u>Topological</u> interfaces map
- primary fields of CFT1 to those of CFT2
- D-branes of CFT1 to D-branes of CFT2

Graham, Watts '03
....

Their fusion is <u>non-singular</u>

<u>Conformal</u> interfaces also map D-branes of CFT1 to D-branes of CFT2 ;
But in general, they mix fields *of equal spin*. In some cases, they implement the mixing of operators under **bulk RG flow**.

Brunner, Roggenkamp '07
Gaiotto '12



Their fusion is in general <u>singular</u>

An interesting observation is that all **automorphisms of a CFT** (including T-dualities and mirror symmetry) are implemented by topological defects

Froehlich, Fuchs, Runkel, Schweigert '04

......

Elitzur, Karni, Rabinovici, Sarkissian '13

NB: clearly, automorphisms commute with the Virasoro algebras; What is not obvious is that they are implemented by a **local quench**

Is there some general structure, analogous to OPE, Verlinde algebra *etc* for line operators ?

Two important quantities :

- **Interface entropy**

fixed by locality, viz Cardy condition

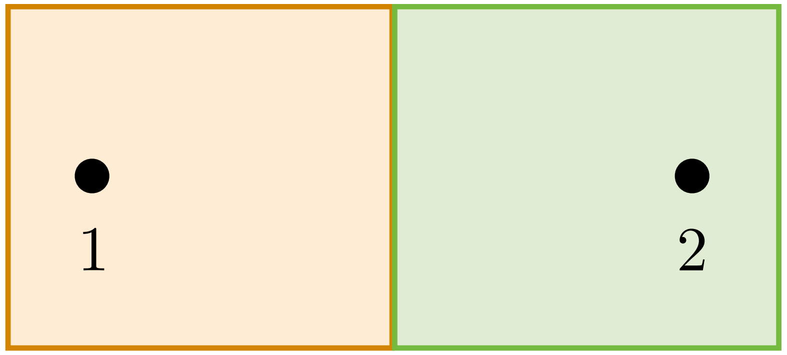
$$\log g \quad : \qquad \mathcal{O} = g\,|0\rangle\langle 0| + \cdots$$

Affleck, Ludwig '91

- **Reflection** (& transmission) **coefficient**

$$\mathcal{R} := \frac{\langle T_1\bar{T}_1 + T_2\bar{T}_2\rangle}{\langle (T_1+\bar{T}_2)(\bar{T}_1+T_2)\rangle}$$

$$\mathcal{T} = 1 - \mathcal{R}$$



$$\mathcal{R} \geq \frac{|c_1 - c_2|}{c_1 + c_2}$$

Quella, Runkel, Watts '06

Free-field examples

2d Ising Model

Folded theory is Z_2- orbifold CFT at $r = 1$

Two (families) of non-trivial defects:

- ***Dirichlet*** $|D, \phi_0\rangle\!\rangle$ with $\phi_0 \in (0, \pi)$

  defective spin-spin couplings : $\tan(\phi_0 - \frac{\pi}{4}) = \frac{\sinh(J(1-b))}{\sinh(J(1+b))}$

  $$g = 1 \ , \quad \mathcal{R} = \cos^2(2\phi_0)$$

  Topological at $\phi_0 = \frac{\pi}{4} \ , \frac{3\pi}{4} \quad \Longleftrightarrow \quad b = \pm 1$

- ***Neumann*** $|N, \tilde{\phi}_0\rangle\!\rangle$ with $\tilde{\phi}_0 \in (0, \frac{\pi}{2})$

  $$g = \sqrt{2} \ , \quad \mathcal{R} = \cos^2(2\tilde{\phi}_0)$$

  order-disorder defects (respect $Z_2 \times Z_2$ symmetry)

The fusion computed in the free-fermion representation

CB, Brunner, Roggenkamp

gluing:

$$\begin{pmatrix} \psi_{-r} \\ -i\,\bar\psi_r \end{pmatrix} \mathcal{O} = \mathcal{O}\,\Lambda \begin{pmatrix} \psi_{-r} \\ -i\,\bar\psi_r \end{pmatrix}$$

O(1,1) matrix

boost angle

spin-spin couplings : $\det\Lambda = +1$ $\qquad \pm e^{\gamma} = \cot\phi_0$

order-disorder : $\det\Lambda = -1$ $\qquad e^{\gamma} = \cot\tilde\phi_0$

Algebra:

$$(a,\Lambda)\star(a',\Lambda') = (a\times a', \Lambda\Lambda')$$

$$a \in \{1,\epsilon,\sigma\}$$

Verlinde algebra

$$\epsilon\times\epsilon = 1\ ,\quad \epsilon\times\sigma=\sigma\ ,\quad \sigma\times\sigma = 1+\epsilon$$

c=1 circle theory

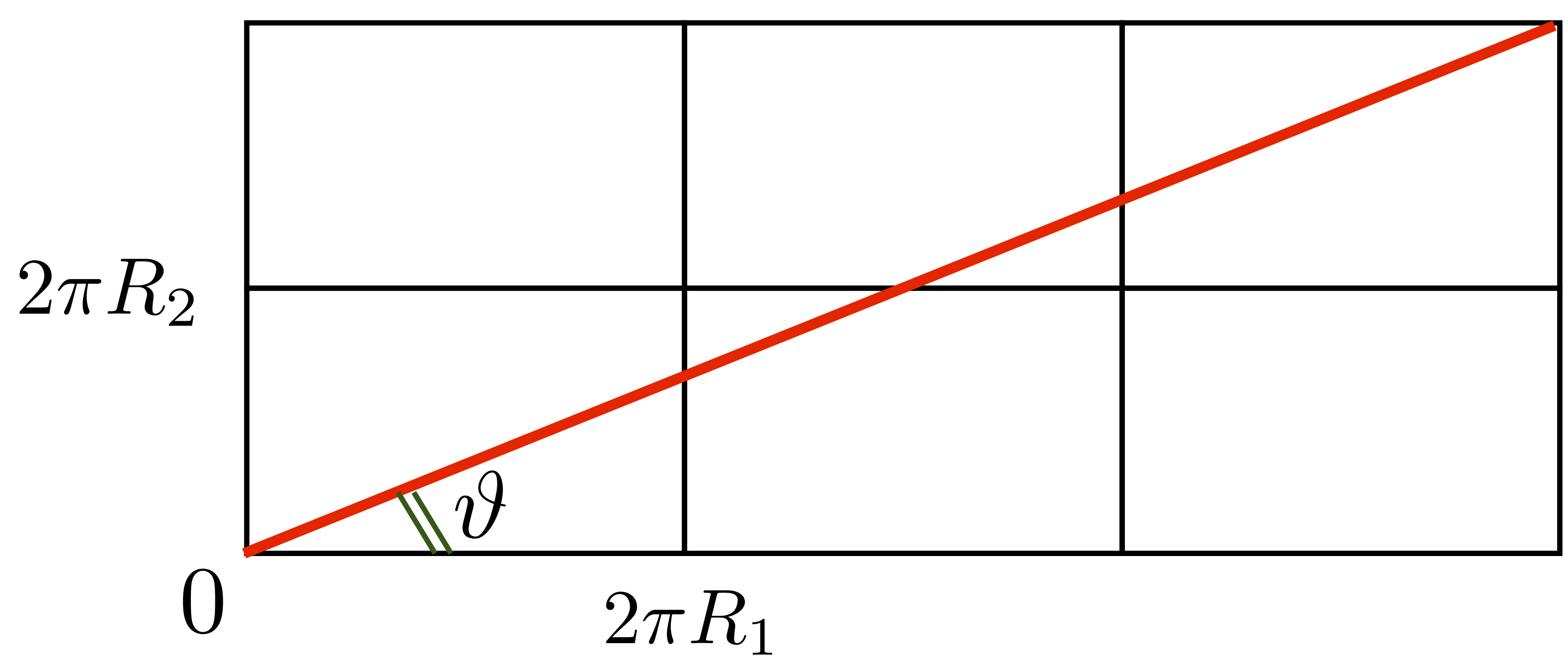
The folded theory is the **c=2** CFT with **orthogonal-torus** target

Two families of $U(1)^2$ preserving interfaces:

- ***D1-branes***

$$|n_1, n_2; R_1, R_2\rangle\!\rangle \qquad g = \sqrt{\frac{n_1^2 R_1^2 + n_2^2 R_2^2}{2R_1 R_2}} = \sqrt{\frac{n_1 n_2}{\sin(2\vartheta)}}$$

Topological if $\tan\vartheta = \dfrac{n_2 R_2}{n_1 R_1} = \pm 1$



- ***D0-D2 bound states***

  Obtained by T-dualizing one of the two dimensions, *e.g.*

$$R_1 \to \frac{1}{2R_1}$$

NB1: $\exists$ **topological interfaces** relating any $\frac{R_1}{R_2} \in Q$ or $2R_1R_2 \in Q$

They minimize g for given (n_1, n_2).

NB2: The (**group-like**) automorphism defects have $(n_1, n_2) = (\pm 1, \pm 1)$ and $g = 1$. Entropies are additive under topological fusion.

NB3: The (**“deformed identity”**) $n_1 = n_2 = 1$ interfaces transport the CFT over its moduli space, while acting trivially on the intger-charge lattice.

The fusion of these interfaces requires in general the subtraction of a divergent Casimir energy; it was computed in

CB, Brunner '07

Fuchs, Gaberdiel, Runkel, Schweigert '07

We have now computed it for arbitrary toroidal CFTs, for which the above geometric language is less useful. The technical tool are the unfolded boundary states -- they are unambiguous, but their expressions are too tedious for this talk.

NB4: There exist non-symmetric interfaces, obtained by fusing deformed identities with $SU(2)$ automorphisms of the $R = \frac{1}{\sqrt{2}}$ CFT.

The algebra of these interfaces has not been computed.

3. Rational extension of *O(d,d, Z)*

A famous table : M theory on $\mathbb{R}^{10-d} \times T^{d+1}$

10-d	U group	T group
9	$SL(2) \times O(1,1)$	$O(1,1)$
8	$SL(3) \times SL(2)$	$O(2,2)$
7	$O(5,5)$	$O(3,3)$
6	$SL(5)$	$O(4,4)$
5	$E_{6(6)}$	$O(5,5)$
4	$E_{7(7)}$	$O(6,6)$

2-derivative supergravity has continuous symmetry

$$G(\mathbb{Z}) \subset G(\mathbb{R})$$

M theory is only invariant under "integer" subgroups

Is there anything in between ?

Consider the $u(1)^{2d}$ - preserving interfaces between toroidal CFTs

<u>gluing conditions</u> (for **physical charges**):

$$\begin{pmatrix} j_r \\ -\tilde{j}_{-r} \end{pmatrix} \mathcal{O} = \mathcal{O}\, \Lambda \begin{pmatrix} j_r \\ -\tilde{j}_{-r} \end{pmatrix} \quad \text{with} \quad \Lambda \in O(d,d,\mathbb{R})$$

For **topological** interfaces: $\Lambda \in O(d) \times O(d)$

Charge lattices : $\Gamma_j = U_j\, \mathbb{Z}^{d,d}$ for $j = 1, 2$

Moduli dependence

So **integer charges** must transform by the matrix

$$\hat{\Lambda} = U_1^{-1} \Lambda U_2 \;\in\; O(d,d,\mathbb{Q})$$

Discrete data

To respect quantization of charge, the action must be preceded by a

projector onto the appropriate sublattice of charges:

$$\Pi_{\hat{\Lambda}} |\hat{\gamma}\rangle := \begin{cases} |\hat{\gamma}\rangle & \text{if} \quad \hat{\Lambda}\hat{\gamma} \in \mathbb{Z}^{d,d} \,, \\ 0 & \text{otherwise} \end{cases}$$

For $\hat{\Lambda} \in O(d,d,\mathbb{Z})$: interfaces that generate **T-duality** group.

Otherwise, the transformation is non-invertible.

Useful to define the index of the interface:

$$\mathrm{ind}(\hat{\Lambda}) = \text{Volume of sublattice unit cell}$$

Then topological interfaces have $g = \sqrt{\mathrm{ind}(\hat{\Lambda})}$

From the normalization of vertex operators, one computes that the effective string coupling constant gets rescaled as follows:

$$\frac{\lambda_c}{\sqrt{\mathrm{Vol}_d}} =: \lambda_{\mathrm{eff}} \to \lambda_{\mathrm{eff}} \sqrt{\mathrm{ind}(\hat{\Lambda})}$$

$\therefore$ Non-invertible interfaces change the effective Newton constant !

Simplest case: d=1

$$O(1,1,\mathbb{Q}) : \quad \hat\Lambda = \begin{pmatrix} n_2/n_1 & 0 \\ 0 & n_1/n_2 \end{pmatrix} \quad \text{or} \quad \hat\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} n_2/n_1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$$

includes identity

includes T duality

Fundamental-string charge lattice :

$$\Gamma_j = \left\{ \begin{pmatrix} N/2R_j + MR_j \\ -N/2R_j + MR_j \end{pmatrix} \Big| N, M \in \mathbb{Z} \right\} = \left\{ U_j \begin{pmatrix} N \\ M \end{pmatrix} \Big| N, M \in \mathbb{Z} \right\}$$

Topological condition: $\Lambda = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \in O(1) \times O(1) \Longrightarrow \dfrac{R_1}{R_2} = \left| \dfrac{n_2}{n_1} \right|$

for the $\det\Lambda = 1$ branch (and likewise for $\det\Lambda = -1$).

transform:

moduli , *integer charges,* *coupling*

$$R \to \frac{n_2}{n_1} R \qquad (N, M) \to (\frac{n_2}{n_1} N, \frac{n_1}{n_2} M) \qquad \lambda_{\text{eff}} \to \sqrt{|n_1 n_2|}\, \lambda_{\text{eff}}$$

Sublattice: $(n_1 \tilde{N}, n_2 \tilde{M})$ index = $|n_1 n_2|$

invariant:

masses, *physical charges,* *field eqs all orders in* α'

$$\left| \frac{N}{R} \pm \frac{MR}{\alpha'} \right|^2 + level \qquad (\frac{N}{R}, \frac{MR}{\alpha'}) \qquad \cdots$$

From the effective supergravity point of view :

..., Maharana-Schwarz, ...

$$S = M_{\rm Planck}^2 \int d^{10-d}x \sqrt{-g} \left[\frac{1}{8} {\rm Tr}(\partial_\mu M^{-1} \partial^\mu M) - \frac{1}{4} (F_{\mu\nu})^T (M^{-1}) F^{\mu\nu} \right] ,$$

where $M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}$ is a $2d \times 2d$ matrix

obeying $M \hat{\eta} M = \hat{\eta}$ with $\hat{\eta} = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix} \right)$.

This is invariant under the $O(d,d,\mathbb{R})$ transformations:

$$F_{\mu\nu} \mapsto \hat{\Lambda} F_{\mu\nu} \qquad M \mapsto \hat{\Lambda} M \hat{\Lambda}^T$$

with $\hat{\Lambda}^T \hat{\eta} \hat{\Lambda} = \hat{\eta}$.

M parametrizes the homogeneous coset $O(d,d,\mathbb{R})/O(d,\mathbb{R}) \times O(d,\mathbb{R})$

It can be expressed in terms of a *frame matrix*:

$$M = 2\, V^T V \;\; \leftrightarrow M^{-1} = 2\,(V\hat{\eta})^T(V\hat{\eta}) \;.$$

which introduces a gauge invariance under $O(d,\mathbb{R}) \times O(d,\mathbb{R})$ transformations

The **physical** *(canonically-normalized)* gauge fields do not transform.

$$F'_{\mu\nu} = V\hat{\eta}F_{\mu\nu}$$

U

The only unexpected feature is the rescaling of M_{Planck}

This is required for the consistent action on D-branes:

Let $$|\mathcal{B}\rangle\rangle = \sum_{\alpha=1}^{2^d} n_\alpha |\alpha\rangle\rangle$$

elementary D-branes

integer RR charges

$$(n_1, \cdots n_{2^d}) := \hat{\gamma}_D$$

then $$\hat{\gamma}_D \to \sqrt{\mathrm{ind}(\hat{\Lambda})}\, S(\hat{\Lambda})\, \hat{\gamma}_D$$

spinor matrix

puncture $\sim$ disk boundary

The transformation respects **quantization of all RR charges**

i.e. $\sqrt{\mathrm{ind}(\hat{\Lambda})}\, S(\hat{\Lambda}) \in GL(2^d, \mathbb{Z})$

↑

projective representation of the semi-group extension $\{\hat{\Lambda}\pi_{\hat{\Lambda}} | \hat{\Lambda} \in O(d,d|\mathbb{Q})\}$

In the d=1 case: $\begin{pmatrix} N_{D0} \\ N_{D_1} \end{pmatrix} \to \sqrt{|n_1 n_2|} \begin{pmatrix} \sqrt{\frac{n_2}{n_1}} & 0 \\ 0 & \sqrt{\frac{n_1}{n_2}} \end{pmatrix} \begin{pmatrix} N_{D0} \\ N_{D_1} \end{pmatrix}$

Mass = $\dfrac{g_{\mathrm{brane}}}{\lambda_{\mathrm{eff}}}$ is left invariant.

- The above transformations are special cases of **orbifold equivalences**

Bershadsky, Kakushadze, Vafa '98

They provide an exact in α' extension of the continuous supergravity symmetry $O(d, d, \mathbb{R})$.

- They describe critical behavior in quantum-wire systems.
  Could be fun to see whether they can be realized in experiment.

NB*: I have skipped explicit calculations, but here is the full interface operator in the bosonic theory:*

$$I_{12} = \prod_{n\geq 0} I_{12}^{n,\mathrm{bos}}$$

where: $$I_{12}^{n,\mathrm{bos}} = \exp\Big(\frac{1}{n}(j^1_{-n}\mathcal{O}_{11}\tilde{j}^1_{-n} - j^1_{-n}\mathcal{O}_{12}j^2_n - \tilde{j}^1_{-n}\mathcal{O}^t_{21}\tilde{j}^2_n + a^2_n\mathcal{O}^t_{22}\tilde{j}^2_n)\Big)$$

$$\Lambda := \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \qquad \mathcal{O}(\Lambda) = \begin{pmatrix} \Lambda_{12}\Lambda_{22}^{-1} & \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21} \\ \Lambda_{22}^{-1} & -\Lambda_{22}^{-1}\Lambda_{21} \end{pmatrix}.$$

$$I_{12}^{0,\mathrm{bos}} = \sqrt{\mathrm{ind}(\hat{\Lambda})\,|\Lambda_{22}|} \sum_{\hat{\gamma}\in\mathbb{Z}^{d,d}} e^{2\pi i\varphi(\hat{\gamma})}\,|\hat{\Lambda}\hat{\gamma}\rangle\langle\hat{\gamma}|\,\Pi_{\hat{\Lambda}}$$

g-factor

NB: there are analogous expressions for the type-II superstring.

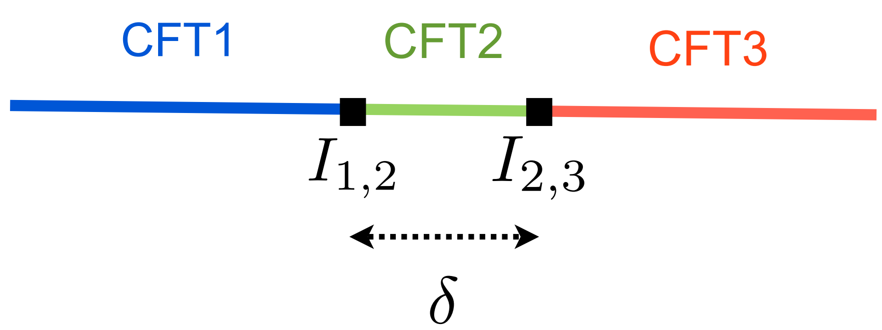
Summary

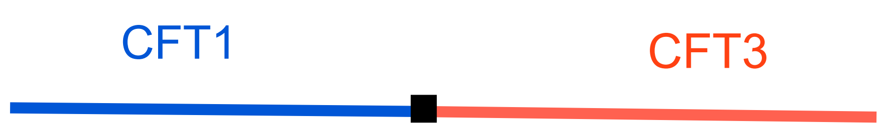
1. Conformal defects and conformal interfaces form a rich set of *non-local* observables of CFT, with applications in condensed-matter and stat. mechanics systems.

2. Their algebra gives a natural non-trivial extension of the worldsheet symmetries of string theory, *but broken by string-loop corrections.*

Whether they will play a (substantial) role in string theory remains to be seen.

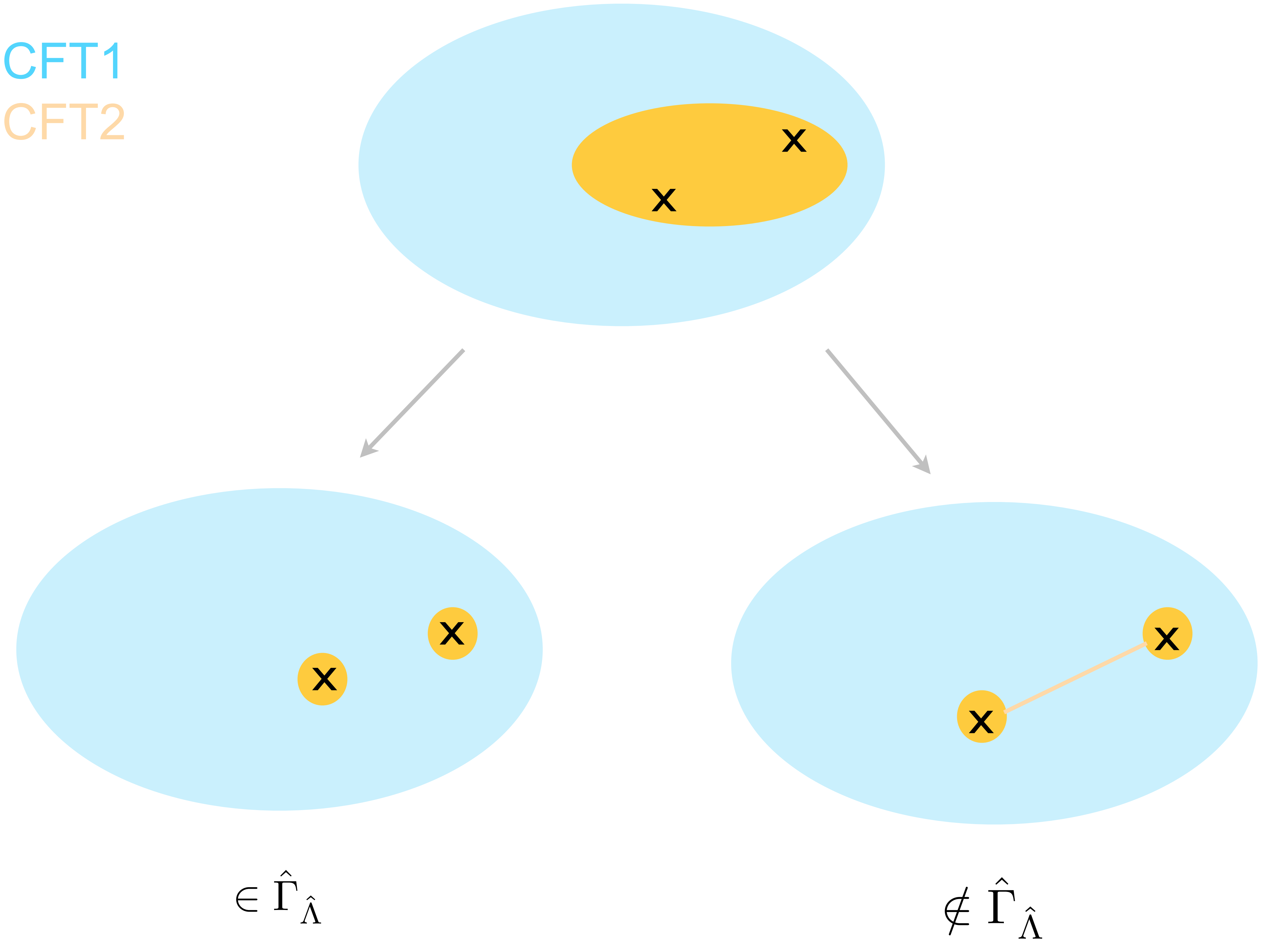
Interfaces can be added and **fused** :





$$I_{1,3} = I_{1,2} \odot I_{2,3}$$

$$I_{12} \odot I_{23} \;:=\; \lim_{\delta \to 0} \mathcal{R}_\delta [I_{12}\, e^{-\delta H}\, I_{23}]$$



CFT1
CFT2
x
x
x
x
x
x
$\in \hat{\Gamma}_{\hat{\Lambda}}$
$\notin \hat{\Gamma}_{\hat{\Lambda}}$